Math 502 Mathematical Logic
Problem Set 5

Due Monday October 12

1) Do Exercise 5.28 from the lecture notes.

2) Suppose (V,E) is a infinite graph such that every subgraph can be colored with four colors so that no adjacent verticies have the same color. Let $I = \{W \subset V : W \text{ is finite}\}$. For $W \in I$ let

$$X_W = \{W' \in I : W \subseteq W'\}.$$

Let

$$\mathcal{F} = \{Y \subseteq I : X_W \subseteq Y \text{ for some } W \in I\}.$$

a) Show that $\mathcal{F}$ is a filter on I.

Let $\mathcal{U} \supseteq \mathcal{F}$ be an ultrafilter on I. For $W \in I$ let let $c_W : W \to \{c_1, c_2, c_3, c_4\}$ be a four-coloring of the subgraph with verticies W.

b) Show that for all $v \in V$ there is a unique i such that

$$\{W \in I : v \in W \wedge c_W(v) = c_i\} \in \mathcal{U}$$

and that $v \mapsto c_i$ give a four-coloring of V.

3) Let $\mathcal{M}$ be a structure. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$ and let $\mathcal{M}^*$ be the ultraproduct $\prod_{i \in I} \mathcal{M}/\mathcal{U}$.

a) Suppose for $n \in \mathbb{N}$ we have a formula $\psi_n(v)$ and non-empty definable sets $Y_n = \{b \in \mathcal{M}^* : \mathcal{M}^* \models \psi_n(b)\}$ such that

$$Y_1 \supseteq Y_2 \supseteq Y_3 \ldots.$$

Prove that $\bigcap_{n \in \mathbb{N}} Y_n$ is non-empty.

b) This is a special property of ultraproducts. Give a simple example that in an arbitrary structure $\mathcal{N}$ the intersection of a countable descending chain of non-empty definable sets could be empty.

c)† More generally, suppose $n \in \mathbb{N}$ we have a formula $\psi_n(v, w_1, \ldots, w_{m_n})$ and $a_1, \ldots, a_{m_n} \in \mathcal{M}^*$ and the definable set

$$X_n = \{b \in \mathcal{M}^* : \mathcal{M}^* \models \psi_n(b, a_1, \ldots, a_{m_n})\}$$

is non-empty for each n. Suppose

$$X_1 \supseteq X_2 \supseteq X_3 \supseteq \ldots.$$

Prove that $\bigcap_{n \in \mathbb{N}} X_n$ is non-empty.