Math 502 Mathematical Logic
Problem Set 6

Due: Friday October 23

1) Let $\mathcal{L}_r = \{+, -, \cdot, 0, 1\}$. Recall that RCF is the $\mathcal{L}_r$-theory of real closed fields.

a) Prove that RCF is model complete, i.e., show that if $F \subseteq K$ are real closed fields, $\overline{a} \in F$, ϕ is an $\mathcal{L}_r$-formula, then $F \models \phi(\overline{a})$ if and only if $K \models \phi(\overline{a})$. [Hint: Note that with the unique orderings $(F, <) \subseteq (K, <)$ and use quantifier elimination for RCOF.]

b) Suppose $\phi(\overline{v})$ is an $\mathcal{L}_r$-formula. Prove there is a quantifier free $\mathcal{L}_r$-formula $\psi(\overline{v}, \overline{w})$ such that

$$\mathrm{RCF} \models \phi(\overline{v}) \leftrightarrow \exists \overline{w}\ \psi(\overline{v}, \overline{w}).$$

[Hint: use quantifier elimination in $\mathcal{L}_{or}$ and show every quantifier free $\mathcal{L}_{or}$-formula is equivalent to an existential $\mathcal{L}_r$-formula.]

In fact, in any model complete theory every formula is equivalent to an existential formula.

2)† (optional) Prove that for any n, d there is are m and d' such that if K is a real closed $f(X_1, \ldots, X_n)$ is rational function over K where numerator and denominator have degree at most d such that $f(\overline{x}) \geq 0$ for all x then there are rational functions $g_1, \ldots, g_m \in K(\overline{X})$ where numerator and denominator have degree at most d' such that $f = \sum g_i^2$. [Hint: Suppose not and do a compactness argument to violated Hilbert's 17th Problem].

3) Write register machine program to compute $\max(x, y)$.

4) Write a register machine program to compute

$$f(x, y) = \begin{cases} 0 & \text{if } y = 0 \\ \lfloor x/y \rfloor & \text{if } y \neq 0 \end{cases}$$

where $\lfloor r \rfloor$ is the greatest integer $\leq r$ for $r \in \mathbb{R}$.

5) a) Suppose $P(\overline{x}, y)$ is a primitive recursive predicate and $g(\overline{x})$ is a primitive recursive function. Define $f(\overline{x}) = 0$ if there is no $n \leq g(\overline{x})$ such that $P(\overline{x}, n) = 1$. Otherwise $f(\overline{x})$ is the least $n \leq g(\overline{x})$ such that $P(\overline{x}, n) = 1$. Prove that f is primitive recursive.

b) Prove that the function f from problem 3 is primitive recursive.