

Math 502 Mathematical Logic
Problem Set 7

Due: Monday November 23

Notation: ϕ_0, ϕ_1, \dots is our usual listing of partial recursive functions and $W_e = \text{dom } \phi_e$.

- 1) Let $A \subseteq \mathbb{N}$ be an infinite recursively enumerable set. Prove that there is a total recursive injective $f : \mathbb{N} \rightarrow \mathbb{N}$ such that A is the image of f .
- 2) Suppose $f : \mathbb{N} \rightarrow \mathbb{N}$ is total recursive. Prove that

$$A = \bigcup_{n \in \mathbb{N}} W_{f(n)}$$

is recursively enumerable.

- 3) a) (Reduction) Suppose A and B are recursively enumerable. Prove that there are recursively enumerable A_0 and B_0 in such that:
 - i) $A_0 \subseteq A$ and $B_0 \subseteq B$;
 - ii) $A_0 \cap B_0 = \emptyset$;
 - iii) $A_0 \cup B_0 = A \cup B$.
- b) (Separation) Suppose A and B are Π_1^0 and $A \cap B = \emptyset$. Prove that there is a recursive C such that $A \subseteq C$ and $C \cap B = \emptyset$. [HINT: Use part a)]
- c) Show that, properly formalized, we all ways have reduction for Σ_n^0 sets and separation for Π_n^0 -sets
- 4) Prove that the following sets are Σ_3^0
 - a) $Cof = \{e : \mathbb{N} \setminus W_e \text{ is finite}\}$.
 - b) $\{(i, j) : W_i \subseteq_* W_j\}$ where $A \subseteq_* B$ if $A \setminus B$ is finite.
 - c) $\{(i, j) : \text{there is a recursive } C \text{ such that } W_i \subseteq C \text{ and } W_j \cap C = \emptyset\}$
- 5) Prove that $\{e : W_e \neq \emptyset\}$ is Σ_1^0 -complete.