Math 502 Mathematical Logic
Problem Set 8

Due: Wednesday December 9
Questions marked with a dagger† are optional.

Let $\mathcal{L} = \{+, \cdot, <, 0, 1\}$.

1) Let $\phi(v_1, \ldots, v_k)$ be a Δ_0-formula and let

$$A = \{\overline{n} \in \mathbb{N}^k : \mathbb{N} \models \phi(\overline{n})\}.$$

Prove that A is a primitive recursive predicate (i.e., the characteristic function of A is primitive recursive).

2) Suppose $f(\overline{x}, y)$ is a partial function with a Σ_1-definable graph. Define $g(\overline{x})$ to be the least y such that $f(\overline{x}, y) = 0$ and $(\overline{x}, z) \in \mathrm{dom}(f)$ for all $z < y$. Using the notation from §6

$$g(\overline{x}) = \mu y f(\overline{x}, y) = 0.$$

Prove that the graph of g is Σ_1-definable. Use this to give a second proof that every partial recursive function has a Σ_1-definable graph.

3) We will outline the proof that if $f(\overline{x})$ is a primitive recursive function there is a Σ_1-formula $\phi(\overline{x}, y)$ representing f in PA^- such that

$$PA \vdash \forall \overline{x} \exists! y \; \phi(\overline{x}, y).$$

a) Show that this is true for the basic functions and that set of functions where this is true is closed under compositions.

To prove closure under primitive recursive functions we need to know that everything we do coding sequences with the β-function is provable in PA. This involve formalizing the proof of the Chinese Remainder Theorem in PA so that you can, for example prove the following statement that more or less says if I can code the sequence $(a_0, \ldots, a_{w-1})$ then for any a_w we can code the sequence $(a_0, \ldots, a_w)$.

$$\forall u \forall v \forall w \forall a \exists u' \exists v' \left(\forall i < w \beta(u, v, w)_i = \beta(u', v', w+1)_i \wedge \beta(u', v', w+1)_w = a \right).$$

Suppose

$$f(\overline{x},0)=g(\overline{x})$$

$$f(\overline{x},y+1)=h(\overline{x},y,f(\overline{x},y))$$

and we have already constructed the formulas ϕ_g and ϕ_h needed for g and h.

Say that $(a_0,\ldots a_w)$ is a *computation sequence* for $f(\overline{x},w)$ if $\phi_g(\overline{x},a_0)$ and $\phi_h(\overline{x},i,a_i,a_{i+1})$ for all $i<w$, i.e., $a_i=f(\overline{x},i)$

c) Show that

$PA\vdash \forall\overline{x}\forall w\exists u\exists v\ \beta(u,v,w+1)$ is a computation sequence for $f(\overline{x},w)$

and

$PA\vdash \forall u\forall v\forall w\forall\overline{x}\forall u'\forall v'\forall w'$ if $\beta(u,v,w+1)$ and $\beta(u',v',w'+1)$ are computation sequences for $f(\overline{x},w)$ and $f(\overline{x},w')$ respectively and $w\leq w'$, then $\forall i<w\ \beta(u,v,w+1)_i=\beta(u',v',w'+1)_i$.

i.e., PA proves that there are unique computation sequences of every length and the sequence of length w' extends the sequence of length w for $w<w'$.

d) Find a Σ_1-formula representing $f(\overline{x},y)=z$ by saying there is a computation sequence $(a_0,\ldots,a_y)$ and $a_y=z$. Prove this formula has the desired property.

4)† Show there is a Σ_1-formula ψ representing the Ackermann function such that $PA\vdash\forall x\exists!y\psi(x,y)$.