Due: Wednesday September 28

Problems marker with a † are more difficult or require a bit more algebra and optional.

1) Let $T$ and $T'$ be $L$-theories with $T' \subseteq T$. We say that $T'$ axiomatizes $T$ if every model of $T'$ is a model of $T$. Suppose $T'$ axiomatizes $T$. Prove that $T \models \phi$ if and only if $T' \models \phi$ for all $L$-sentences $\phi$.

2) Let $\phi$ be a formula where the only free variable is $v$. Let $\psi$ be a sentence.
[Recall that if $\Gamma$ is a set of formulas. Then $\Gamma \models \phi$ if and only if whenever $M$ is a structure and $\sigma$ is an assignment making all of the sentences in $\Gamma$ true, then $M \models \sigma(\phi)$.

a) Show that if $\psi \models \phi$ then $\psi \models \forall v \phi$.

b) Suppose $\phi \models \psi$ then $\exists v \phi \models \psi$.

3) Let $L = \{ +, 0 \}$.

a) Prove that $\text{Th}(\mathbb{Z}) \neq \text{Th}(\mathbb{Q})$.

b)† Prove that $\text{Th}(\mathbb{Z}) \neq \text{Th}(\mathbb{Z} \oplus \mathbb{Z})$.

4) a) Let $L$ be the language $\{ +, 0 \}$ and consider the structure $R$ with universe $\mathbb{R}$ where $+$ is interpreted as the usual addition and $0$ as zero. Show that there is no formula $\phi(v, w)$ such that $R \models \phi(a, b)$ if and only if $a < b$ for all $a, b \in \mathbb{R}$. [Hint: Find $a$ and $b$ and an automorphism $F$ of $\mathbb{R}$ such that $a < b$ but $F(a) > F(b)$.

b) Let $L$ be the language of rings $\{ +, \cdot, 0, 1 \}$ and let $R$ be $\mathbb{R}$ with the usual interpretation. Find a formula $\phi(v, w)$ such that $a < b$ iff $R \models \phi(a, b)$.

5) If $\phi$ is a sentence, the spectrum of $\phi$ is the set of all natural numbers $n$ such that there is a model of $\phi$ with exactly $n$ elements.

a) Let $L = \{ E \}$ where $E$ is a binary relation. Write down a sentence $\phi$ asserting that $E$ is an equivalence relation and every equivalence class has exactly three elements. Show that the spectrum of $\phi$ is $\{ n > 0 : 3 \text{ divides } n \}$.
b) Let $\mathcal{L} = \{P, Q, f\}$ where $P$ and $Q$ are unary predicates and $f$ is a binary function. Let $\phi$ be the conjunction of:

\[ \exists x \exists y \ x \neq y \land P(x) \land P(y) \]
\[ \exists x \exists y \ x \neq y \land Q(x) \land Q(y) \]
\[ \forall z \exists x \exists y \ P(x) \land Q(y) \land f(x, y) = z \]
\[ \forall x_1 \forall x_2 \forall y_1 \forall y_2 \ [(P(x_1) \land P(x_2) \land Q(y_1) \land Q(y_2) \land f(x_1, y_1) = f(x_2, y_2)) \rightarrow (x_1 = x_2 \land y_1 = y_2)] \]

Show that the spectrum of $\phi = \{n > 3 : n \text{ is not prime}\}$.\(^1\)

c) Find a sentence with the spectrum $\{n > 0 : n \text{ is a square}\}$.

d)† Find a sentence with the spectrum $\{p^n : p \text{ prime } n > 0\}$.

e)†† Find a sentence with spectrum $\{p : p \text{ is prime}\}$.

f)††† Prove that $X \subseteq \mathbb{N} \setminus \{0\}$ is a spectrum if and only if $X$ is in recognizable in nondeterministic exponential time.\(^2\)

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\(^1\)Another sentence with the same spectrum is the sentence in the language of rings asserting that we have a commutative ring with zero divisors.

\(^2\)The question about whether the complement of a spectrum is a spectrum is an old open question. By this result it is equivalent to the question about whether nondeterministic exponential time is co-nondeterministic exponential time.