Math 502 Metamathematics I
Problem Set 1

Due: Friday September 11

1) We say that a formula ϕ is *existential* if it is of the form

$$\exists v_{i_1}\exists v_{i_2}\ldots\exists v_{i_n}\ \psi$$

where ψ is quantifier free and we say that ϕ is *universal* if it is of the form

$$\forall v_{i_1}\forall v_{i_2}\ldots\forall v_{i_n}\ \psi.$$

where ψ is quantifier free.

Suppose $j:\mathcal{M}\to\mathcal{N}$ is an embedding and $\sigma:V\to M$ is an assignment.

a) Show that $\mathcal{M}\models_\sigma \phi \Rightarrow \mathcal{N}\models_{j\circ\sigma}\phi$ for all existential ϕ.

b) Show that $\mathcal{N}\models_{j\circ\sigma}\phi \Rightarrow \mathcal{M}\models_\sigma \phi$ for all universal ϕ.

c) Give examples showing than neither a) nor b) can be strengthened to $\Leftrightarrow$.

2) The set of *positive* $\mathcal{L}$-formulas is the smallest set P of formulas containing the atomic formulas such that:

i) if ϕ and ψ are in P so are $\phi\wedge\psi$ and $\phi\vee\psi$;

ii) if ϕ is in P, the so are $\exists v_i\ \phi$ and $\forall v_i\ \phi$.

a) Suppose $j:\mathcal{M}\to\mathcal{N}$ is a surjective homomorphism. Show that

$$\mathcal{M}\models_\sigma \phi \Rightarrow \mathcal{N}\models_{j\circ\sigma}\phi$$

for all positive formulas ϕ and all assignment $\sigma:V\to M$.

b) Give an example showing that a) can not be strengthened to $\Leftrightarrow$.

c) Give an example showing that we need the assumption that j is surjective in a).

d) Suppose G is a divisible abelian group [see Example 2.5 in the notes] and $j:G\to H$ is a surjective homomorphism. Use a) to show that H is also a divisible abelian group.