Math 502 Metamathematics I
Problem Set 3

Due: Friday September 18

Do Exercises 2.16 and 2.19 from the notes.

Problems marked with a dagger (†) are either more difficult or require more algebra. They are optional.

1) Let $\mathcal{L} = \{+, 0\}$.
a)Prove that $\mathrm{Th}(\mathbb{Z}) \neq \mathrm{Th}(\mathbb{Q})$.
b)† Prove that $\mathrm{Th}(\mathbb{Z}) \neq \mathrm{Th}(\mathbb{Z} \oplus \mathbb{Z})$.

2) Let $\mathcal{L}$ be the language $\{+, 0\}$ and consider the structure $(\mathbb{R}, +, 0)$. Show that there is no formula $\phi(v, w)$ such that $\mathbb{R} \models \phi(a, b)$ if and only if $a < b$ for all $a, b \in \mathbb{R}$. [**Hint:** Find a and b and an automorphism F of $\mathbb{R}$ such that $a < b$ but $F(a) > F(b)$.][1]

3) If ϕ is a sentence, the *spectrum* of ϕ is the set of all natural numbers n such that there is a model of ϕ with exactly n elements.

a) Let $\mathcal{L} = \{E\}$ where E is a binary relation. Write down a sentence ϕ asserting that E is an equivalence relation and every equivalence class has exactly three elements. Show that the spectrum of ϕ is $\{n > 0\colon 3 \text{ divides } n\}$.

b) Let $\mathcal{L} = \{P, Q, f\}$ where P and Q are unary predicates and f is a binary function. Let ϕ be the conjuction of:
$\exists x \exists y\ x \neq y \wedge P(x) \wedge P(y)$
$\exists x \exists y\ x \neq y \wedge Q(x) \wedge Q(y)$
$\forall z \exists x \exists y\ P(x) \wedge Q(y) \wedge f(x, y) = z$
$\forall x_1 \forall x_2 \forall y_1 \forall y_2\ [(P(x_1) \wedge P(x_2) \wedge Q(y_1) \wedge Q(y_2) \wedge f(x_1, y_1) = f(x_2, y_2)) \rightarrow (x_1 = x_2 \wedge y_1 = y_2)]$

Show that the spectrum of ϕ is $\{n > 3 : n \text{ is not prime}\}$.[Note: Another sentence with the same spectrum is the sentence in the language of rings asserting that we have a commutative ring **with** zero divisors.]

[1]Note that in $(\mathbb{R}, +, \cdot, 0, 1)$ we can define $<$ since

$$x < y \Leftrightarrow \exists z\ (z \neq 0 \wedge x + z^2 = y).$$

c) Find a sentence with the spectrum $\{n > 0 : n \text{ is a square }\}$.

d) Find a sentence with the specturm $\{p^n : p \text{ prime } n > 0\}$.

e)†† Find a sentence with spectrum $\{p : p \text{ is prime}\}$.

Open Problem: Prove that if X is a spectrum so is $\{n \in \mathbb{N} : n > 0 \text{ and } n \notin X\}$.

For those of you with a background in computational complexity theory might be interested in trying to prove that $X \subseteq \mathbb{N} \setminus \{0\}$ is a spectrum if and only if X is in recognizable in nondeterministic exponential time. The open problem is then equivalent to the question about whether nondeteministic exponential time is co-nondeteministic exponential time.