Math 502 Metamathematics I
Problem Set 5

Due: Friday October 8

1) Let $\mathcal{L} = \{E\}$ where $E$ is a binary relation. Let $T_0$ be the axioms for equivalence relations. Suppose $T \supseteq T_0$ is an $\mathcal{L}$-theory such that for all $n$ there is $\mathcal{M} \models T$ with an equivalence class of size at least $n$. Then there is $\mathcal{M} \models T$ with an infinite equivalence class. Conclude that the class of all equivalence relations where every class is finite is not an elementary class.

2) Let $\mathcal{L} = \{R\}$ where $R$ is a binary relation. Recall that a graph is an $\mathcal{L}$-structure $\mathcal{M}$ where $R^\mathcal{M}$ is symmetric and irreflexive. We say that a graph is connected if for each $x \neq y$ we can find a path from $x$ to $y$. Prove that the class of connected graphs is not an elementary class.

3) Let $\mathcal{L}$ be the language with one binary relation symbol $\prec$. Let $T$ be an $\mathcal{L}$-theory extending the theory of linear orders such that $T$ has infinite models. Show that there is $\mathcal{M} \models T$ and an order preserving embedding $\sigma : \mathbb{Q} \to \mathcal{M}$ of the rational numbers into $\mathcal{M}$. For example there is $\mathcal{M} \models \text{Th}(\mathbb{Z}, \prec)$ in which the rational order embeds. [Hint: Add constants $c_q$ for all $q \in \mathbb{Q}$ and let $T^* = T \cup \{c_q < c_r : q, r \in \mathbb{Q}, q < r\}$.]

4) We say that a group $(G, +, \prec)$ is archimedian if for all $x, y \in G$ with $x, y > 0$ there is a positive integer $m$ such that $x \prec my$. Show that there are non-archimedian models of $\text{Th}(\mathbb{Q}, +, \prec)$.

5) Let $\mathcal{L} = \{s\}$, where $s$ is a unary function symbol. Let $T$ be the $\mathcal{L}$-theory that asserts that $s$ is a bijection with no cycles (i.e., $s^{(n)}(x) \neq x$ for $n = 1, 2, \ldots$). For which cardinals $\kappa$ is $T \kappa$-categorical? Is $T$ complete?