Problem Set 8

Due: Friday November 20

Questions marked with a dagger† are optional.

Recall that $\phi_0, \phi_1, \ldots$ is our usual listing of partial recursive functions and $W_e = \text{dom } \phi_e$.

1) Let $A \subseteq \mathbb{N}$ be an infinite recursively enumerable set. Prove that there is a total recursive injective $f : \mathbb{N} \to \mathbb{N}$ such that $A$ is the image of $f$.

2) Suppose $f : \mathbb{N} \to \mathbb{N}$ is total recursive. Prove that $A = \bigcup_{n \in \mathbb{N}} W_{f(n)}$ is recursively enumerable.

3)† a) (Reduction) Suppose $A$ and $B$ are $\Sigma^0_n$. Prove that there are $A_0$ and $B_0$ in $\Sigma^0_n$ such that:
   i) $A_0 \subseteq A$ and $B_0 \subseteq B$;
   ii) $A_0 \cap B_0 = \emptyset$;
   iii) $A_0 \cup B_0 = A \cup B$.

   b) (Separation) Suppose $A$ and $B$ are $\Pi^0_n$ and $A \cap B = \emptyset$. Prove that there is $C \in \Delta^0_n$ such that $A \subseteq C$ and $C \cap B = \emptyset$. [HINT: Use part a)]

4) Let $\text{Cof} = \{e : \neg W_e \text{ is finite}\}$. Prove that $\text{Cof}$ is $\Sigma^0_3$.

5) Prove that $\{e : W_e \neq \emptyset\}$ is $\Sigma^0_1$-complete.

Two more Compactness Problems These two problems must be turned in by the end of the semester. For the following problems let $\mathcal{L} = \{+, \cdot, 0, 1\}$

C1) Prove that there is $\mathcal{M} \models \text{Th}(\mathbb{N})$ and $a \in \mathcal{M}$ such that $\mathcal{M} \models p \text{ divides } a$

for every prime $p \in \mathbb{N}$.

C2) a) Let $T = \{\phi : \text{if } F \text{ is a finite field, then } F \models \phi\}$. Prove there is $K \models T$ where $K$ has characteristic zero.

   b)†† Prove that if $K \models T$, then for each $d$ there is a unique extension $L \supseteq K$ such that the degree of $L/K$ is $d$.