Math 504 Set Theory I
Problem Set 2

Due Friday February 1

1) Suppose X is a transitive set. Prove that (X, ϵ) is a model of the axiom of extensionality.

2) Let C be a set of ordinals and let $\beta = \bigcup_{\alpha \in C} \alpha$.

a) Prove that β is an ordinal.
b) Prove that $\alpha \leq \beta$ for all $\alpha \in C$.
c) Suppose γ is an ordinal such that $\alpha \leq \gamma$ for all $\alpha \in C$. Prove that $\beta \leq \gamma$.

We have shown that β is the least upper bound for the set C. We write $\beta = \sup C$.

3) For ordinals α and β we inductively define $\alpha + \beta$ as follows.
i) $\alpha + 0 = \alpha$;
ii) $\alpha + (\beta + 1) = (\alpha + \beta) + 1$;
iii) $\alpha + \beta = \sup\{\alpha + \gamma : \gamma < \beta\}$ for β a limit ordinal.

a) Is $1 + \omega = \omega + 1$?
b) Prove that $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$ for all ordinals α, β, γ.
c) Let α and β be ordinals. Let $X = (\{0\} \times \alpha) \cup (\{1\} \times \beta)$ and order X lexicographically, i.e.,

$$(m, x) < (n, y) \leftrightarrow m < n \text{ or } m = n \text{ and } x < y.$$

Prove that X has order type $\alpha + \beta$.