Math 504 Set Theory I
Problem Set 3

Due Monday February 11

1) Recall that we defined the function $\alpha \mapsto \aleph_\alpha$ as follows:

$$\aleph_\alpha = \begin{cases} \omega & \text{if } \alpha = 0 \\ \aleph_\beta^+ & \text{if } \alpha = \beta + 1 \\ \sup\{\aleph_\beta : \beta < \alpha\} & \text{if } \alpha \text{ is a limit ordinal} \end{cases} .$$

a) Prove that for every cardinal κ there is an ordinal α with $\kappa = \aleph_\alpha$.

b) Suppose κ is weakly inaccessible (i.e., κ is a regular limit cardinal), then $\kappa = \aleph_\kappa$.

2) Prove that if $f : A \to B$ is surjective, there is an injective $g : B \to A$ such that $f \circ g$ is the identity.

3) Prove that

$$^{A}(^{B}C) \approx {}^{(A \times B)}C.$$

Conclude that $(\kappa^\lambda)^\mu = \kappa^{\lambda\mu}$ for all cardinals κ, λ, μ. [Hint: If $f : A \times B \to C$, let $G_f : A \to^B C$ such that

$$G_f(a)(b) = f(a, b).$$

Prove that $f \mapsto G_f$ is a bijection.]

4) Prove the following rules for cardinal exponentiation:

i) If $\alpha \le \beta$, then $\aleph_\alpha^{\aleph_\beta} = 2^{\aleph_\beta}$.

ii) If there is $\gamma < \alpha$ with $\aleph_\gamma^{\aleph_\beta} \ge \aleph_\alpha$, then $\aleph_\alpha^{\aleph_\beta} = \aleph_\gamma^{\aleph_\beta}$.

iii) If $\alpha > \beta$, $\aleph_\gamma^{\aleph_\beta} < \aleph_\alpha$ for all $\gamma < \alpha$ and either $\aleph_\alpha$ is regular or $\mathrm{cf}(\aleph_\alpha) > \aleph_\beta$, then

$$\aleph_\alpha^{\aleph_\beta} = \aleph_\alpha.$$

iv) If $\alpha > \beta$, $\aleph_\gamma^{\aleph_\beta} < \aleph_\alpha$ for all $\gamma < \alpha$ and $\mathrm{cf}(\aleph_\alpha) \le \aleph_\beta < \aleph_\alpha$, then

$$\aleph_\alpha^{\aleph_\beta} = \aleph_\alpha^{\mathrm{cf}(\aleph_\alpha)}.$$