Due Monday February 11

1) Recall that we defined the function $\alpha \mapsto \aleph_\alpha$ as follows:

$$
\aleph_\alpha = \begin{cases} 
\omega & \text{if } \alpha = 0 \\
\aleph_\beta^+ & \text{if } \alpha = \beta + 1 \\
\sup\{\aleph_\beta : \beta < \alpha\} & \text{if } \alpha \text{ is a limit ordinal}
\end{cases}
$$

a) Prove that for every cardinal $\kappa$ there is an ordinal $\alpha$ with $\kappa = \aleph_\alpha$.

b) Suppose $\kappa$ is weakly inaccessible (i.e., $\kappa$ is a regular limit cardinal), then $\kappa = \aleph_\kappa$.

2) Prove that if $f : A \to B$ is surjective, there is an injective $g : B \to A$ such that $f \circ g$ is the identity.

3) Prove that $A^{(B \times C)} \approx (A \times B)^C$.

Conclude that $(\kappa^\lambda)^\mu = \kappa^{\lambda\mu}$ for all cardinals $\kappa, \lambda, \mu$. [Hint: If $f : A \times B \to C$, let $G_f : A \to B \times C$ such that $G_f(a)(b) = f(a, b)$.

Prove that $f \mapsto G_f$ is a bijection.]

4) Prove the following rules for cardinal exponentiation:

i) If $\alpha \leq \beta$, then $\aleph_\alpha^{\aleph_\beta} = 2^{\aleph_\beta}$.

ii) If there is $\gamma < \alpha$ with $\aleph_\gamma^{\aleph_\beta} \geq \aleph_\alpha$, then $\aleph_\alpha^{\aleph_\beta} = \aleph_\gamma^{\aleph_\beta}$.

iii) If $\alpha > \beta$, $\aleph_\gamma^{\aleph_\beta} < \aleph_\alpha$ for all $\gamma < \alpha$ and either $\aleph_\alpha$ is regular or $\text{cf}(\aleph_\alpha) > \aleph_\beta$, then $\aleph_\alpha^{\aleph_\beta} = \aleph_\alpha$.

iv) If $\alpha > \beta$, $\aleph_\gamma^{\aleph_\beta} < \aleph_\alpha$ for all $\gamma < \alpha$ and $\text{cf}(\aleph_\alpha) \leq \aleph_\beta < \aleph_\alpha$, then $\aleph_\alpha^{\aleph_\beta} = \aleph_\alpha^{\text{cf}(\aleph_\alpha)}$. 

1