Math 504 Set Theory I
Problem Set 4

Due Monday February 25

1) (ZF^-) Suppose $\mathcal{M}$ is a class and for all x if $x \subseteq \mathcal{M}$, then $x \in \mathcal{M}$. Prove that $\mathbb{W}F \subseteq \mathcal{M}$. [Hint: Prove $\mathbb{V}_\alpha \subseteq \mathcal{M}$ for all α.]

2) (ZFC^-) Let R be a binary relation on X. Prove that R is well founded if and only if there is no $f : \omega \to X$ such that $f(n+1)\ R\ f(n)$ for all $n \in \omega$.

3) (ZFC) For any infinite cardinal κ, let $H(\kappa) = \{x : |TC(x)| < \kappa\}$, where $TC(x)$ is the transitive closure of X. [i.e., $TC(x)$ is the smallest transitive y with $x \subseteq y$.] If $x \in H(\kappa)$, we say x is *hereditarily of cardinality* $< \kappa$.

a) Prove that if $y \in TC(x)$, then $TC(y) \subseteq TC(x)$.

b) Show that $H(\kappa)$ is transitive.

c) Show that $H(\kappa) \subseteq \mathbb{V}_\kappa$, for any infinite cardinal κ. [Hint: For $x \in H(\kappa)$, let $\beta = \{\text{rank}(y) : y \in TC(x)\}$. Show that β is an ordinal, $\beta < \kappa$ and $\beta = \text{rank}(x)$.]

d) Show $H(\aleph_0) = \mathbb{V}_\omega$.

e) Show $H(\aleph_1) \neq \mathbb{V}_{\omega_1}$.

f)We say that κ is *strongly inaccessible* if $\kappa > \aleph_0$, κ is regular and $2^\lambda < \kappa$ for all $\lambda < \kappa$.

Suppose κ is regular. Prove that $H(\kappa) = \mathbb{V}_\kappa$ if and only if $\kappa = \aleph_0$ or κ is strongly inaccessible.

4) Let $\mathcal{M}$ be a transitive class. Suppose $\psi(x, y, \bar{z})$ is absolute for $\mathcal{M}$. Let $\phi(y, \bar{z})$ be

$$\forall x \in y\ \psi(x, y, \bar{z}).$$

Prove that ϕ is absolute for $\mathcal{M}$.

5) The collection of Σ_1-formulas is the smallest collection of formulas containing the Δ_0-formulas such that:

i) if ϕ is Σ_1, then $\exists v_i \phi$ is Σ_1;

ii) if ϕ is Σ_1, then $\forall v_i \in v_j \phi$ is Σ_1;

iii) if ϕ and ψ are in Σ_1, then $\phi \wedge \psi$ and $\phi \vee \psi$ are in Σ_1.

Similarly we the Π_1-formulas is the smallest collection of formulas containing the Δ_0-formulas such that:

i) if ϕ is Π_1, then $\forall v_i \phi$ is Π_1;

ii) if ϕ is Π_1, then $\exists v_i \in v_j \phi$ is Π_1;

iii) if ϕ and ψ are in Π_1, then $\phi \wedge \psi$ and $\phi \vee \psi$ are in Π_1.

Note that if $\phi \in \Sigma_1$, then $\neg\phi$ is equivalent to a Π_1-formula.

a) Suppose $\mathcal{M}$ is a transive class, $\bar{a} \in \mathcal{M}$ and $\phi(\bar{v})$ is Σ_1. Show that if $\mathcal{M} \models \phi(\bar{a})$, then $\mathbb{V} \models \phi(\bar{a})$.

b) Suppose $\mathcal{M}$ is a transive class, $\bar{a} \in \mathcal{M}$ and $\phi(\bar{v})$ is Π_1. Show that if $\mathbb{V} \models \phi(\bar{a})$, then $\mathcal{M} \models \phi(\bar{a})$.

We say that a formula ϕ is Δ_1^T if there is a Σ_1-formula ψ_1 and a Π_1-formula ψ_2 such that

$$T \vdash \phi \leftrightarrow \psi_1 \leftrightarrow \psi_2.$$

c) Prove that if $\mathcal{M}$ is a transitive class and $\mathcal{M}, \mathbb{V} \models T$, then any Δ_1^T-formula is absolute for $\mathcal{M}$.

d) Using c) prove that "R is a well order of A" is absolute for transitive models of ZF-P.