Math 504 Set Theory I
Problem Set 7

Due Friday April 4

1) a) Suppose $\mathbb{L} \models \kappa$ is a cardinal. Show that $\mathbb{L}_\kappa \models$ comprehension.

b) Suppose $\mathbb{L} \models \kappa$ is a regular cardinal. Then $\mathbb{L}_\kappa \models$ Replacement.

c) Suppose $\mathbb{L} \models \kappa$ is a limit cardinal. Then $\mathbb{L}_\kappa \models$ Power Set.

2) Assume $\mathbb{V} = \mathbb{L}$. Let $X \prec \mathbb{L}_{\omega_1}$.

a) Show $\omega \in X$.

Fix $x \in X$.

b) Show that there is $f \in \mathbb{L}_{\omega_1}$ such that $f : \omega \to x$ is onto.

c) Let $f : \omega \to x$ be $<_{\mathbb{L}}$-least in $\mathbb{L}_{\omega_1}$ such that f is surjective. Argue that $f \in X$ and conclude that $x \subseteq X$.

d) Conclude that X is transitive and must be equal to $\mathbb{L}_\alpha$ for some $\alpha \leq \omega_1$. [Hint: Recall that the Mostowski Collapse is the identity on transitive sets.]

Note: The Condensation Lemma tells us that $X \cong \mathbb{L}_\beta$ for some $\beta \leq \omega_1$. But we have, in this case, proved the much stronger result that X is already some $\mathbb{L}_\beta$.