Math 506 Model Theory I
Problem Set 1

Due: Friday September 5

Read: For review you should read Chapter 1 and Section §2.1 of the text

1) Suppose T has arbitrarily large finite models. Show that T has an infinite model.

2) Let $\mathcal{L} = \{E\}$ where E is a binary relation. Let T_0 be the axioms for equivalence relations. Suppose $T \supseteq T_0$ is an $\mathcal{L}$-theory such that for all n there is $\mathcal{M} \models T$ with an equivalence class of size at least n. Then there is $\mathcal{M} \models T$ with an infinite equivalence class. Conclude that the class of all equivalence relations where every class is finite is not an elementary class.

3) Let $\mathcal{L} = \{R\}$ where R is a binary relation. Recall that a *graph* is an $\mathcal{L}$-structure $\mathcal{M}$ where $R^{\mathcal{M}}$ is symmetric and irreflexive. We say that a graph is *connected* if for each $x \neq y$ we can find a path from x to y. Prove that the class of connected graphs is not an elementary class.

4) Let $\mathcal{L} = \{+, \cdot, <, 0, 1\}$. Prove there is $\mathcal{M} \models \mathrm{Th}(\mathbb{N})$ with $a_1, a_2, \ldots \in M$ such that $a_1 >^{\mathcal{M}} a_2 >^{\mathcal{M}} a_3 > \ldots$.

5) We say that a group $(G, +, <)$ is *archimedian* if for all $x, y \in G$ with $x, y > 0$ there is an integer m such that $|x| < m|y|$. Show that there are non-archimedian fields elementarily equivalent to the field of real numbers.

6) Let $\mathcal{L}$ be the language with one binary relation symbol $<$. Let T be an $\mathcal{L}$-theory extending the theory of linear orders such that T has infinite models. Show that there is $\mathcal{M} \models T$ and an order preserving embedding $\sigma : \mathbb{Q} \to M$ of the rational numbers into M. For example there is $\mathcal{M} \models \mathrm{Th}(\mathbb{Z}, <)$ in which the rational order embeds.[Hint: Add constants c_q for all $q \in \mathbb{Q}$ and consider $T^* = T \cup \{c_q < c_r : q, r \in \mathbb{Q}, q < r\}$.]