Math 506 Model Theory I
Problem Set 4

Due Friday March 4

From the Text: 2.5.17, 3.4.12, 3.4.13, 3.4.20, 3.4.21

The following problems are optional bonus problems. [may be turned in any time]

1) Show that for each uncountable κ there are 2^κ non-isomorphic linear orders of cardinality κ.

We fix two countable linear orders $B \cong \mathbb{Q} + 2 + \mathbb{Q}$ and $C \cong \mathbb{Q} + 3 + \mathbb{Q}$ [i.e., B looks like a DLO followed by two points followed by a DLO].

Suppose $X \subseteq \kappa$. For $\alpha < \kappa$ let A^X_α be B if $\alpha \in X$ and C if $\alpha \notin X$. Let

$$A^X = \bigcup_{\alpha<\kappa} \alpha \times A^X_\alpha$$

which we order lexicographically (i.e., $(\alpha, a) < (\beta, b)$ if $\alpha < \beta$ or ($\alpha = \beta$ and $a < b$)).

Prove that if $X \neq Y$ then $A^X \ncong A^Y$.

2) Let G be a divisible ordered abelian group. Define $\sim$ on $\{g \in G : g > 0\}$ by $g \sim h$ if and only if there are $n, m \in \mathbb{N}$ such that $g < mh$ and $h < ng$.

a) Prove that $\sim$ is an equivalence relation and that if $0 < g < h$, $g_1 \sim g$, and $h_1 \sim h$, then $g_1 < h_1$. Thus there is a natural order on the equivalence classes of G.

b) Let L be the set of equivalence classes. We call $(L, <)$ the ladder of G. Prove that if G, H are isomorphic ordered abelian groups, then the corresponding ladders are isomorphic.

c) Let $(L, <)$ be any non-empty linear ordering. Let $G(L)$ be the $\mathbb{Q}$-vector space with basis $(x_l : l \in L)$. We can order $G(L)$. Suppose $x \in G(L)$ be nonzero. We can write

$$x = n_1 x_{l_1} + \ldots + n_m x_{l_m}$$

where $n_i \in \mathbb{Q} \setminus \{0\}$ and $l_1 > l_2 > \ldots > l_m$. Prove that $(G(L), <)$ is an ordered divisible abelian group.

d) Prove that the ladder of $G(L)$ is isomorphic to L. Thus $G(L) \cong G(L_1)$ if and only if $L \cong L_1$.

e) Conclude that for each uncountable cardinal κ there are 2^κ non-isomorphic divisible ordered groups of cardinality κ.