Math 506 Model Theory I
Problem Set 5

Due Friday March 18

From the Text: 4.5.2, 4.5.6, 4.5.11

1) Consider the dense linear order \((\mathbb{Q}, <)\).

Let \(A = \{0, 1, 2, \ldots\} \cup \{\frac{1}{2}, \frac{1}{3}, \ldots\}\).

a) Which types in \(S^\mathbb{Q}_1(A)\) are isolated?

b) Describe the elements of \(S^\mathbb{Q}_2(\mathbb{N})\).

The following problems are optional bonus problems. [may be turned in any time]

B1) Show that under Martin’s Axiom, if \(T\) is a theory in a countable language and \(p_i\) is an isolated type for each \(i \in I\) where \(|I| < 2^{\aleph_0}\), then there is a model of \(T\) omitting each \(p_i\). (see Ex. 4.5.14)

B2) Suppose \(K\) is a formally real field. We say that \(I \subset K[X_1, \ldots, X_n]\) is real if whenever \(f_1^2 + \ldots + f_m^2 \in I\), then \(f_1, \ldots, f_m \in I\).

a) Prove that if \(I\) is a real prime ideal, then the fraction field of \(K[X_1, \ldots, X_n]/I\) is a formally real field.

b) Prove that if \(K\) is real closed and \(I\) is a real prime ideal, then \(V_K(I) = \{x \in K^n : f(x) = 0\} \neq \emptyset\).

(This is the Real Nullstellensatz).

c) Suppose \(K\) is real closed. For \(p \in S^K_n(K)\) let

\[I_p = \{f \in K[X_1, \ldots, X_n] : "f(\bar{v}) = 0" \in p\}\]

Prove that \(I_p\) is a real prime ideal and that \(p \mapsto I_p\) is a bijection onto the set of real prime ideals.