Math 506 Model Theory I
Problem Set 5

Due Friday March 18

From the Text: 4.5.2, 4,5,6, 4.5.11

1) Consider the dense linear order $(\mathbb{Q}, <)$.
Let $A = \{0, 1, 2, \ldots, \} \cup \{\frac{1}{2}, \frac{1}{3}, \ldots\}$.
a) Which types in $S_1^{\mathbb{Q}}(A)$ are isolated?
b) Describe the elements of $S_2^{\mathbb{Q}}(\mathcal{N})$.

The following problems are optional bonus problems. [may be turned in any time]

B1) Show that under Martin's Axiom, if T is a theory in a countable language and p_i is an isolated type for each $i \in I$ where $|I| < 2^{\aleph_0}$, then there is a model of T omitting each p_i. (see Ex. 4.5.14)

B2) Suppose K is a formally real field. We say that $I \subset K[X_1, \ldots, X_n]$ is *real* if whenever $f_1^2 + \ldots + f_m^2 \in I$, then $f_1, \ldots, f_m \in I$.

a) Prove that if I is a real prime ideal, then the fraction field of $K[X_1, \ldots, X_n]/I$ is a formally real field.

b) Prove that if K is real closed and I is a real prime ideal, then

$$V_K(I) = \{x \in K^n : f(x) = 0 \text{ for all } f \in I\} \neq \emptyset.$$

(This is the Real Nullstellensatz).

c) Suppose K is real closed. For $p \in S_n^K(K)$ let

$$I_p = \{f \in K[X_1, \ldots, X_n] : \text{“}f(\overline{v}) = 0\text{”} \in p\}.$$

Prove that I_p is a real prime ideal and that $p \mapsto I_p$ is a bijection onto the set of real prime ideals.