Math 506 Model Theory I
Problem Set 7

Due: Wednesday April 20

Do problem 1) below and the following problems from the text:

4.5.16 k) (you've already done earlier parts. In this problem and then next it might simplify to note that in the definition of κ-stability we need only look at one-types.)

4.5.22

4.5.30

4.5.45 (Assume that T has infinite models.)

1) Given an $\mathcal{L}$-structure $\mathcal{M}$ and $A \subset M$ we say that b is algebraic over A in $\mathcal{M}$, if there is a formula $\phi(v, \overline{a})$ with parameters $\overline{a} \in A$. Such that $M \models \phi(b, \overline{a})$ and $\{c \in M : \mathcal{M} \models \phi(c, \overline{a})\}$ is finite, i.e., there is a finite A-definable set $X \subseteq M$ such that $b \in A$.

a) Show that if $A \subset M$ and $\mathcal{M} \prec \mathcal{N}$, then $b \in N$ is algebraic over A in $\mathcal{N}$ if and only if $b \in M$ and b is algebraic over A in $\mathcal{M}$.

We let $\mathrm{acl}(A)$ denote the elements algebraic over A. By a) we see that this does not depend on the ambient model $\mathcal{M}$.

b) Prove the following easy properties of algebraic closure

i) $\mathrm{acl}(A) \supseteq A$;

ii) If $A \subseteq B$, then $\mathrm{acl}(A) \subseteq \mathrm{acl}(B)$;

iii) $\mathrm{acl}(\mathrm{acl}(A)) = \mathrm{acl}(A)$;

iv) If $b \in \mathrm{acl}(A)$, then there is $A_0 \subseteq A$ finite such that $b \in \mathrm{acl}(A_0)$.

c) Suppose $\mathcal{M}$ is saturated, $A \subseteq M$ and $|A| < |M|$. Prove that $b \in \mathrm{acl}(A)$ if and only if there is a finite set B such that $\sigma(b) \in B$ for any automorphism σ of $\mathcal{M}$ such that $\sigma|A$ is the identity.

Do one at least one of d), e) , f)

d) Show that if $\mathcal{M} \models \mathrm{DLO}$ and $A \subseteq M$, then $\mathrm{acl}(A) = A$.

e) Show that if $\mathcal{M} \models \mathrm{ACF}$ and $A \subseteq M$, then $\mathrm{acl}(A)$ is the algebraic closure of subfield generated by A.

f) Let G be a torsion free divisible abelian group and let $A \subseteq G$. Show that if $A \neq \emptyset$, then

$$\mathrm{acl}(A) = \{m_1 a_1 + \ldots + m_n a_n : m_i \in \mathbb{Q}, a_i \in A, n = 0, \ldots\}.$$