Math 507: Model Theory II

Exercises

ω-stable groups

1) Let $\mathcal{L} = \{+, 0, G_0, G_1, \ldots)$. Let T be the theory asserting that we are in an abelian group where every element has order 2, G_0 is the whole group and G_{i+1} is an index 2 subgroup of G_i for all i. Prove that T is superstable but not ω-stable. How many countable models does T have?

2) Give full details of the proof that in a superstable group there is no infinite descending sequence of definable subgroups $G_0 > G_1 > \ldots$ where $[G_i : G_{i+1}]$ is infinite for all i.

3) Let $\mathcal{L} = \{+, 0, G_0, G_1, \ldots\}$. Let T be the theory asserting that we are in a divisible abelian group, each G_i is a divisible subgroup and G_{i+1} is an infinite index subgroup of G_i for all i. Prove that T is stable but not superstable.

4) Let G be a group. Suppose H_1 and H_2 are finite index subgroups of G, prove that $H_1 \cap H_2$ is also a finite index subgroup.

5) Suppose $G \prec H$ and $\phi(v)$ defines G^0. Show that $\phi(v)$ defines H^0.

6) Suppose G is stable and $(H_a : a \in A)$ is a definable family of subgroups closed under finite intersection, then $\bigcap_{a \in A} H_a$ is definable.

7) Show that if G is connected and $H \triangleleft G$ is definable, then G/H is connected.

8) Suppose G is connected and $\sigma : G \to G$ is a definable group homomorphism with finite kernel, show that σG is surjective.

9) Let $\psi(v)$ define $Stab(p)$, let $G \prec H$ and let q be the nonforking extension of p to H, show that $\psi(v)$ defines $Stab(q)$ in H.

10) Suppose a and b realize generic types of G and $a \perp_G b$. Show that ab realizes a generic of G.

11) Suppose $p, q \in S_1(G)$ are generic. Show that there is $g \in G$ such that $q = gp$ (i.e., G acts transitively on the generic types).

12) Suppose $(K, +, -, \cdot, 0, 1)$ is an infinite field and $\mathrm{Th}(K)$ has quantifier elimination (in the language of rings). Prove that $\mathrm{Th}(K)$ is ω-stable and conclude that K is algebraically closed.

13) Suppose $X \subseteq G$ is indecomposable and $g \in G$. Show that gX and X^g are indecomposable.

14) Suppose G is ω-stable and $A \subset G$ is definable. Prove that there are disjoint indecomposable $X_1, \ldots, X_n$ such that $A = X_1 \cup \ldots \cup X_n$. [Hint: Build a finite branching tree of definable sets X_σ and definable subgroups H_σ. Such that if X_σ is not indecomposable, $1 < |X_\sigma/H_\sigma| = n < \infty$ and let $X_{\sigma,0}, \ldots, X_{\sigma,n-1}$ are the cosets of X_σ/H_σ and $H_\sigma \subset H_\tau$ for $\tau \subset \sigma$.]

15) a) Show that if G is a finite Morley rank group and $A \subset G$ is definable and infinite with $|A| < |G|$, then there is a normal $H \lhd G$ such that $|H| < |G|$. [Hint: Use 14 to show that without loss of generality A is indecomposable and $1 \in A$. Consider the group generated by $(A^g : g \in G)$.]

b) Prove that a simple group of finite Morley rank is $\aleph_1$-categorical.

16) Suppose that $(K, +, \cdot, \ldots)$ is a field of finite Morley rank and $X \subseteq K$ is an infinite definable set.

a) Show that there are $a_1, \ldots, a_n \in K$ such that $K = a_1X + a_2X + \ldots + a_nX$. [Hint: By 14) we may assume that X is indecomposable. Let $x \in X$ and $Y = X - x$. Show that the additive subgroup A generated by $\{aY : a \in K\}$ is definable. Argue that $A = K$.]

b) Show that if the language is countable, then the theory of K is categorical in all uncountable powers.

17) Let F be an infinite field, and let G be a group of automorphisms of F such that the action of G on F has finite Morley rank. Show that $G = \{1\}$. [Hint: Without loss of generality, G is Abelian. Using Exercise **??**, F has characteristic $p > 0$ and for all $\sigma \in G - \{1\}$, $Fix(\sigma)$, the fixed field of σ, is finite. Show that if $\sigma \in G - \{1\}$ then for all $n > 1$, $\sigma^n \neq 1$ and $|Fix(\sigma^n)| > |Fix(\sigma)|$. Thus, if $G \neq \{1\}$, then G is infinite. On the other hand, if G is infinite and σ is generic, so is σ^n. Derive a contradiction.]

18) (NIP DCC) Work in a structure with NIP. Suppose we have a formula $\phi(x, \overline{y})$ and $\mathcal{H}$ is a family of definable subroups such that each $H \in \mathcal{H}$ is defined by $\phi(x, \overline{a})$ for some $\overline{a}$. Prove that there is a number m such that if $H_1, \ldots, H_n \in \mathcal{H}$, then there are $i_1, \ldots, i_m \leq n$ such that

$$H_1 \cap H_2 \cap \ldots \cap H_n = H_{i_1} \cap \ldots \cap H_{i_m}.$$

[Hint: If not then for any n we can find $H_1, \ldots, H_n$ such that for all j there is $b_j \cap_{i \neq j} H_i$ with $b_j \notin H_j$. For $I \subset \{1, \ldots, n\}$ consider $b_I = \prod_{i \in I} b_i$. For which i is $b_I \in H_i$?]

19) (Another stable DCC) a) Work in a stable structure and let $\mathcal{H}$ be as in 18). Show that $\bigcap_{H\in\mathcal{H}} H$ is definable.

b) Give an example of an unstable NIP theory where a) fails.

20) Suppose we have a stable faithful action of a group G on a strongly minimal set X.

a) Show that there is $a_1, \ldots, a_n \in X$ such that $Stab(a_1)\cap\ldots\cap Stab(a_n) = \{1\}$. [Hint: Use 19).]

b) Conclude that $g \mapsto (g(a_1), \ldots, g(a_n))$ is injective.

c) Conclude that G has Morley rank at most n. [Hint: recall that we know X^n has Morley rank n.]

21) Let T be stable (or even simple) in a countable language. Suppose $(b_\alpha : \alpha < \omega_1)$ is a Morley sequence over A. Then for any c there is $\alpha < \omega_1$ such that $c \mathop{\smile\!\!\!\!\!\!\;|}_A b_\alpha$. [Also argue that if T is ω-stable we can replace ω_1 with ω.]

22) Let $\mathbb{M}$ be ω-stable, X and infinite $\emptyset$-definable set and $p \in S(A)$ a stationary, complete type.

a) Suppose there is $\bar{b}$ and some c realizing p such that $c \in \mathrm{dcl}(\bar{b}, X)$ and $c \mathop{\smile\!\!\!\!\!\!\;|}_A \bar{b}$. Show that p is X-intenal. [Hint: Let $B = (\bar{b}_0, \bar{b}_1, \ldots)$ be a Morley sequence for $\mathrm{tp}(\bar{b}/A)$. Let d realize p. Use 21) to find i such that $d \mathop{\smile\!\!\!\!\!\!\;|}_A \bar{b}_i$. Conclude that $d \in \mathrm{dcl}(\bar{b}_i, X)$ and $p(\mathbb{M}) \subset \mathrm{dcl}(B, X)$.]

b) Extend the result to non-stationary types.

Geometry of Strongly Minimal Sets

23) a) Prove Lemma 8.1.3.

b) Prove Lemma 8.1.4

c) Prove Lemma 8.1.6

24) a) Prove that if (X, cl) is a modular pregeometry, then any localization is modular.

b) Prove that (X, cl) is locally modular if and only if whenever A and B are closed sets with dim $(A \cap B) > 0$ we have

$$\dim\,(A, B) = \dim\,(A) + \dim\,(B) - \dim\,(A \cap B).$$

25) Suppose K is an algebraically closed field $x, a_0, \ldots, a_n$ are algebraically independent $\sum_{i=0}^{n} a_i x^i = y$. Prove that y is not algebraic over $k(x)$ for k any proper subfield of $\mathrm{acl}(a_0, \ldots, a_n)$ of transcendecne degree at most n. Use

this fact to give a second proof that algebraically closed fields are not locally modular.

26) Suppose D is strongly minimal and $\phi(x,y,\overline{a})$ defines a strongly minimal subset of D^2. If $\mathrm{tp}(\overline{b}) = \mathrm{tp}(\overline{a})$. Let $C_{\overline{b}} = \{(x,y) \in D^2 : \phi(x,y,\overline{b}\}$. Show that there is an m formula such that if $\mathrm{tp}(\overline{b}_1) = \mathrm{tp}(\overline{b}_2) = \mathrm{tp}(\overline{a})$, then

$$C_{\overline{b}_1} \triangle C_{\overline{b}_2} \text{ is finite } \Leftrightarrow |C_{\overline{b}_1} \cap C_{\overline{b}_2}| \leq m.$$

27) Suppose M is constructible over A and $|M \setminus A| = \aleph_1$. Prove that there is a construction $(a_\alpha : \alpha < \omega_1)$ of M over A. [Hint: Recall that any finite set is contained in a finite sufficient set.]

28) Let $\mathcal{L} = \{E_\alpha : \alpha < \omega_1\}$. Let T be the theory asserting that each E_α is an equivalence relation, $E_\beta \subset E_\alpha$ for $\alpha < \beta$, and if $\alpha < \beta$, then each E_α class is split into infinitely many E_β-classes. Prove that T has quantifier elimination and that T is λ-stable, whenever $\lambda^{\aleph_1} = \lambda$.

29) Let T be as in 28). Let $A \subset \mathbb{M}$ and let $\phi(v,\overline{a})$ be a satisfiable $\mathcal{L}_A$-formula that is not satisfied in A. Assume further that ϕ is a conjunction of atomic and negated atomic formulas.

a) Suppose there is $\alpha < \omega_1$ such that there is $a^* \in \mathbb{A}$ with $\phi(v,\overline{a}) \models vE_\alpha a^*$. Show that there is a maximal such α. Let $\theta(v,\overline{a})$ is the formula

$$vE_\alpha a^* \wedge \bigwedge_{a \in \overline{a}} \neg(vE_{\alpha+1}a).$$

Show that $\theta(v,\overline{a}) \models \phi(v,\overline{a})$ and $\theta(v,\overline{a})$ isolates a type over A.

b) If there is no $a^* \in \overline{a}$ and $\alpha < \omega_1$ such that $\phi(v,\overline{a}) \models vE_\alpha a$, show that

$$\bigwedge_{a \in \overline{a}} \neg(vE_0 a)$$

isolates a type over A consistent with $\phi(v,\overline{a})$.

c) Conclude that the isolated types in $S_1(A)$ are dense for any A.

You can also do the following problems from the text:

5.5.9
6.6.33
6.6.34
6.6.35