Internality and the Binding Group

Let T be an ω-stable theory, $\mathbb{M} \models T$ be a monster model and $X \subset \mathbb{M}$ an infinite $\emptyset$-definable set.

Definition 0.1 Let $A \subset \mathbb{M}$ be small and let $\gamma(v)$ be a partial type over A. We say that γ is *X-internal* if there is a small $B \subset \mathbb{M}$ such that every realization of γ in $\mathbb{M}$ is in $\mathrm{dcl}(B, X)$.

We say that a definable set Y is X-internal if the formula defining it is.

Lemma 0.2 *Suppose γ is X-internal. There are $\phi_1, \ldots, \phi_m \in \gamma$ and $f_1(\overline{x}, \overline{b}_1), \ldots, f_n(\overline{x}, \overline{b}_n)$ such that every element of $\bigwedge \phi_i(\mathbb{M})$ is $f_j(\overline{x}, \overline{b}_j)$ for some j and some $\overline{x} \in X$.*

Proof Let $\Gamma(v)$ be the partial type

$$\gamma(v) \cup \{\forall \overline{x} \in X \;\; f(\overline{x}) \neq v : f \text{ a partial function defined over } B\}.$$

Since γ is X-internal, $\Gamma(v)$ is not realized in $\mathbb{M}$ and, hence, inconsistent. The finite inconsistent subset gives $\phi_1, \ldots, \phi_m \in \gamma$ and $f_1(\overline{x}, \overline{b}_1), \ldots, f_n(\overline{x}, \overline{b}_n)$ such that

$$\bigwedge_{i=1}^{m} \phi_i(v) \rightarrow \bigwedge_{j=1}^{n} \exists \overline{x} \in X \;\; f_j(\overline{x}, \overline{b}_j) = v.$$

$\square$

Corollary 0.3 *Let Y be a non-empty X-internal definable set. There is a finite set B and a B-definable function F such that $F : X^n \rightarrow Y$ is surjective.*

Proof $f_1(\overline{x}, \overline{b}_1), \ldots, f_n(\overline{x}, \overline{b}_n)$ such that every element of Y is $f_j(\overline{x}, b_j)$ for some j and some $\overline{x} \in X$. By padding we can assume that $\overline{b}_1 = \ldots \overline{b}_n$, call the common value $\overline{b}$, and all f_j has l variables. Choose $a_1, \ldots, a_n \in X$ distinct and $d \in Y$. Let $F : X^{l+1} \rightarrow Y$ be the function

$$F(u, \overline{x}) = \begin{cases} f_j(\overline{x}, \overline{b}) & \text{if } u = a_j \text{ and } f_j(\overline{x}, \overline{b}) \in Y \\ d & \text{otherwise} \end{cases}.$$

$\square$

Lemma 0.4 *If $\alpha : A \cup X \rightarrow \mathbb{M}$ is partial elementary such that α is the identity on X, then there is $\sigma \in \mathrm{Aut}(\mathbb{M}/X)$ extending α.*

Proof We prove this by a back-and-forth argument. The key step is to show that if $b \in \mathbb{M}$ then we can extend α to A, b. We can find a small $X_0 \subset X$ such that $\mathrm{tp}(A, b/X)$ is definable over X_0. There is $c \in \mathbb{M}$ such that $\mathrm{tp}(\alpha(A), c/X_0) = \mathrm{tp}(A, b/X_0)$.

Claim $\mathrm{tp}(\alpha(A), c/X) = \mathrm{tp}(A, b/X)$.

For and $\overline{a} \in A$ and $\phi(\overline{u}, v, \overline{w})$

$$\mathbb{M} \models \forall \overline{x} \in X \ (\phi(\overline{a}, b, \overline{x}) \leftrightarrow d\phi(\overline{x}))$$

where $d\phi$ is a formula with parameters from X_0. Since $\mathrm{tp}(\alpha(A), c/X_0) = \mathrm{tp}(A, b/X_0)$,

$$\mathbb{M} \models \forall \overline{x} \in X \ (\phi(\alpha(\overline{a}), c, \overline{x}) \leftrightarrow d\phi(\overline{x}))$$

It follows that we can extend α to a partial elementary map by sending b to c. $\square$

Exercise 0.5 Suppose X is definable and $\alpha : X \to X$ is partial elementary, then α extends to an automorphism of $\mathbb{M}$.

If Y is a definable set we let $\mathrm{Aut}(Y/X)$ be the group of partial elementary permutations of $Y \cup X$ fixing X pointwise.

Theorem 0.6 (Binding Group Theorem) *Suppose Y is X-internal. There is a group G definable in $\mathbb{M}^{\mathrm{eq}}$ isomorphic to $\mathrm{Aut}(Y/X)$ and there is a definable action of G on Y isomorphic to action of $\mathrm{Aut}(Y/X)$ on Y.*

Proof There is $\overline{b}$ and a definable function such that $\overline{x} \mapsto F(\overline{x}, \overline{b})$ such that

$$\mathbb{M} \models \forall \overline{x} \in X \ F(\overline{x}, \overline{b}) \in Y \ \wedge \forall y \in Y \exists \overline{x} \in X \ F(\overline{x}, \overline{b}) = y.$$

By ω-stability, the isolated types over X are dense, thus without loss of generality, i.e., changing $\overline{b}$ if necessary, we may assume that $\mathrm{tp}(\overline{b}/X)$ is isolated by a formula $\psi(\overline{v})$ with parameters from X.

Suppose $\psi(\overline{c})$ we can define a permutation $\sigma_{\overline{c}} : Y \to Y$ by $\sigma_{\overline{c}}(y_1) = y_2$ if and only if

$$\exists \overline{x} \in X \ F(\overline{x}, \overline{b}) = y_1 \wedge F(\overline{x}, \overline{c}) = y_2.$$

We can find $\tau \in \mathrm{Aut}(\mathbb{M}/X)$ such that $\tau(\overline{b}) = \overline{c}$. Then $\tau Y = \sigma_{\overline{c}}$. Thus $\sigma_{\overline{c}} \in \mathrm{Aut}(Y/X)$.

Moreover, if $\sigma \in \mathrm{Aut}(Y/X)$ we can extend σ to $\widehat{\sigma} \in \mathrm{Aut}(\mathbb{M}/X)$. Let $\overline{c} = \widehat{\sigma}(b)$. Then $\sigma = \sigma_{\overline{c}}$. Thus $\overline{c} \mapsto \sigma_{\overline{c}}$ is surjection from $\psi(\mathbb{M})$ onto $\mathrm{Aut}(Y/X)$ and $\overline{c}, y \mapsto \sigma_{\overline{c}}(y)$ is a definable action.

There is a definable relation R on $\psi(\mathbb{M})^3$ such that $R(c_1, c_2, c_3)$ if and only if $\sigma_{\bar{c}_1} \circ \sigma_{\bar{c}_2} = \sigma_{\bar{c}_3}$, i.e.,

$$\forall y \forall z \forall w[(\sigma_{\bar{c}_2}(y) = w \wedge \sigma_{\bar{c}_1}(w) = z) \to \sigma_{\bar{c}_3}(y) = z].$$

Let $\sim$ be the definable equivalence relation on $\psi(\mathbb{M})$ such that $\bar{c} \sim \bar{c}_1$ if and only if $\sigma_{\bar{c}} = \sigma_{\bar{c}}$. Let G be $\psi(\mathbb{M})/\sim$ and define $\bar{c}_1/\sim \ \cdot \bar{c}_2/\sim = \ \bar{c}/\sim$ if and only if $R(\bar{c}_1, \bar{c}_2, \bar{c})$. Then $(G, \cdot)$ is isomorphic to $\mathrm{Aut}(Y/X)$ and the induced action of G on Y is definable. $\square$