# Lectures on Infinitary Model Theory

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## Contents

Part I

# Classical Results in Infinitary Model Theory

## 1 Infinitary Languages

Let $\tau$ be a vocabulary, i.e., a set of function symbols, relation symbols and constant symbols. In the logic $\mathcal{L}_{\infty,\omega}(\tau)$ we build formulas using the symbols from $\tau$, equality, Boolean connectives $\neg$, $\bigwedge$ and $\bigvee$, quantifiers $\forall$ and $\exists$, variables $\{v_\alpha : \alpha \text{ an ordinal}\}$.[1]

- terms and atomic formulas are defined as in first order logic;
- if $\phi$ is a formula, then so is $\neg\phi$;
- if $X$ is a set of formulas, then so are

$$\bigvee_{\phi\in X} \phi \text{ and } \bigwedge_{\phi\in X} \phi;$$

- if $\phi$ is a formula, so are $\exists v_\alpha\ \phi$ and $\forall v_\alpha\ \phi$.

We extend the usual definition of satisfaction by saying

$$\mathcal{M} \models \bigvee_{\phi\in X} \phi \text{ if and only if } \mathcal{M} \models \phi \text{ for some } \phi \in X$$

and

$$\mathcal{M} \models \bigwedge_{\phi\in X} \phi \text{ if and only if } \mathcal{M} \models \phi \text{ for all } \phi \in X.$$

For notational simplicity, we use the symbols $\wedge$ and $\vee$ as abbreviations for binary conjunctions and disjunctions. Similarly we will use the abreviations $\rightarrow$ and $\leftrightarrow$ when helpful.

We can inductively define the notions of *free variable*, *subformula*, *sentence*, *theory* and *satisfiability* in the usual ways.

[1] When no confusion arises we omit the $\tau$ and write $\mathcal{L}_{\infty,\omega}$ instead of $\mathcal{L}_{\infty,\omega}(\tau)$.

**Exercise 1.1** Suppose $\phi$ is an $\mathcal{L}_{\infty,\omega}$-sentence and $\psi$ is a subformula of $\phi$. Prove that $\psi$ has only finitely many free variables.

Let $\kappa$ be an infinite cardinal. In the logic $\mathcal{L}_{\kappa,\omega}(\tau)$ we form formulas in a similar way but we only use variables $\{v_\alpha : \alpha < \kappa\}$ and restrict infinite conjunctions and disjunctions to $\bigvee_{\phi\in X}\phi$ and $\bigwedge_{\phi\in X}\phi$ where $|X| < \kappa$. Thus $\mathcal{L}_{\omega,\omega}$ is just the usual first order logic. Throughout these notes we be focusing primarily on $\mathcal{L}_{\omega_1,\omega}$, the logic where we allow countable conjunctions and disjunctions.[2]

**Exercise 1.2** Show that if $\kappa$ is a regular cardinal and $\phi$ is a sentence of $\mathcal{L}_{\kappa,\omega}$, then $\phi$ has fewer than $\kappa$ subformulas. Show that this fails for singular cardinals. This is one reason it is customary to restrict attention to $\mathcal{L}_{\kappa,\omega}$ for $\kappa$ a regular cardinal.

**Definition 1.3** We say $\mathcal{M} \equiv_{\infty,\omega} \mathcal{N}$ if

$$\mathcal{M} \models \phi \text{ if and only if } \mathcal{N} \models \phi$$

for all $\mathcal{L}_{\infty,\omega}$ sentences $\phi$. The notion $\mathcal{M} \equiv_{\kappa,\omega} \mathcal{N}$, is defined analogously.

**Exercise 1.4** Show that if $\mathcal{M} \cong \mathcal{N}$, then $\mathcal{M} \equiv_{\infty,\omega} \mathcal{N}$.

When studying the model theory of infinitary logics there is one fundamental and inescapable fact:

**THE COMPACTNESS THEOREM FAILS FOR INFINITARY LANGUAGES.**

**Exercise 1.5** Let $\tau$ be the vocabulary with constant symbols $d, c_0, c_1, \ldots$ and let $\Gamma$ be the set of sentences

$$\{d \neq c_i : i \in \omega\} \cup \{\forall v \bigvee_{i\in\omega} v = c_i\}.$$

Show that every finite subset of $\Gamma$ is satisfiable, but $\Gamma$ is not satisfiable. Thus the Compactness Theorem fails for $\mathcal{L}_{\infty,\omega}$ and even $\mathcal{L}_{\omega_1,\omega}$.

[2]Logically it would make sense to call this logic $\mathcal{L}_{\aleph_1,\aleph_0}$ but we will follow the historical precedent and refer to it as $\mathcal{L}_{\omega_1,\omega}$.

The failure of compactness will lead to many new phenomena and force us to to find new approaches and develop new tools. [3]

**Exercise 1.6** a) Give an example of structures $\mathcal{M}_0, \mathcal{M}_1, \ldots$ and $\phi \in \mathcal{L}_{\omega_1,\omega}$ such that $\mathcal{M}_i \models \phi$ for all $i$, but if $\mathcal{U}$ is a nonprincipal ultrafilter on $\omega$ then $\prod M_i/\mathcal{U} \models \neg\phi$.

b) Show that if $\mathcal{U}$ is a $\sigma$-complete ultrafilter on $I$, then

$$\prod_{i \in I} \mathcal{M}_i/\mathcal{U} \models \phi \Leftrightarrow \{i \in I : \mathcal{M}_i \models \phi\} \in \mathcal{U}$$

for $\phi \in \mathcal{L}_{\omega_1,\omega}$.

If compactness fails, why then do we study the model theory infinitary languages? One reason is that we get new insights about first order model theory. But the simplest answer is that there are many natural classes that are axiomatized by $\mathcal{L}_{\omega_1,\omega}$-sentences.

**Exercise 1.7** Show that the following classes are $\mathcal{L}_{\omega_1,\omega}$-axiomatizable for appropriate choices $\tau$.

i) torsion abelian groups;

ii) finitely generated groups;

iii) non-finitely generated groups;

iv) linear orders isomorphic to $(\mathbb{Z}, <)$;

v) archimedian fields;

vi) connected graphs;

vii) recursively saturated models of PA;

vii) $\omega$-models of ZFC (i.e., models of ZFC where the integers are standard);

ix) models of $T$ omitting $p$, where $T$ is a first order theory and $p$ is a type.

**Exercise 1.8** Show by compactness that none on these classes is axiomatizable in first order logic.

[3]Though we will not focus on it, another fruitful approach is to look for more general forms of the compactness theorem that hold in particular settings. One example of this is Barwise Compactness for countable admissible fragments. We will see an avatar of these results in Theorem 11.9, but the interested reader should consult [6]. Another important example is studying compactness in languages $\mathcal{L}_{\kappa,\kappa}$ where $\kappa$ is a large cardinal. See [20] or [19] for further information.

Examples ii), iv) and v) immediately show the failure of the Upward Löwenheim–Skolem Theorem in infinitary languages. There are no uncountable linear orders isomorphic to $(\mathbb{Z}, <)$ and any archimedean ordered field is isomorphic to a subfield of the real numbers and hence has cardinality at most $2^{\aleph_0}$. We give another example in Exercise 1.27.

**Exercise 1.9** Show by induction that for ordinal $\alpha$ there is an $\mathcal{L}_{\infty,\omega}$-sentence $\Phi_\alpha$ describing $(\alpha, <)$ up to isomorphism.

**Exercise 1.10** Let $\kappa$ be a regular cardinal. Show there are $\alpha, \beta < (2^{2^\kappa})^+$ such that $(\alpha, <) \equiv_{\kappa,\omega} (\beta, <)$.

Taken together these two exercises give examples of structures $\mathcal{M}, \mathcal{N}$ with $\mathcal{M} \equiv_{\kappa,\omega} \mathcal{N}$ but $\mathcal{M} \not\equiv_{\infty,\omega} \mathcal{N}$ for all $\kappa$.[4]

We will also, from time to time, look at *pseudoelementary classes* which are simply reducts of elementary classes.

**Definition 1.11** We say that a class $\mathcal{K}$ of $\tau$-structures is a $\mathrm{PC}_{\omega_1,\omega}$-*class* if there is a vocabulary $\tau^* \supseteq \tau$ and $\phi \in \mathcal{L}_{\omega_1,\omega}(\tau^*)$ such that

$$\mathcal{K} = \{\mathcal{M} : \text{ there is a } \tau\text{-structure } \mathcal{M}^* \text{ expanding } \mathcal{M} \text{ with } \mathcal{M}^* \models \phi\}$$

i.e., $\mathcal{K}$ is the class of $\tau$-reducts of models of $\phi$.

Similarly we say that $\mathcal{K}$ is a PC-*class* if $\phi$ is a first order sentence and $\mathcal{K}$ is a $\mathrm{PC}_\delta$-*class* if $\phi$ is a first order theory.

**Exercise 1.12** Show that the following classes are $\mathrm{PC}_{\omega_1,\omega}$.

a) orderable groups

b) free groups

c) 1-trainsitive linear orders, i.e. linear orders where for any $a, b$ there is an order automorphism taking $a$ to $b$;

d) fields that are pure transcendental extensions of $\mathbb{Q}$;

e) incomplete ordered fields;

f) ordered fields with an integer part (i.e., an ordered subring with no element between 0 and 1 such that every element of the field is within distance at most one of some element of the subring);

**Exercise 1.13** [Silver] Let $\tau = \{U\}$ where $U$ is unary and let $\mathcal{K} = \{\mathcal{M} : |\mathcal{M}| \leq 2^{|U(\mathcal{M})|} \wedge |\mathcal{M} \setminus U(\mathcal{M})| = |\mathcal{M}|\}$.

[4] For a specific example: suppose $\kappa < \lambda$ are uncountable cardinals. Show that $\kappa \equiv_{\omega_1,\omega} \lambda$, but $\kappa \not\equiv_{\infty,\omega} \lambda$. See [27] for details.

a) Show that $\mathcal{K}$ is PC-class.

b) Show that $\mathcal{K}$ is $\kappa$-categorical if and only if $\kappa = \beth_\alpha$ for some limit ordinal $\alpha$.

This example shows that the straightforward generalization of Morley's Categoricity Theorem to PC-classes fails

**Exercise 1.14** Let $\mathcal{K}$ be a $\mathrm{PC}_{\omega_1,\omega}$-class. Show that

$$\mathcal{K}_0 = \{\mathcal{M} \in \mathcal{K} : \mathcal{M} \text{ countable}\}$$

is also a $\mathrm{PC}_{\omega_1,\omega}$-class.

## 1.1 Fragments and Downward Löwenheim–Skolem

We will prove a useful and natural version of the Downward Löwenheim–Skolem Theorem. The next exercise shows that even here we will need to be careful.

**Exercise 1.15** Give an example of a countable vocabulary $\tau$ and an $\mathcal{L}_{\omega_1,\omega}$-theory $T$ such that every model of $T$ has cardinality at least $2^{\aleph_0}$.

We will often restrict our attention to subcollections of the set of all $\mathcal{L}_{\omega_1,\omega}$-formulas. We will look at collections of formulas with some natural closure properties. One of these will be formal negation an operation that shows how we inductively move a negation inside a quantifier or a Boolean operation. Closure under formal negation isn't needed in this proof, but will be in later arguments.

**Definition 1.16** For each formula $\phi$ we define $\sim\phi$, a *formal negation* of $\phi$ as follows:

i) for $\phi$ atomic, $\sim\phi$ is $\neg\phi$;

ii) $\sim(\neg\phi)$ is $\phi$;

iii)

$$\sim \bigwedge_{\phi\in X} \phi \text{ is } \bigvee_{\phi\in X} \sim\phi \text{ and } \sim \bigvee_{\phi\in X} \phi \text{ is } \bigwedge_{\phi\in X} \sim\phi;$$

iv) $\sim\exists v\phi$ is $\forall v \sim\phi$ and $\sim\forall v\phi$ is $\exists v\sim\phi$.

**Exercise 1.17** Show by induction that

$$\mathcal{M} \models \neg\phi \text{ if and only if } \mathcal{M} \models \sim\phi$$

for all $\phi \in \mathcal{L}_{\infty,\omega}$.

**Definition 1.18** We say that a set of $\mathcal{L}_{\infty,\omega}$-formulas $\mathbb{A}$ is a *fragment* if there is an infinite set of variables $V$ such that if $\phi \in \mathbb{A}$, then all variables occurring in $\phi$ are in $V$ and $A$ satisfies the following closure properties:

i) all atomic formulas using only the constant symbols of $\tau$ and variables from $V$ are in $\mathbb{A}$;

ii) if $\phi \in \mathbb{A}$ and $\psi$ is a subformula of $\phi$, then $\psi \in \mathbb{A}$;

iii) If $\phi \in \mathbb{A}$, $v$ is free in $\phi$, and $t$ is a term where every variable is in $V$, then the formula obtained by substituting $t$ into all free occurrences of $v$ is in $\mathbb{A}$;

iv) $\mathbb{A}$ is closed under $\sim$;

v) $\mathbb{A}$ is closed under $\neg$, $\wedge$, $\vee$, $\exists v$, and $\forall v$ for $v \in V$.

**Exercise 1.19** a) Suppose $\kappa$ is regular (in particular this holds for $\mathcal{L}_{\omega_1,\omega}$). Prove that if $T$ is a set of $\mathcal{L}_{\kappa,\omega}$-sentences with $|T| < \kappa$, then there is $\mathbb{A}$ a fragment of $\mathcal{L}_{\kappa,\omega}$ such that $T \subseteq \mathbb{A}$ and $|\mathbb{A}| < \kappa$.

b) Show that there is a smallest such fragment.

**Exercise 1.20** Let $T$ and $\mathbb{A}$ be as above. Show that every formula in $\mathbb{A}$ has only finitely many free variables. Give an example showing that even though are only finitely many free variables, there may be infinitely many bound occurrences.

**Exercise 1.21** We say that a formula is in *negation normal form* if the $\neg$ only occurs applied to atomic formulas. Show by induction that for any fragment $\mathbb{A}$ and any $\mathcal{L}_{\infty,\omega}$-formula $\phi \in \mathbb{A}$, there is $\psi \in \mathbb{A}$ such that $\psi$ is equivalent to $\phi$ and $\psi$ is in negation normal form.

Conjunctive and disjunctive normal form does not work as well.

**Exercise 1.22** Let $\tau = \{P_{i,j} : i, j \in \omega\} \cup \{c\}$ where each $P_{i,j}$ is a unary predicate and $c$ is a constant symbol. Consider the $\mathcal{L}_{\omega_1,\omega}$-sentence

$$\bigwedge_{i\in\omega}\bigwedge_{j\neq k}(P_{i,j}(c) \to \neg P_{i,k}(c)) \wedge \bigwedge_{i\in\omega}\bigvee_{j\in\omega} P_{i,j}(c).$$

Show that there is an equivalent sentence in disjunctive normal form in $\mathcal{L}_{(2^{\aleph_0})^+,\omega}$. (See also Exercise 3.6.)

We write $\mathcal{M} \equiv_{\mathbb{A}} \mathcal{N}$ and $\mathcal{M} \prec_A \mathcal{N}$ for elementary equivalence and elementary submodels with respect to formulas in $\mathbb{A}$ (where $\mathbb{A}$ could be $\mathcal{L}_{\omega_1,\omega}$ or $\mathcal{L}_{\infty,\omega}$).

**Theorem 1.23 (Downward Löwenheim–Skolem)** *Let $\mathbb{A}$ be a fragment of $\mathcal{L}_{\infty,\omega}$ such that any formula $\phi \in \mathbb{A}$ has at most finitely many free variables. Let $\mathcal{M}$ be a $\tau$-structure with $X \subseteq \mathcal{M}$. There is $\mathcal{N} \prec_{\mathbb{A}} \mathcal{M}$ with $X \subseteq \mathcal{N}$ and $|\mathcal{N}| \leq \max(|\mathbb{A}|, |X|)$.*

*In particular, if $\tau$ is countable and $\mathbb{A}$ is a countable fragment of $\mathcal{L}_{\omega_1,\omega}(\tau)$, then every $\tau$-structure has a countable $\mathbb{A}$-elementary submodel.*

The proof is a simple generalization of the proof in first-order logic. It is outlined in the following Exercise.

**Exercise 1.24** a) Prove there is $\tau^* \supseteq \tau$ and $\mathbb{A}^* \supseteq \mathbb{A}$ and $\mathcal{M}^*$ an $\tau^*$-expansion of $\mathcal{M}$ such that $|\tau^*|, |\mathbb{A}^*| = |\mathbb{A}|$ and for each $\mathbb{A}^*$-formula $\phi(\overline{v}, w)$ with free variables from $v_1, \ldots, v_n, w$, there is an $n$-ary function symbol $f_\phi$ such that

$$\mathcal{M}^* \models \forall \overline{v}\ (\exists w\ \phi(\overline{v}, w) \rightarrow \phi(\overline{v}, f_\phi(\overline{v})))$$

b) Prove that if $\mathcal{N}$ is a $\tau^*$-substructure of $\mathcal{M}^*$, then $\mathcal{N} \prec_{\mathbb{A}^*} \mathcal{M}^*$.

c) Prove that there is a $\tau^*$-substructure $\mathcal{N}$ of $\mathcal{M}^*$ with $X \subseteq \mathcal{N}$ and $|\mathcal{N}| \leq \max(|\mathbb{A}^*|, |X|)$.

**Exercise 1.25** For $\tau$ and $\mathbb{A}$ define the appropriate notion of *built-in Skolem functions*. Prove that for any $\tau$ and $\mathbb{A}$ and $\mathbb{A}$-theory $T$ there are $\tau^* \supseteq \tau$, $\mathbb{A}^* \supseteq \mathbb{A}$ and $T^* \supseteq T$ with $|\mathbb{A}^*| = |T^*| = |\mathbb{A}|$ where $T^*$ has built-in Skolem functions and any model of $T$ has and expansion to a model of $T^*$.

**Exercise 1.26** Let $\mathbb{A}$ be a fragment of $\mathcal{L}_{\infty,\omega}$ where every formula has finitely many free variables and let $\mathcal{K}$ be a $\mathrm{PC}_{\omega_1,\omega}$-class. Show that if $\mathcal{M} \in \mathcal{K}$ is a $\tau$-structure and $X \subset \mathcal{M}$, then there is $\mathcal{N} \prec_{\mathbb{A}} \mathcal{M}$ such that $|\mathcal{N}| = \max(|X|, |\mathbb{A}|))$ and $\mathcal{N} \in \mathcal{K}$.

**Exercise 1.27** As mentioned above, in first order logic, the Upward Löwenheim-Skolem is an easy consequence of Compactness. In infinitary logics it is generally false. Let $\tau = \{U, S, E, c_1, \ldots, c_n, \ldots\}$, where $U$ and $S$ are unary predicates, $E$ is a binary predicate and each $c_i$ is a constant. Let $\phi$ be the conjunction of

i) $\forall x\ U(x) \leftrightarrow \neg S(x)$,

ii) $\forall x \forall y\ (E(x, y) \rightarrow U(x) \wedge S(y))$;

iii) $c_i \neq c_j$, for $i \neq j$;

iv) $U(c_i)$ for all $i$,

v) $\forall y \forall z\ ([S(y) \wedge S(z) \wedge\ \forall x\ ((E(x, y) \leftrightarrow E(x, z))] \rightarrow y = z)$

vi) $\forall x (U(x) \rightarrow \bigvee_{i=1}^{\infty} x = c_i)$.

Prove that every model of $\phi$ has size at most $2^{\aleph_0}$.

Elementary chains behave as they do in first order logic.

**Exercise 1.28** Suppose $\mathbb{A}$ is a fragment of $\mathcal{L}_{\infty,\omega}$, $(I,<)$ is a linear order and $(\mathcal{M}_i : i \in I)$ is an elementary chain of $\tau$-structures, i.e., $\mathcal{M}_i \prec_{\mathbb{A}} \mathcal{M}_j$ for $i<j$. Let $\mathcal{M} = \bigcup \mathcal{M}_\alpha$. Then $\mathcal{M}_i \prec_{\mathbb{A}} \mathcal{M}$ for all $i \in I$.

**Exercise 1.29** Give an example showing that PC classes need not be preserved by elementary chains.

**Exercise 1.30** Let $\tau$ be countable. Suppose $(\mathcal{M}_\alpha : \alpha < \omega_1)$ is a chain of countable $\tau$-structures such that $\mathcal{M}_\beta = \bigcup_{\alpha<\beta} \mathcal{M}_\alpha$ for $\beta$ a limit ordinal. Let $\mathcal{M} = \bigcup_{\alpha<\omega_1} \mathcal{M}_\alpha$. Suppose$\phi \in \mathcal{L}_{\omega_1,\omega}$ and $\mathcal{M} \models \phi$. Show that $\{\alpha : \mathcal{M}_\alpha \models \phi\}$ is closed unbounded.

## 1.2 $\mathcal{L}_{\omega_1,\omega}$ and omitting first order types

In Exercise 1.7 we noted that the class of models of a first order theory omitting a type is expressible in $\mathcal{L}_{\omega_1,\omega}$. The next result, due to Chang, shows that any class axiomatizable by an $\mathcal{L}_{\omega_1,\omega}$-sentence is the reduct of a class of models of a first order theory omitting a set of types.

**Theorem 1.31** *Let $\tau$ be a countable vocabulary and let $T$ be a countable set of $\mathcal{L}_{\omega_1,\omega}$-sentences. There is a countable vocabulary $\tau^* \supseteq \tau$, a first order $\tau^*$-theory $T^*$ and a set of partial types $\Gamma$ such that:*

*i) if $\mathcal{M} \models T^*$ and $\mathcal{M}$ omits all types in $\Gamma$, then the $\tau$-reduct of $\mathcal{M}$ is a model of $T$;*

*ii) every model of $T$ has an $\tau^*$-expansion that is a model of $T^*$ omitting all types in $\Gamma$.*

**Proof** Let $\mathbb{A}$ be the smallest fragment containing all sentences in $T$. We expand $\tau$ to $\tau^*$ so for each formula $\phi$ in $\mathbb{A}$ with free variables from $v_1,\ldots,v_n$ we have an $n$-ary relation symbol $R_\phi$. $T^*$ is formed by taking the following sentences:

i) if $\phi$ is atomic add

$$\forall \overline{v}\ (R_\phi \leftrightarrow \phi);$$

ii) if $\phi$ is $\neg\theta$ add

$$\forall \overline{v}\ (R_\phi \leftrightarrow \neg R_\theta);$$

iii) if $\phi$ is $\bigwedge_{\theta\in X}\theta$ add

$$\forall\overline{v}\ (R_\phi \to R_\theta)$$

for all $\theta \in X$ and let $\gamma_\phi$ be the type

$$\{\neg R_\phi\} \cup \{R_\theta : \theta \in X\};$$

iv) if $\phi$ is $\bigvee_{\theta\in X}\theta$, add

$$\forall\overline{v}\ R_\theta \to R_\phi$$

for all $\theta$ in $X$ and let $\gamma_\phi$ be the type

$$\{R_\phi\} \cup \{\neg R_\theta : \theta \in X\};$$

v) if $\phi$ is $\exists w\ \theta$, then add

$$\forall\overline{v}\ (R_\phi \leftrightarrow \exists w R_\theta);$$

vi) if $\phi$ is $\forall w\ \theta$, then add

$$\forall\overline{v}\ (R_\phi \leftrightarrow \forall\overline{v}\forall w\ R_\theta);$$

vii) for each sentence $\phi \in T$, add $R_\phi$ to $T^*$.

Let $\Gamma$ be the collection of types $\gamma_\phi$ described above.[5]

**Exercise 1.32** a) Suppose $\mathcal{M} \models T^*$ and $\mathcal{M}$ omits every type in $\Gamma$. Prove that

$$\mathcal{M} \models \forall\overline{v}\ \phi \leftrightarrow R_\phi$$

for all $\phi \in \mathbb{A}$.

Conclude that the $\tau$-reduct of a model of $T^*$ is a model of $T$.

b) Prove that every $\mathcal{M} \models T$ has an expansion that is a model of $T^*$ omitting all types in $\Gamma$ by interpreting $R_\phi$ as $\phi$.

This completes the proof. □

We will examine a refinement of this result in Theorem 2.32

[5]As described here for each sentence $\phi$ we have added a 0-ary predicate symbol. If you are not comfortable with this approach we could rephrase this by adding a single constant $c$. Then for each sentence $\phi$ we could add unary predicate $R_\phi$ and make assertions about $R_\phi(c)$.

# 2 Back and Forth

## 2.1 Karp's Theorem

We next work on several characterizations of $\equiv_{\infty,\omega}$ due to Karp.

**Definition 2.1** Let $\mathcal{M}$ and $\mathcal{N}$ be $\tau$-structures. A *partial isomorphism system* between $\mathcal{M}$ and $\mathcal{N}$ is a non-empty collection $P$ of partial $\tau$-embeddings $f : A \to \mathcal{N}$ where $A \subseteq \mathcal{M}$ such that:

i) for all $f \in P$ and $a \in \mathcal{M}$ there is $g \in P$ such that $g \supseteq f$ and $a \in \mathrm{dom}(g)$;

ii) for all $g \in P$ and $b \in \mathcal{N}$ there is $g \in P$ such that $g \supseteq f$ and $b \in \mathrm{img}(g)$.

We write $\mathcal{M} \cong_p \mathcal{N}$ if there is a partial isomorphism system between $\mathcal{M}$ and $\mathcal{N}$.

**Exercise 2.2** If $\mathcal{M}$ and $\mathcal{N}$ are countable and $\mathcal{M} \cong_p \mathcal{N}$, show by a back-and-forth argument that $\mathcal{M} \cong \mathcal{N}$.

**Definition 2.3** Let $\mathcal{M}$ and $\mathcal{N}$ be $\tau$-structures. The game $G(\mathcal{M},\mathcal{N})$ is played as follows. At stage $n$, Player I plays $a_n \in \mathcal{M}$ or $b_n \in \mathcal{N}$. In the first case Player II responds with $b_n \in \mathcal{N}$ and in the later case Player II responds with $a_n \in \mathcal{M}$. Player II wins the play of the game if the map $a_n \mapsto b_n$ is a partial $\tau$-embedding.

**Theorem 2.4** *The following are equivalent:*

*i)* $\mathcal{M} \equiv_{\infty,\omega} \mathcal{N}$*;*

*ii)* $\mathcal{M} \cong_p \mathcal{N}$*;*

*iii) there is a partial isomorphism system $P$ between $\mathcal{M}$ and $\mathcal{N}$ where every $p \in P$ has finite domain;*

*iv) Player II has a winning strategy in $G(\mathcal{M},\mathcal{N})$.*

**Proof**

iii) $\Rightarrow$ ii) is clear.

ii) $\Rightarrow$ i) Let $P$ be a system of partial isomorphisms. We prove that for all $\phi(v_1, \ldots, v_n) \in \mathcal{L}_{\infty,\omega}$, $f \in P$ and $a_1, \ldots, a_n \in \mathrm{dom}(f)$, then

$$\mathcal{M} \models \phi(\overline{a}) \Leftrightarrow \mathcal{N} \models \phi(f(\overline{a})).$$

We prove this by induction on formulas. This is clear for atomic formulas and the induction step is obvious for $\neg$, $\bigwedge$ and $\bigvee$. Suppose $\phi(\overline{v})$ is $\exists w\ \psi(\overline{v}, w)$.

If $\mathcal{M} \models \psi(\overline{a}, c)$. There is $g \in P$ with $g \supseteq f$ and $c \in \mathrm{dom}(P)$. By induction, $\mathcal{N} \models \psi(f(\overline{(a)}), f(c))$, so $\mathcal{N} \models \phi(f(\overline{a}))$.

On the other hand, if $\mathcal{N} \models \psi(f(\overline{a}), d)$. There is $g \in P$ with $g \supseteq f$ and $c \in \mathrm{dom}(g)$ such that $g(c) = d$. By induction, $\mathcal{M} \models \psi(\overline{a}, c)$. Thus $\mathcal{M} \models \phi(\overline{a})$.

This proves the claim. In particular, if $\phi$ is a sentence $\mathcal{M} \models \phi \Leftrightarrow \mathcal{N} \models \phi$.

iv)$\Rightarrow$ iii) Let $\eta$ be a winning strategy for Player II. Let $P$ be the set of all maps $f(a_i) = b_i$ where $a_1, \ldots, a_n$, $b_1, \ldots, b_n$ are the results of some play of the game where at each stage Player I has played either $a_n$ or $b_n$ and Player II has responded using $\eta$. Since $\eta$ is a winning strategy for Player II, each such $f$ is a partial $\tau$-embedding. Since Player I can at any stage play any element from $\mathcal{M}$ or $\mathcal{N}$, $P$ satisfies i) and ii) in the definition of a partial isomorphism system.

i) $\Rightarrow$ iv) We need one fact.

**claim** Suppose $(\mathcal{M}, \overline{a}) \equiv_{\infty,\omega} (\mathcal{N}, \overline{b})$ and $c \in M$, there is $d \in N$ such that

$$(\mathcal{M}, \overline{a}, c) \equiv_{\infty,\omega} (\mathcal{N}, \overline{b}, d).$$

Suppose not. Then for all $d \in \mathcal{N}$ there is $\phi_d$ such that $\mathcal{M} \models \phi_d(\overline{a}, c)$ and $\mathcal{N} \models \neg\phi_d(\overline{b}, d)$. But then

$$\mathcal{M} \models \exists v \bigwedge_{d \in N} \phi_d(\overline{a}, v)$$

and

$$\mathcal{N} \not\models \exists v \bigwedge_{d \in N} \phi_d(\overline{b}, v)$$

a contradiction.

We now describe Player II's strategy. Player II always has a play to ensure $(\mathcal{M}, a_1, \ldots, a_n) \equiv_{\infty,\omega} (\mathcal{N}, b_1, \ldots, b_n)$. As long as Player II does this the resulting map will be a partial $\mathcal{L}$-embedding. □

**Exercise 2.5** If $\mathcal{M} \equiv_{\infty,\omega} \mathcal{N}$, then for any countably generated substructure $\mathcal{M}_0 \subseteq \mathcal{N}$ there is an isomorphic $\mathcal{N}_0 \subseteq \mathcal{N}$.

**Exercise 2.6** We say that an abelian group is $\aleph_1$-*free* if every countable subgroup is free.

Prove that if $G$ is free abelian and $H \equiv_{\infty,\omega} G$, then $H$ is $\aleph_1$-free. We will show the converse in Corollary 2.23.

**Exercise 2.7** For any abelian group $A$ we say that a subgroup $A_0$ is *thin* if there is a subgroup $A_1$ such that $A = A_0 \oplus A_1$ and $A_1$ is isomorphic to $A$.

Let $G$ be the free abelian group on countably many generators. Let $P$ be the set of isomorphisms between thin subgroups of $G$ and thin subgroups of $\mathbb{Z}^\omega$.

a) Prove that $P$ is a partial isomorphism system. Conclude that $G \equiv_{\infty,\omega} \mathbb{Z}^\omega$;

b) The group $\mathbb{Z}^\omega$ is known not to be free abelian (see for example ??). Conclude that the class of free abelian groups is $\mathrm{PC}_{\omega_1,\omega}$ but not $\mathcal{L}_{\infty,\omega}$-axiomatizable. In Appendix A.4 we will show that there are $2^{\aleph_1}$ non-isomorphic $\aleph_1$-free abelian groups of cardinality $\aleph_1$.

**Exercise 2.8** Prove that $\mathcal{M} \equiv_{\infty,\omega} \mathcal{N}$ if and only if there is a forcing extension of the universe $\mathbb{V}[G]$ where $\mathcal{M} \cong \mathcal{N}$.

For another view, we begin by defining a deceptively simple sequence of equivalence relations. For each ordinal $\alpha$, we will have a relation

$$(\mathcal{M}, \overline{a}) \sim_\alpha (\mathcal{N}, \overline{b})$$

where $\mathcal{M}$ and $\mathcal{N}$ are $\tau$-structures, $\overline{a} \in \mathcal{M}^n$ and $\overline{b} \in \mathcal{N}^n$ and $n = 0, 1, 2, \ldots$.

- $(\mathcal{M}, \overline{a}) \sim_0 (\mathcal{N}, \overline{b})$ if $\mathcal{M} \models \phi(\overline{a})$ if and only if $\mathcal{N} \models \phi(\overline{b})$ for all atomic $\tau$-formulas $\phi$.

- For all ordinals $\alpha$, $(\mathcal{M}, \overline{a}) \sim_{\alpha+1} (\mathcal{N}, \overline{b})$ if for all $c \in \mathcal{M}$ there is $d \in \mathcal{N}$ such that $(\mathcal{M}, \overline{a}, c) \sim_\alpha (\mathcal{N}, \overline{b}, d)$ and for all $d \in \mathcal{N}$ there is $c \in \mathcal{M}$ such that $(\mathcal{M}, \overline{a}, c) \sim_\alpha (\mathcal{N}, \overline{b}, d)$.

- For all limit ordinals $\beta$, $(\mathcal{M}, \overline{a}) \sim_\beta (\mathcal{N}, \overline{b})$ if and only if $(\mathcal{M}, \overline{a}) \sim_\alpha (\mathcal{N}, \overline{b})$ for all $\alpha < \beta$.

We show that $\sim_\alpha$ captures the more complicated notion of equivalence for sentences of quantifier rank at most $\alpha$.

**Definition 2.9** Define $\mathrm{qr}(\phi)$ the *quantifier rank* of $\mathcal{L}_{\infty,\omega}$-formulas as follows:

- If $\phi$ is atomic, then $\mathrm{qr}(\phi) = 0$;

- $\mathrm{qr}(\neg\phi) = \mathrm{qr}(\phi)$;

- $\mathrm{qr}(\bigwedge_{\phi \in X} \phi) = \mathrm{qr}(\bigvee_{\phi \in X} \phi) = \sup_{\phi \in X} \mathrm{qr}(\phi)$;

- $\mathrm{qr}(\exists v\ \phi) = \mathrm{qr}(\forall v\ \phi) = \mathrm{qr}(\phi) + 1$.

**Definition 2.10** We say $\mathcal{M} \equiv_\alpha \mathcal{N}$ if and only

$$\mathcal{M} \models \phi \Leftrightarrow \mathcal{N} \models \phi$$

for any $\mathcal{L}_{\infty,\omega}$-sentence $\phi$ of quantifier rank at most $\alpha$.

**Exercise 2.11** If $\alpha$ is a limit ordinal, show that $\mathcal{M} \equiv_\alpha \mathcal{N}$ if and only if $\mathcal{M} \equiv_\beta \mathcal{N}$ for all $\beta < \alpha$.

**Exercise 2.12** If $\alpha$ is a limit ordinal, then $\mathcal{M} \equiv_\alpha \mathcal{N}$ if and only if $\mathcal{M} \equiv_\beta \mathcal{N}$ for all $\beta < \alpha$.

**Theorem 2.13** *$\mathcal{M} \equiv_\alpha \mathcal{N}$ if and only if $\mathcal{M} \sim_\alpha \mathcal{N}$.*

We leave the proof as an exercise.

**Exercise 2.14** Prove by induction on $\alpha$ that for all $\tau$-structures $\mathcal{M}, \mathcal{N}$, $\overline{a} \in \mathcal{M}$ and $\overline{b} \in \mathcal{N}$, $(\mathcal{M}, \overline{a}) \sim_\alpha (\mathcal{N}, \overline{b})$ if and only if

$$\mathcal{M} \models \phi(\overline{a}) \Leftrightarrow \mathcal{N} \models \phi(\overline{b})$$

for all formulas $\phi(\overline{v})$ of quantifier rank at most $\alpha$.

## 2.2 Scott's Theorem

If $\tau$ is any vocabulary and $\mathcal{M}$ is an $\tau$-structure we define a sequence of $\mathcal{L}_{\infty,\omega}$-formulas $\Phi^{\mathcal{M}}_{\overline{a},\alpha}(\overline{v})$ for $\overline{a} \in \mathcal{M}^{<\omega}$ and $\alpha$ is an ordinal as follows:

$$\Phi^{\mathcal{M}}_{\overline{a},0}(\overline{v}) = \bigwedge_{\psi \in X} \psi(\overline{v}),$$

where $X = \{\psi : \mathcal{M} \models \psi(\overline{a})$ and $\psi$ is atomic or the negation of an atomic $\tau$-formula$\}$.

If $\alpha$ is a limit ordinal, then

$$\Phi^{\mathcal{M}}_{\overline{a},\alpha}(\overline{v}) = \bigwedge_{\beta < \alpha} \Phi^{\mathcal{M}}_{\overline{a},\beta}(\overline{v}).$$

If $\alpha = \beta + 1$, then

$$\Phi^{\mathcal{M}}_{\overline{a},\alpha}(\overline{v}) = \bigwedge_{b \in \mathcal{M}} \exists w\ \Phi^{\mathcal{M}}_{\overline{a},b,\beta}(\overline{v}, w) \wedge \forall w \bigvee_{b \in \mathcal{M}} \Phi^{\mathcal{M}}_{\overline{a},b,\beta}(\overline{v}, w).$$

**Lemma 2.15** *Let $\mathcal{M}$ and $\mathcal{N}$ be $\tau$-structures, $\overline{a} \in \mathcal{M}^l$, and $\overline{b} \in \mathcal{N}^l$. Then, $(\mathcal{M}, \overline{a}) \sim_\alpha (\mathcal{N}, \overline{b})$ if and only if $\mathcal{N} \models \phi^{\mathcal{M}}_{\overline{a},\alpha}(\overline{b})$.*

**Proof** We prove this by induction on $\alpha$. Because $(\mathcal{M}, \overline{a}) \sim_0 (\mathcal{N}, \overline{b})$ if and only if they satisfy the same atomic formulas, the lemma holds for $\alpha = 0$.

Suppose that $\gamma$ is a limit ordinal and the lemma is true for all $\alpha < \gamma$. Then

$$\begin{aligned}(\mathcal{M}, \overline{a}) \sim_\gamma (\mathcal{N}, \overline{b}) &\Leftrightarrow (\mathcal{M}, \overline{a}) \sim_\alpha (\mathcal{N}, \overline{b}) \text{ for all } \alpha < \gamma \\ &\Leftrightarrow \mathcal{N} \models \Phi^{\mathcal{M}}_{\overline{a},\alpha}(\overline{b}) \text{ for all } \alpha < \gamma \\ &\Leftrightarrow \mathcal{N} \models \Phi^{\mathcal{M}}_{\overline{a},\gamma}(\overline{b}).\end{aligned}$$

Suppose that the lemma is true for $\alpha$. First, suppose that $\mathcal{N} \models \phi^{\mathcal{M}}_{\overline{a},\alpha+1}(\overline{b})$. Let $c \in \mathcal{M}$. Because

$$\mathcal{N} \models \bigwedge_{x \in \mathcal{M}} \exists w\ \Phi^{\mathcal{M}}_{\overline{a},x,\alpha}(\overline{b}, w),$$

there is $d \in N$ such that $\mathcal{N} \models \Phi^{\mathcal{M}}_{\overline{a},c,\alpha}(\overline{b}, d)$. By induction, $(\mathcal{M}, \overline{a}, c) \sim_\alpha (\mathcal{N}, \overline{b}, d)$. If $d \in \mathcal{N}$, then because

$$\mathcal{N} \models \forall w \bigvee_{c \in \mathcal{M}} \Phi^{\mathcal{M}}_{\overline{a},c,\alpha}(\overline{b}, w)$$

there is $c \in \mathcal{M}$ such that $\mathcal{N} \models \Phi^{\mathcal{M}}_{\overline{a},c,\alpha}(\overline{b}, d)$ and $(\mathcal{M}, \overline{a}, c) \sim_\alpha (\mathcal{N}, \overline{b}, d)$. Thus $(\mathcal{M}, \overline{a}) \sim_{\alpha+1} (\mathcal{N}, \overline{b})$.

Suppose, on the other hand, that $(\mathcal{M}, \overline{a}) \sim_{\alpha+1} (\mathcal{N}, \overline{b})$. Suppose that $c \in \mathcal{M}$, then there is $d \in \mathcal{N}$ such that $(\mathcal{M}, \overline{a}, c) \sim_\alpha (\mathcal{N}, \overline{b}, d)$ and $\mathcal{N} \models \Phi^{\mathcal{M}}_{\overline{a},c,\alpha}(\overline{b}, d)$. Similarly, if $d \in \mathcal{N}$, then there is $c \in \mathcal{M}$ such that $\mathcal{N} \models \Phi^{\mathcal{M}}_{\overline{a},c,\alpha}(\overline{b}, d)$. Thus, $\mathcal{N} \models \Phi^{\mathcal{M}}_{\overline{a},\alpha+1}(\overline{b})$, as desired. $\square$

A similar induction shows that even more is true. The formula $\Phi^{\mathcal{M}}_{\overline{a},\alpha}$ depends only on the $\sim_\alpha$-class of $(\mathcal{M}, \overline{a})$.

**Exercise 2.16** Prove that if $(\mathcal{M}, \overline{a}) \sim_\alpha (\mathcal{N}, \overline{b})$ if and only if $\Phi^{\mathcal{M}}_{\overline{a},\alpha}(\overline{v}) = \Phi^{\mathcal{N}}_{\overline{b},\alpha}(\overline{v})$

**Lemma 2.17** *For any infinite $\tau$-structure $\mathcal{M}$, there is an ordinal $\alpha < |M|^+$ such that if $\overline{a}, \overline{b} \in \mathcal{M}^l$ and $(\mathcal{M}, \overline{a}) \sim_\alpha (\mathcal{M}, \overline{b})$, then $(\mathcal{M}, \overline{a}) \sim_\beta (\mathcal{M}, \overline{b})$ for all $\beta$. The least such $\alpha$ is called the* Scott rank *of $\mathcal{M}$ and denoted* $\mathrm{sr}(\mathcal{M})$.

**Proof** Let $\Gamma_\alpha = \{(\overline{a}, \overline{b}) : \overline{a}, \overline{b} \in \mathcal{M}^l$ for some $l = 0, 1, \ldots$ and $(\mathcal{M}, \overline{a}) \not\sim_\alpha (\mathcal{M}, \overline{b})\}$. Clearly, $\Gamma_\alpha \subseteq \Gamma_\beta$ for $\alpha < \beta$.

**claim 1**: If $\Gamma_\alpha = \Gamma_{\alpha+1}$, then $\Gamma_\alpha = \Gamma_\beta$ for all $\beta > \alpha$.

We prove this by induction on $\beta$. If $\beta$ is a limit ordinal and the claim holds for all $\gamma < \beta$, then it also holds for $\beta$. Suppose that the claim is true for $\alpha \leq \beta$ and we want to show that it holds for $\beta + 1$. Suppose that $(\mathcal{M}, \overline{a}) \sim_\beta (\mathcal{M}, \overline{b})$ and $c \in \mathcal{M}$. Because $(\mathcal{M}, \overline{a}) \sim_{\alpha+1} (\mathcal{M}, \overline{b})$, there is $d \in \mathcal{N}$ such that $(\mathcal{M}, \overline{a}, c) \sim_\alpha (\mathcal{M}, \overline{b}, d)$. By our inductive assumption, $(\mathcal{M}, \overline{a}, c) \sim_\beta (\mathcal{M}, \overline{b}, d)$. Similarly, if $d \in \mathcal{M}$, then there is $c \in \mathcal{M}$ such that $(\mathcal{M}, \overline{a}, c) \sim_\beta (\mathcal{M}, \overline{b}, d)$. Thus, $(\mathcal{M}, \overline{a}) \sim_{\beta+1} (\mathcal{M}, \overline{b})$ as desired.

**claim 2**: There is an ordinal $\alpha < |M|^+$ such that $\Gamma_\alpha = \Gamma_{\alpha+1}$.

Suppose not. Then, for each $\alpha < |\mathcal{M}|^+$, choose $(\overline{a}_\alpha, \overline{b}_\alpha) \in \Gamma_{\alpha+1} \setminus \Gamma_\alpha$. Because $\Gamma_\alpha \subseteq \Gamma_\beta$ for $\alpha < \beta$, the function $\alpha \mapsto (\overline{a}_\alpha, \overline{b}_\alpha)$ is one-to-one. Because there are only $|\mathcal{M}|$ finite sequences from $\mathcal{M}$ this is impossible. $\square$

A more subtle notion of Scott rank is sometimes useful

**Definition 2.18** For $\overline{a} \in \mathcal{M}$ define $r(\overline{a})$ as the least ordinal $\alpha$ such that for all $\overline{b}$

$$(\mathcal{M}, \overline{a}) \sim_\alpha (\mathcal{M}, \overline{b}) \Rightarrow (\mathcal{M}, \overline{a}) \sim_{\alpha+1} (\mathcal{M}, \overline{b}).$$

Then

$$sr(\mathcal{M}) = \sup\{r(\overline{a}) : \overline{a} \in \mathcal{M}\}.$$

We also define

$$\mathrm{SR}(\mathcal{M}) = \sup\{r(\overline{a}) + 1 : \overline{a} \in \mathcal{M}\}.$$

**Exercise 2.19** Calculate SR for the following structures.

a) $(\omega, <)$
b) $(\omega + 1, <)$;
c) $(\omega + \omega, <)$;
d) $(\mathbb{Z}, +, 0)$.

We will see later that at limit ordinals $\alpha$ there can be interesting distinctions between $\mathrm{SR}(\mathcal{M}) = \alpha$ and $\mathrm{SR}(\mathcal{M}) = \alpha + 1$ while in both cases $\mathrm{sr}(\mathcal{M}) = \alpha$.

**Exercise 2.20** Formulate and prove a version of Lemma 2.17 for finite $\tau$-structures.

**Exercise 2.21** a) Show that for all $\beta < \omega_1$ and $n \in \omega$ there is an $\mathcal{L}_{\omega_1,\omega}(\tau)$-formula $S^n_\beta(\overline{v}, \overline{w})$ where $|\overline{v}| = |\overline{w}| = n$ such that

$$\mathcal{M} \models S^n_\beta(\overline{a}, \overline{b}) \Leftrightarrow (\mathcal{M}, \overline{a}) \sim_\alpha (\mathcal{N}, \overline{b})$$

for any $\tau$-structure $\mathcal{M}$.

b) Show that for all $\alpha < \omega_1$ there is $\sigma_\alpha \in \mathcal{L}_{\omega_1,\omega}(\tau)$ such that $\mathcal{M} \models \sigma_\alpha$ if and only if $SR(\mathcal{M}) \geq \alpha$.

We conclude this section with Scott's Isomorphism Theorem

Let $\mathcal{M}$ be an infinite $\tau$-structure of cardinality $\kappa$, and let $\alpha$ be the Scott rank of $\mathcal{M}$. Let $\Phi^{\mathcal{M}}$ be the sentence

$$\Phi^{\mathcal{M}}_{\emptyset,\alpha} \wedge \bigwedge_{l=0}^{\infty} \bigwedge_{\overline{a} \in M^l} \forall \overline{v} (\Phi^{\mathcal{M}}_{\overline{a},\alpha}(\overline{v}) \to \Phi^{\mathcal{M}}_{\overline{a},\alpha+1}(\overline{v})).$$

Because all of the conjunctions and disjunctions in $\Phi^{\mathcal{M}}_{\overline{a},\beta}$ are of size $\kappa$, $\Phi^{\mathcal{M}}_{\overline{a},\beta} \in \mathcal{L}_{\kappa^+,\omega}$ for all ordinals $\beta < \kappa^+$. Thus $\Phi^{\mathcal{M}}$ is an $\mathcal{L}_{\kappa^+,\omega}$-sentence. We call $\Phi^{\mathcal{M}}$ the *Scott sentence* of $\mathcal{M}$. If $\tau$ is a countable vocabulary and $\mathcal{M}$ is countable $\tau$-structure, then $\Phi^{\mathcal{M}} \in \mathcal{L}_{\omega_1,\omega}$.

**Theorem 2.22 (Scott's Isomorphism Theorem)** *Let $\mathcal{M}$ and $\mathcal{N}$ be $\tau$-structures, and let $\Phi^{\mathcal{M}} \in \mathcal{L}_{\infty,\omega}$ be the Scott sentence of $\mathcal{M}$. Then, $\mathcal{N} \equiv_{\infty,\omega} \mathcal{M}$ if and only if $\mathcal{N} \models \Phi^{\mathcal{M}}$.*

*In particular, if $\mathcal{M}$ and $\mathcal{N}$ are countable then $\phi \in \mathcal{L}_{\omega_1,\omega}$ and, if $\mathcal{N} \models \Phi^{\mathcal{M}}$, then $\mathcal{N} \cong \mathcal{M}$.*

**Proof** Clearly if $\mathcal{N} \equiv_{\infty,\omega} \mathcal{M}$, then $\mathcal{N} \models \Phi^{\mathcal{M}}$.

For the other direction, suppose $\mathcal{N} \models \Phi^{\mathcal{M}}$. We will show that $\mathcal{N} \cong_p \mathcal{M}$. Let $\alpha$ be the Scott rank of $\mathcal{M}$. Let $P$ be the set of partial functions $f$ with finite domain from $\mathcal{M}$ to $\mathcal{N}$ such that if $\overline{a}$ is the domain of $f$ and $f(\overline{a}) = \overline{b}$ then $\mathcal{N} \models \Phi^{\mathcal{M}}_{\overline{a},\alpha}(\overline{b})$. Since $\mathcal{N} \models \Phi^{\mathcal{M}}$, the empty map is in $P$ so $P \neq \emptyset$.

Suppose we have $f$ in $P$ with domain $\overline{a}$ with $f(\overline{a}) = \overline{b}$. Let $c \in \mathcal{M}$. Since $\mathcal{N} \models \Phi^{\mathcal{M}} \wedge \Phi^{\mathcal{M}}_{\overline{a},\alpha}(\overline{b})$, $\mathcal{N} \models \Phi^{\mathcal{M}} \wedge \Phi^{\mathcal{M}}_{\overline{a},\alpha+1}(\overline{b})$. But $\Phi^{\mathcal{M}}_{\overline{a},\alpha+1}(\overline{v}) \to \exists w\ \Phi^{\mathcal{M}}_{\overline{a},c,\alpha}(\overline{v}, w)$. Thus there is $d \in \mathcal{N}$ such that $\mathcal{N}$ models $\Phi^{\mathcal{M}}_{\overline{a},c,\alpha}(\overline{b}, d)$ and we can extend $f$ by sending $c$ to $d$.

On the other hand suppose we are given $\widehat{d} \in \mathcal{N}$. Since $\mathcal{N} \models \Phi^{\mathcal{M}}_{\overline{a},\alpha+1}(\overline{b})$,

$$\mathcal{N} \models \forall w \bigvee_{m \in \mathcal{M}} \Phi^{\mathcal{M}}_{\overline{a},m,\alpha}(\overline{b}, w).$$

Thus there is $\widehat{c} \in \mathcal{N}$ such that $\mathcal{N} \models \Phi^{\mathcal{M}}_{\overline{a},\widehat{c},\alpha}(\overline{b}, d)$ and we can extend $f$ be sending $\widehat{c}$ to $\widehat{d}$.

Thus $P$ is a partial isomorphism system and $\mathcal{M} \cong_p \mathcal{N}$. By Theorem 2.4 $\mathcal{M} \equiv_\infty \mathcal{N}$ and, by Exercise 2.2, if $\mathcal{M}$ and $\mathcal{N}$ are countable, then $\mathcal{M} \cong \mathcal{N}$. □

We use this to prove a converse to Exercise 2.6.

**Corollary 2.23** *Let $G$ be the free abelian group on countably many generators and let $H$ be an $\aleph_1$-free group that is not free on a finite set of generators. Then $G \equiv_{\infty,\omega} H$.*

**Proof** Let $\Phi^G \in \mathcal{L}_{\omega_1,\omega}$ be the Scott sentence of $G$. Let $\mathbb{A}$ be a countable fragment of $\mathcal{L}_{\infty,\omega}$ containing $\Phi^G$ and a sentence asserting $H$ is not free on finitely many generators. By Downward Löwenheim–Skolem we can find $H_1 \prec_{\mathbb{A}} H$ containing countably many independent elements with $|H_1| = \aleph_0$. Since $H$ is $\aleph_1$-free, $H$ is free abelian on countably many generators and hence isomorphic to $G$. But then $H_1 \models \Phi^G$ and hence $H \models \Phi^G$. Thus $H \equiv_{\infty,\omega} G$. □

A stronger version of this result is proved in Exercise 2.57.

**Exercise 2.24** Show that the class of $\aleph_1$-free abelian groups is $\mathcal{L}_{\omega_1,\omega}$-axiomatizable.

### Complete Sentences

**Definition 2.25** A satisfiable sentence $\phi \in \mathcal{L}_{\infty,\omega}$ is *complete* if for all $\psi \in \mathcal{L}_{\infty,\omega}$

$$\phi \models \psi \text{ or } \phi \models \neg\psi.$$

**Corollary 2.26** *For any $\mathcal{M}$ the Scott sentence $\Phi^{\mathcal{M}}$ is complete.*

**Exercise 2.27** For any satisfiable sentence $\psi \in \mathcal{L}_{\omega_1,\omega}$ there is a complete sentence $\phi$ such that $\phi \models \psi$.

**Exercise 2.28** A satisfiable sentence $\phi \in \mathcal{L}_{\omega_1,\omega}$ is complete if and only if it is $\aleph_0$-categorical.

**Exercise 2.29** Suppose $\psi \in \mathcal{L}_{\omega_1,\omega}$ is $\kappa$-categorical for some infinite cardinal $\kappa$. Show that if $\mathcal{M},\mathcal{N} \models \psi$ of cardinality at least $\kappa$ then $\mathcal{M} \equiv_{\infty,\omega} \mathcal{N}$. (Compare this to Exercise 5.10.)

**Exercise 2.30** Give an example of an $\aleph_1$-categorical $\phi \in \mathcal{L}_{\omega_1,\omega}$ that is not complete.

**Exercise 2.31** Let $\sigma$ be a complete $\mathcal{L}_{\omega_1,\omega}$-sentence. and let $\mathcal{M}$ be the unique countable model. Show that the following are equivalent.

i) There is an uncountable model of $\sigma$.

ii) There is a non-surjective $\mathcal{L}_{\omega_1,\omega}$-elementary embedding of $\mathcal{M}$ into itself.

iii) There is a non-surjective $\mathbb{A}$-elementary embedding of $\mathcal{M}$ into itself, where $\mathbb{A}$ is a countable fragment of $\mathcal{L}_{\omega_1,\omega}$-containing $\phi$.

Gao [14] has shown these conditions are equivalent to the non-existence of a complete left-invariant metric on $\mathrm{Aut}(\mathcal{M})$ the automorphism group of $\mathcal{M}$.

For complete sentences in $\mathcal{L}_{\omega_1,\omega}$ we can give an improvement of Chang's Theorem 1.31.

**Theorem 2.32** *Suppose $\tau$ is countable and $\phi$ be a complete sentence of $\mathcal{L}_{\omega_1,\omega}(\tau)$. There is $\tau^* \supseteq \tau$ and $T^*$ a complete first order $\tau^*$-theory such that $\mathcal{M} \models \phi$ if and only $\mathcal{M}$ is the $\tau$-reduct of an atomic model of $T^*$.*

**Proof** Let $\mathcal{M}$ be the unique countable model of $\phi$ and let $\alpha$ be its Scott rank. Let $\mathbb{A}$ be a countable fragment containing all formulas $\Phi^{\mathcal{M}}_{\overline{a},\alpha}$ for $\overline{a} \in \mathcal{M}$. Now follow the proof of Theorem 1.31 to obtain $\tau^* = \tau \cup \{R_\psi : \psi \in \mathbb{A}\}$ and $\Gamma$ a collection of $\tau^*$-types such that models of $\Phi^{\mathcal{M}}$ are exactly the reducts of models of $T^*$ that omit all types in $\Gamma$. Let $T$ be the complete $\tau^*$-theory of $\mathcal{M}$. As $\mathcal{M}$ omits all of the types in $\Gamma$, no type in $\Gamma$ is implied by a principal type. Thus all types in $\Gamma$ are omitted in any atomic model of $T^*$ and the reduct of an atomic model of $T^*$ is a model of $\phi$.

For any $\mathcal{N} \models \phi$ we can take its natural expansion as a $\tau^*$-structure. For any $\overline{a} \in \mathcal{N}$ there is $\overline{b} \in \mathcal{M}$ such that $R_{\Phi^{\mathcal{M}}_{\overline{b},\alpha}}(\overline{v})$ isolates $\mathrm{tp}^{\mathcal{N}}(\overline{a})$. Thus the expansion is an atomic model of $T^*$. □

In later sections we will be looking at uncountable structures that are models of complete sentences. We would like to give a characterization of when this happens.

**Definition 2.33** Let $\mathbb{A}$ be a fragment of $\mathcal{L}_{\omega_1,\omega}(\tau)$. If $\mathcal{M}$ is a $\tau$-structure and $a_1, \ldots, a_n \in \mathcal{M}$ then the $\mathbb{A}$-*type* of $\overline{a}$ is

$$\mathrm{tp}^{\mathcal{M}}(\overline{a}, \mathbb{A}) = \{\phi(\overline{v}) \in \mathbb{A} : \mathcal{M} \models \phi(\overline{a})\}$$

and $S_n(\mathbb{A}, \phi)$ is the set of $\mathbb{A}$-types of $n$-tuples realized in some model of $T$.

**Exercise 2.34** Let $\tau$ be a countable vocabulary and let $\mathcal{M}$ be a countable $\tau$-structure.

a) Suppose there is a countable $\mathcal{N}$ such that $\mathcal{M} \equiv_{\infty,\omega} \mathcal{N}$. Show that $\mathcal{M}$ realizes only countably many $\mathcal{L}_{\omega_1,\omega}$-types.

For the remainder of the Exercise assume $\mathcal{M}$ realizes countably many $\mathcal{L}_{\omega_1,\omega}$-types.

b) Prove that $\mathcal{M}$ has countable Scott rank.

c) Prove that there is a countable $\mathcal{N} \equiv_{\infty,\omega} \mathcal{M}$.

## 2.3 Countable Approximations

In this section we give another characterization of $\equiv_{\infty,\omega}$ due to Kueker [26]. We begin with some set theoretic preliminaries. Kueker, and independently, Jech, introduced the following generalization of the the filter of closed unbounded subsets of a regular cardinal.

For any set $X$ let $\mathcal{P}_{\omega_1}(X)$ be the collection of countable subsets of $X$.

**Definition 2.35** $A \subseteq \mathcal{P}_{\omega_1}(X)$ is *closed* if $\bigcup a_n \in X$ for any $a_1 \subset a_2 \subset \ldots$ in $X$ and $A \subseteq \mathcal{P}_{\omega_1}(X)$ is *unbounded* if for all $a \subset X$ countable, there is $b \in A$ with $a \subseteq b$.

**Lemma 2.36** *Suppose $A_0, A_1, \ldots \in \mathcal{P}_{\omega_1}(X)$ are closed and unbounded. Then so is $\bigcap A_i$ is closed and unbounded.*

**Proof** It is clear that $\bigcap A_i$ is closed, so we need only show it is unbounded.

Let $a \in \mathcal{P}_{\omega_1}(X)$. Let $\langle \cdot, \cdot \rangle : \omega^2 \to \omega$ be a bijective pairing function such that $\langle m, i \rangle < \langle m, j \rangle$ if $i < j$. We build

$$a = a_{-1} \subset a_0 \subset a_1 \subset \ldots \in \mathcal{P}_{\omega_1}(X)$$

such that if $n = \langle i, j \rangle$ we choose $a_n \in A_i$ with $a_{n-1} \subset a_n$. This is always possible since each $A_i$ is unbounded. Then

$$b = \bigcap_{n \in \omega} a_n = \bigcap_{j \in \omega} a_{\langle i,j \rangle} \in A_i$$

for all $i \in \omega$. Thus $b \in \bigcap A_i$ and $\bigcap A_i$ is unbounded. $\square$

Let $D(X) = \{A \subseteq \mathcal{P}_{\omega_1} : A$ has a closed, unbounded subset$\}$.

**Exercise 2.37** Show that $D(X)$ is a countably complete filter on $\mathcal{P}_{\omega_1}(X)$.

We also have closure under diagonal intersections.

**Lemma 2.38** *Suppose $I \subseteq X$ and $A_i$ is closed and unbounded for all $i \in I$ then so is*

$$\triangle A_i = \{a \in \mathcal{P}_{\omega_1}(X) : a \in A_i \text{ for all } i \in a \cap I\}.$$

**Proof** Suppose $a_0 \subset a_1 \subset \ldots$ are in $\triangle A_i$ and $a = \bigcup a_i$. Let $j \in a \cap I$. Then there is $n_0$ such that $j \in a_n$ for $n \geq n_0$. But then $a_n \in A_j$ for $n \geq n_0$ and, since $A_j$ is closed, $a \in A_i$. Thus $a \in \triangle A_i$ and $\triangle A_i$ is closed.

Let $a \in \mathcal{P}_{\omega_1}(X)$. Let $a_0 = a$. Given $a_n$, we know from Lemma 2.36 that $\bigcap_{i \in a_n \cap I} A_i$ is closed and unbounded. Thus there is

$$a_n \subset a_{n+1} \in \bigcap_{i \in a_n \cap I} A_i.$$

Let $a = \bigcup a_n$. Suppose $i \in a$. There is $n_0$ such that $i \in a_n$ for $n \geq n_0$. Thus $a_{n+1} \in A_i$ for $n \geq n_0$ and, since $A_i$ is closed, $a \in A_i$. Thus $a \in \triangle A_i$ and $\triangle A_i$ is unbounded. $\square$

Kueker also gave a useful game theoretic formulation of membership in $D(X)$.

**Exercise 2.39** For $A \subseteq \mathcal{P}_{\omega_1}(X)$ consider the game $G(A)$ where Players I and II alternate playing elements of $X$.

| Player I | Player II |
|---|---|
| $c_0$ | |
| | $d_0$ |
| $c_1$ | |
| | $d_1$ |
| $\vdots$ | $\vdots$ |

Player II wins if $\{c_i, d_i : i \in \omega\}$ is in $A$. Prove that Player II has a winning strategy in $G(A)$ if and only if $A \in D(X)$.

We use the filter $D(X)$ as a notion of "almost all". Let $X$ be a set. Let $\tau$ be a countable vocabulary and let $\mathcal{M}$ be a $\tau$-structure. We say that $X$ is *large enough to approximate* $\mathcal{M}$ if the universe of $\mathcal{M}$ is contained in $X$.

**Exercise 2.40** Suppose $X$ is large enough to approximate $\mathcal{M}$. Show that $\{a \in \mathcal{P}_{\omega_1}(X) : \mathcal{M} \cap a \text{ is a substructure of } \mathcal{M}\} \in D(X)$.

Let $\mathcal{M}^a$ denote $\mathcal{M} \cap a$. We say that $\mathcal{M}^a$ is a substructure *almost everywhere.*

**Definition 2.41** Suppose $\phi \in \mathcal{L}_{\infty,\omega}$ and $a \in \mathcal{P}_{\omega_1}(X)$ we define the approximation $\phi^a$ to $\phi$ inductively as follows:

- $\phi^a = \phi$ if $\phi$ is atomic;
- $(\neg\phi)^a = \neg\phi^a$;
- $(\forall v\ \phi)^a = \forall v\ \phi^a$ and $(\exists v\ \phi)^a = \exists v\ \phi^a$;
- $\left(\bigwedge_{i\in I} \phi_i\right)^a = \bigwedge_{i\in I\cap a} \phi_i^a$ and $\left(\bigvee_{i\in I} \phi_i\right)^a = \bigvee_{i\in I\cap a} \phi_i^a$.

We say that $X$ is *large enough to approximate* $\phi$ if $I \subset X$ whenever $\bigvee_{i\in I} \psi_i$ or $\bigwedge_{iinI} psi_i$ is in the smallest fragment of $\mathcal{L}_{\infty,\omega}$ containing $\phi$.

**Exercise 2.42** Show that if $\phi \in \mathcal{L}_{\infty,\omega}$ and $\psi$ is the canonical formula in negation normal form (see Exercise 1.21) such that $\models \phi \leftrightarrow \psi$, then for any $\mathcal{M}$ if $X$ is large enough to approximate $\phi$ and $\mathcal{M}$, then

$$\models \phi^a \leftrightarrow \psi^a \text{ a.e..}$$

We work in an $X$ large enough to approximate $\mathcal{M}$ and $\phi$. Intuitively, we think of $X$ as some sufficiently rich substructure of the universe of sets $\mathbb{V}$ containing $\mathcal{M}$ and $\phi$.

**Theorem 2.43** *For any $\mathcal{M}$ and $\phi \in \mathcal{L}_{\infty,\omega}$*

$$\mathcal{M} \models \phi \textit{ if and only if } \mathcal{M}^a \models \phi^a \textit{ a.e.}$$

*i.e.,* $\{a \in \mathcal{P}_{\omega_1}(X) : \mathcal{M}^a \models \phi^a\} \in D(X)$.

**Proof** First note that, since $D(X)$ is a filter, it is impossible that

$$\mathcal{M}^a \models \phi^a \text{ a.e. and } \mathcal{M}^a \models \neg\phi^a \text{ a.e..}$$

Thus it suffices to prove $\Rightarrow$ direction.

Next note that, by Exercise 2.42, we may assume that $\phi$ is in negation normal form. We prove by induction that if $\psi$ is a subformula of $\phi$, $\bar{b} \in \mathcal{M}$ and $\mathcal{M} \models \psi(\bar{b})$, then $\mathcal{M}^a \models \psi^a(\bar{b})$ a.e.. Note that $\bar{b} \in \mathcal{M}^a$ a.e., so this makes sense.

- If $\psi$ is atomic or negated atomic this is clear since $\mathcal{M}^a$ is a substructure of $\mathcal{M}$. Note that these are the only negations we need to deal with.

- If $\psi$ is $\bigvee_{i \in I} \theta_i(\bar{b})$, then there is $j \in I$ such that $\mathcal{M} \models \theta_j(\bar{b})$. Note that $j \in a$ for almost every $a \in P_{\omega_1}(X)$. By induction $\mathcal{M}^a \models \theta_j^a(\bar{b})$ a.e.. Thus $\mathcal{M}^a \models \psi^a(\bar{b})$ a.e..

- Suppose $\psi$ is $\bigwedge_{i \in I} \theta_i(\bar{b})$ and $\mathcal{M} \models \psi(\bar{b})$. By induction,

  $$A_i = \{a : \mathcal{M}^a \models \theta_i^a(\bar{b})\} \in D(X)$$

  for all $i \in I$. By Lemma 2.38, $\triangle A_i \in D(X)$. Let $a \in \triangle D(X)$. If $i \in a \cap I$, then $a \in A_i$ so $\mathcal{M}^a \models \theta_i^a(\bar{b})$. Thus $\mathcal{M}^a \models \psi^a$ a.e..

- If $\psi$ is $\exists v\ \theta(v, \bar{b})$, this is similar to the $\bigvee$-case.

- if $\psi$ is $\forall v\ \theta(v, \bar{b})$ this is similar to the $\bigwedge$-case.

□

**Exercise 2.44** Fill in the details in the $\exists v\ \theta$ and $\forall v\ \theta$ cases.

**Corollary 2.45** *i) If $\phi \in \mathcal{L}_{\omega_1,\omega}$, then $\mathcal{M} \models \phi$ if and only if $\mathcal{M}^a \models \phi$ a.e..*
*ii) If $\mathcal{M}$ is countable and $\phi \in \mathcal{L}_{\infty,\omega}$, $\mathcal{M} \models \phi$ if and only if $\mathcal{M} \models \phi^a$ a.e..*

**Theorem 2.46** *i) $\mathcal{M} \equiv_{\infty,\omega} \mathcal{N} \Leftrightarrow \mathcal{M}^a \cong \mathcal{N}^a$ a.e..*
*ii) $\mathcal{M} \not\equiv_{\infty,\omega} \mathcal{N} \Leftrightarrow \mathcal{M}^a \not\cong \mathcal{N}^a$ a.e..*

**Proof** Note that filter $D(X)$ is not an ultrafilter, thus ii) is not an immediate consequence of i). Though, since it is a filter, it is enough to prove the $\Rightarrow$ directions of both i) and ii).

Suppose $\mathcal{M} \not\equiv_{\infty,\omega} \mathcal{N}$. Let $\phi \in \mathcal{L}_{\infty,\omega}$ such that $\mathcal{M} \models \phi$ and $\mathcal{N} \models \neg\phi$. By the previous theorem, $\mathcal{M}^a \models \phi^a$ a.e. and $\mathcal{N}^a \models \neg\phi^a$ a.e. Since $D(X)$ is a filter $\{a : \mathcal{M}^a \models \phi^a \text{ and } \mathcal{N}^a \models \neg\phi^a\} \in D(X)$ and, thus, $\{a : \mathcal{M}^a \not\cong \mathcal{N}^a\} \in D(X)$.

Suppose $\mathcal{M} \equiv_{\infty,\omega} \mathcal{N}$. By Theorem 2.4 there is a non-empty system $P$ of partial isomorphism with finite domains from $\mathcal{M}$ to $\mathcal{N}$. Let $A = \{a \in \mathcal{P}(\omega_1(X))$ : such that $\{f \in P : \mathrm{dom}(f) \subset \mathcal{M}^a$ and $\mathrm{img}(f) \subset \mathcal{N}^a\}$ is a nonempty system of partial isomorphism between $\mathcal{M}^a$ and $\mathcal{N}^a\}$.

**Exercise 2.47** Show that $A \in D(X)$ and prove that $\mathcal{M}^a \cong \mathcal{N}^a$ for $a \in A$.

□

**Corollary 2.48** *If $\mathcal{M}$ is countable, then $\mathcal{M} \equiv_{\infty,\omega} \mathcal{N}$ if and only if $\mathcal{M} \cong \mathcal{N}^a$ a.e..*

The corollary leads to another proof that an abelian group is $\equiv_{\infty,\omega}$ to a free abelian group if and only if every countable subgroup is free.

## 2.4 Larger Infinitary Languages

For completeness, we will briefly mention the the logics $\mathcal{L}_{\kappa,\lambda}$. In these logics we allow a more general quantification rule. We will say very little in this section and refer to reader to [11] for a more detailed survey.

Suppose $\phi$ is a formula and $\vec{v}$ is a sequence of variables freely occuring in $\phi$ with $|\vec{v}| < \lambda$, then $\exists \vec{v}\ \phi$ is a formula.

Thus we are allowed existential and, by taking negations in the usual way, universal quantification over sequences of variables of cardinality less than $\lambda$.

**Exercise 2.49** Show there is $\phi \in \mathcal{L}_{\omega_1,\omega_1}$ such that $\mathcal{M} \models \phi$ if and only if $|M| > \aleph_0$.

**Exercise 2.50** Show there is an $\mathcal{L}_{\omega_1,\omega_1}$-sentence $\phi$ such that $(A,<) \models \phi$ if and only if $(A,<)$ is a well-order.

**Exercise 2.51** Show that is an $\mathcal{L}_{\omega_1,\omega_1}$-sentence $\phi$ such that $(A,<) \models \phi$ if and only if $(A,<) \cong (\mathbb{R},<)$.

**Exercise 2.52** Show that there is an $\mathcal{L}_{\omega_1,\omega_1}$ sentence $\phi$ such that $\phi$ has a model of cardinality $\kappa$ if and only if the cofinality of $\kappa$ is $\omega$.

**Exercise 2.53** Recall that a linear ordering $(A,<)$ is $\aleph_1$-*like* if $|A| = \aleph_1$ and $|\{b : b < a\}| \leq \aleph_0$ for all $a \in A$. Show that the class of $\aleph_1$-like linear orders is $\mathcal{L}_{\omega_1,\omega_1}$-axiomatizable.

**Definition 2.54** We say $\mathcal{M} \cong_p^\kappa \mathcal{N}$ if and only if there is a nonempty system of partial embeddings $P$ with the property that:

i) if $f \in P$, $A \subset \mathcal{M}$ and $|A| < \kappa$, then there is $g \in P$, $f \subseteq g$ and $A \subseteq \mathrm{dom}(g)$ and

ii) if $f \in P$, $B \subset \mathcal{N}$ and $|B| < \kappa$, then there is $g \in P$, $f \subseteq g$ and $B \subseteq \mathrm{img}(g)$.

Our original notion $\cong_p$ is $\cong_p^{\aleph_0}$.

**Exercise 2.55** a) Prove $\mathcal{M} \cong_p^\kappa \mathcal{N} \Rightarrow \mathcal{M} \equiv_{\infty,\kappa} \mathcal{N}$ by induction on complexity of formulas.

b) Prove $\mathcal{M} \equiv_{\infty,\kappa} \mathcal{N} \Rightarrow \mathcal{M} \cong_p^\kappa \mathcal{N}$. [Hint: Let $P$ be the set of $f : A \to \mathcal{N}$ where $|A| < \kappa$ and $f$ is $\mathcal{L}_{\infty,\kappa}$-elementary.]

**Exercise 2.56** Let $\mathcal{M}$ and $\mathcal{N}$ be $\aleph_1$-like dense linear orders with out end-points. Prove $\mathcal{M} \equiv_{\infty,\omega_1} \mathcal{N}$. [Hint: Consider $P = \{f : A \to B : A$ an initial segment of $\mathcal{M}$, $B$ is an initial segment of $\mathcal{N}$, $\mathcal{M} \setminus A$ and $\mathcal{N} \setminus B$ has no least element and $f$ is order preserving$\}$.]

Note in A.2 we show there are $2^{\aleph_1}$ non-isomorphic $\aleph_1$-like dense linear orders.

**Exercise 2.57** Suppose $G$ and $H$ are uncountable $\aleph_1$-free abelian groups. Prove that $G \equiv_{\infty,\omega_1} H$. [Hint: ?? ]

In Exercise A.4 we show that there are $2^{\aleph_1}$ non-isomorphic $\aleph_1$-free abelian groups of cardinality $\aleph_1$.

# 3 The Space of Countable Models

There are many connections between the model theory of $\mathcal{L}_{\omega_1,\omega}$ and descriptive set theory. Suppose $\tau$ is a countable vocabulary. The first observation is that the set of countably infinite $\tau$-structures can naturally be given the structure of a Polish space. The reader should consult [21] for any unexplained notions or results from descriptive set theory.

## 3.1 Spaces of $\tau$-structures

To each countable vocabulary $\tau$ we will define a Polish space on the set of $\tau$-structures with universe $\omega$. The notation in the general case makes this seem more mysterious then it is, so we first illustrate with a simple example. Let $\tau_0$ be the language with one binary relation symbol $R$. In this case we will let $\mathbb{X}_{\tau_0} = 2^{\omega\times\omega}$. If $\mathcal{M}$ is a $\tau_0$-structure with universe $\omega$, define $f_\mathcal{M} : \omega \times \omega \to 2$ by $f_\mathcal{M}(i,j) = 1$ if and only if $\mathcal{M} \models R(i,j)$. Similarly, for any $f : \omega^2 \to 2$ we can define $\mathcal{M}_f$ as the $\tau_0$-structure on $\omega$ with binary relation $R_f = \{(i,j) : f(i,j) = 1\}$. The maps $\mathcal{M} \mapsto f_\mathcal{M}$ and $f \mapsto \mathcal{M}_f$ are easily seen to be inverses. Thus we can identify the set $\tau_0$-structures with universe $\omega$ with elements of $\mathbb{X}_{\tau_0}$. Of course $\mathbb{X}_{\tau_0}$ is a Polish space, indeed it is homeomorphic to the Cantor space.

Suppose we instead consider $\tau_1$ where we have one unary function symbol. Then we could identify $\tau_1$-structures with $\omega^\omega$.

Let $\tau$ be a countable vocabulary with constant symbols $(c_i : i < \alpha_C)$, relations symbols $(R_i : i < \alpha_R)$ and function symbols $(f_i : i < \alpha_F)$ where $\alpha_C, \alpha_R, \alpha_F \leq \omega$. For each $i < \alpha_R$ and $j < \alpha_F$, let $m(i)$ and $n(j)$ be the arities of $R_i$ and $f_j$ respectively. Define

$$\mathbb{X}_\tau = \omega^{\alpha_C} \times \prod_{i<\alpha_R} 2^{\omega^{m(i)}} \times \prod_{j<\alpha_F} \omega^{\omega^{n(j)}}.$$

Arguing as above, it is easy to see that there is a natural bijection between $\mathbb{X}_\tau$ and the set of all $\tau$-structures with universe $\omega$. The $\omega^{\alpha_C}$ part tells how to interpret the constant symbols, the $\prod_{i<\alpha_R} 2^{\omega^{m(i)}}$ part tells how to interpret the relation symbols and the $\prod_{j<\alpha_F} \omega^{\omega^{n(j)}}$ part tells how to interpret the function symbols. [6]

[6]Note that this definition still makes sense if one or several of $\alpha_C, \alpha_R$ or $\alpha_F$ are zero. In particular if $\tau$ is the empty vocabulary, then $\mathbb{X}_\tau$ is the one element space.

Suppose $\phi(v_{i_1}, \ldots, v_{i_m})$ is an $\mathcal{L}_{\omega_1,\omega}$-formula with free variables from $v_{i_1}, \ldots, v_{i_m}$ and $n_1, \ldots, n_m \in \omega$. Define

$$E_{\phi,\overline{n}} = \{\mathcal{M} \in \mathbb{X}_\tau : \mathcal{M} \models \phi(n_1, \ldots, n_m).\}$$

**Lemma 3.1** *Each $E_{\phi,\overline{n}}$ is a Borel subset of $\mathbb{X}_\tau$.*

**Proof** We prove this by induction on the complexity of $\mathcal{L}_{\omega_1,\omega}$-formulas.

**claim** If $\phi$ is atomic, then $E_{\phi,\overline{n}}$ is clopen.

The formula $\phi$ is either $R(t_1, \ldots, t_l)$ or $t_1 = t_2$ where $t_1, \ldots, t_l$ are terms and $R$ is a relation symbol. If each of the $t_i$ is a variable, this is clear from the construction of $\mathbb{X}_\tau$.

Unwinding the terms, we can find atomic formulas $\psi_1(\overline{v}, \overline{w}), \ldots, \psi_k(\overline{v}, \overline{w})$ each of the form $R(\overline{v})$, $v_i = v_j$, $v_i = c$ or $f(\overline{v}) = v_j$ where $R$ is a relation symbol, $c$ is a constant symbol and $f$ is a function symbol, and

$$\begin{aligned} \phi(\overline{v}) &\Leftrightarrow \exists \overline{w}\ (\psi_1(\overline{v}, \overline{w}) \wedge \ldots \wedge \psi_k(\overline{v}, \overline{w})) \\ &\Leftrightarrow \forall \overline{w}\ [(\psi_1(\overline{v}, \overline{w}) \wedge \ldots \wedge \psi_{k-1}(\overline{v}), \overline{w}) \rightarrow \psi_k(\overline{v}, \overline{w})]. \end{aligned}$$

For example, suppose $\phi(v_1, v_2)$ is $R(v_1, g(f(v_1, v_2)))$. Then

$$\begin{aligned} \phi(v_1, v_2) &\Leftrightarrow \exists w_1 \exists w_2\ [f(v_1, v_2) = w_1 \wedge g(w_1) = w_2 \wedge R(v_1, w_2)] \\ &\Leftrightarrow \forall w_1 \forall w_2 [(f(v_1, v_2) = w_1 \wedge g(w_1) = w_2) \rightarrow R(v_1, w_2)]. \end{aligned}$$

Thus we have $\phi(\overline{v}) \Leftrightarrow \exists \overline{w}\ \psi(\overline{v}, \overline{w}) \Leftrightarrow \forall \overline{w}\ \theta(\overline{v}, \overline{w})$, where each $E_{\psi,\overline{a}}$ and $E_{\theta,\overline{a}}$ is clopen. But then

$$\begin{aligned} E_{\phi,\overline{m}} &= \bigcup_{\overline{a}} E_{\psi,\overline{m},\overline{a}} \text{ an open set} \\ &= \bigcap_{\overline{a}} E_{\theta,\overline{m},\overline{a}} \text{ a closed set} \end{aligned}$$

Thus $E_{\phi,\overline{m}}$ is clopen.

The rest of the induction is easy:

- $E_{\neg\phi,\overline{m}} = \mathbb{X}_\tau \setminus E_{\phi,\overline{m}}$;
- $E_{\bigvee \phi,\overline{m}} = \bigcup E_{\phi,\overline{m}}$;
- $E_{\bigwedge \phi,\overline{m}} = \bigcap E_{\phi,\overline{m}}$;

- $E_{\exists v\phi,\overline{m}} = \bigcup_a E_{\phi,\overline{m},a}$;
- $E_{\forall v\phi,\overline{m}} = \bigcap_a E_{\phi,\overline{m},a}$;

So, by induction on complexity, each $E_{\phi,\overline{m}}$ is Borel. $\square$

**Definition 3.2** If $\phi$ is a sentence of $\mathcal{L}_{\omega_1,\omega}(\tau)$, let

$$\mathrm{Mod}(\phi) = \{\mathcal{M} \in \mathbb{X}_\tau : \mathcal{M} \models \phi\}.$$

Then $\mathrm{Mod}(\phi)$ is a Borel subset of $\mathbb{X}_\tau$ and can be viewed as a standard Borel space. Not every Borel subset of $\mathbb{X}_\tau$ arises as $\mathrm{Mod}(\phi)$ for some $\mathcal{L}_{\omega_1,\omega}$-sentence.

There is a natural action of $S_\infty$,the set of all permutations of $\omega$, on $X_\tau$ by homeomorphisms. If $\mathcal{M} \in \mathbb{X}_\tau$ and $\sigma \in S_\infty$ let $\sigma(\mathcal{M})$ be the $\tau$-structure induced by $\sigma$, i.e.,

$$\sigma(\mathcal{M}) \models \phi(\overline{a}) \Leftrightarrow \mathcal{M} \models \phi(\sigma^{-1}(\overline{a})).$$

We say that $A \subseteq \mathbb{X}_\tau$ is *invariant* if $\sigma(A) = A$ for all $\sigma \in S_\infty$.

**Corollary 3.3** *For any sentence $\phi \in \mathcal{L}_{\omega_1,\omega}(\tau)$, $\mathrm{Mod}(\phi)$ is an invariant Borel subset of $\mathbb{X}_\tau$.*

**Proof** If $\phi$ is a sentence, then $\mathcal{M} \models \phi$ if and only if $\sigma(\mathcal{M}) \models \phi$. $\square$

In Theorem 4.26 we will prove the converse namely that for every invariant Borel set $A \subseteq \mathbb{X}_\tau$ there is an $\mathcal{L}_{\omega_1,\omega}(\tau)$-sentence $\phi$ such that $A = \mathrm{Mod}(\phi)$.

**Corollary 3.4** *For any $\mathcal{M} \in \mathbb{X}_\tau$, $\{\mathcal{N} : \mathcal{N} \cong \mathcal{M}\}$ is invariant Borel.*

**Proof** By Scott's Theorem the isomorphism class of $\mathcal{M}$ is $\mathrm{Mod}(\Phi^{\mathcal{M}})$, where $\Phi^{\mathcal{M}}$ is the Scott sentence of $\mathcal{M}$. $\square$

**Exercise 3.5** Prove that if $\mathcal{K}$ is a $\mathrm{PC}_{\omega_1,\omega}$-class then $\mathcal{K} \cap \mathbb{X}_\tau$ is invariant $\mathbf{\Sigma}^1_1$.

**Exercise 3.6** a) Suppose $\phi \in \mathcal{L}_{\omega_1,\omega}(\tau)$ is in disjunctive normal form, i.e., $\phi$ is $\bigvee_{i\in\omega} \bigwedge_{j\in\omega} \psi_{i,j}$, where each $\psi_{i,j}$ is atomic or negated atomic. Show that $\mathrm{Mod}(\phi)$ is a $\mathbf{\Sigma}^0_2$ subset of $\mathbb{X}_\tau$ (i.e., $\mathrm{Mod}(\phi)$ is an $F_\sigma$-set).

b) Find a vocabulary $\tau$ and an $\mathcal{L}_{\omega_1,\omega}(\tau)$-sentence $\phi$ in conjunctive normal form such that $\phi$ is not equivalent to any $\mathcal{L}_{\omega_1,\omega}(\tau)$-sentence in disjunctive normal form. (Hint: Find $\phi$ such that $\mathrm{Mod}(\phi)$ is $\mathbf{\Pi}^0_2$ but not $\mathbf{\Sigma}^0_2$.)

## 3.2 The Number of Countable Models

One of the nagging long standing open problems in the model theory of first order logic is *Vaught's Conjecture* that if $T$ is a complete first order theory in a countable language, then the number of non-isomorphic models countable of $T$ is either at most $\aleph_0$ or exactly $2^{\aleph_0}$. It is natural to extend the conjecture from first order logic to models of an $\mathcal{L}_{\omega_1,\omega}$-sentence. Thus, *Vaught's Conjecture for* $\mathcal{L}_{\omega_1,\omega}$ asserts that for any sentence $\phi \in \mathcal{L}_{\omega_1,\omega}$ the number of non-isomorphic countable models is either at most $\aleph_0$ or exactly $2^{\aleph_0}$. Of course, if the Continuum Hypothesis holds, Vaught's Conjecture is a trivial consequence, but we will later give a rephrasing of Vaught's Conjecture that does not depend on the continuum hypothesis. The first progress toward Vaught's Conjecture is still the strongest general result known.

**Theorem 3.7 (Morley [40])** *Let $\phi$ be an $\mathcal{L}_{\omega_1,\omega}$-sentence. Then the number of non-isomorphic countable models of $\phi$ is either at most $\aleph_1$ or exactly $2^{\aleph_0}$.*

We will give Morley's original proof below, but first it worth noting that, from the point of view of modern descriptive set theory, Morley's Theorem is an easy consequence of core theorems on equivalence relations. Consider the isomorphism relation $\cong$ on $\mathbb{X}_\tau$. Then $\mathcal{M} \cong \mathcal{N}$ if and only if

$\exists f \in \omega^\omega$ $f$ is a bijection and $f$ preserves all of the constant, relation and function symbols of $\tau$.

This is easily seen to be $\mathbf{\Sigma}^1_1$. For example, let $\tau$ be the vocabulary with a single binary relation symbol $R$. Then $\mathcal{M} \cong \mathcal{N}$ if and only if there exists $f \in \omega^\omega$ such that $f$ is a bijection and

$$\forall m, n\ \mathcal{M} \models R(m,n) \leftrightarrow \mathcal{M} \models R(f(m), f(n)).$$

Thus for any $\phi$, isomorphism is a $\mathbf{\Sigma}^1_1$-equivalence relation on $\mathrm{Mod}(\phi)$.

We can now apply two important theorem equivalence relations from descriptive set theory. For proofs see, for example [13] Theorems 5.35 and 9.1.5.

**Theorem 3.8** *i)* **(Silver [52])** *If $E$ is a $\mathbf{\Pi}^1_1$-equivalence relation on standard Borel space $X$ with uncountably many equivalence classes, then there is a perfect set of $E$-inequivalent elements. In particular, this is true for any Borel equivalence relation.*

*ii)* **(Burgess)** *If $E$ is a $\mathbf{\Sigma}^1_1$-equivalence relation on standard Borel space $X$ with at least $\aleph_2$ equivalence classes, then there is a perfect set of $E$-inequivalent elements.*

For proofs see, for example [13] 5.3.5 and 9.1.5.

Morely's Theorem is, of course, an easy consequence of Burgess' result. It is worth noting that Burgess's result can't be improved. For example define an equivalence relation on $E$ on $\omega^\omega$ such that $xEy$ if and only if either neither codes a well ordering or there both code linear orders and there is an isomorphism between the orders they code. This is easily seen to be a $\Sigma^1_1$-equivalence relation with exactly $\aleph_1$-classes.

In the isomorphism equivalence relation on $\mathrm{Mod}(\phi)$ we have added information for Corollary 3.4 that every equivalence class is Borel. This is also not enough.

**Exercise 3.9** Define an equivalence relation $E$ on $\omega^\omega$ such that $xEy$ if and only if $\omega_1^x = \omega_1^y$ (see §9 for definitions). Show that $E$ is $\mathbf{\Sigma}^1_1$-equivalence relation with exactly $\aleph_1$ classes, all of which are Borel.

Further examples get closer to Vaught's Conjecture. In particular there are $\mathrm{PC}_{\omega_1,\omega}$ classes with exactly $\aleph_1$-countable models.

**Exercise 3.10 (Friedman)** a) Suppose $\mathcal{M}$ is a countable $\omega$-model of $\mathrm{ZFC}^-$ (i.e., ZFC$-$25 2Power set) and the ordinals of $\mathcal{M}$ are ill-founded. Show that there is an ordinal $\alpha$ such that the ordinals of $\mathcal{M}$ have order type $\alpha+(\mathbb{Q}\times\alpha)$.

b) Show that a) gives rise to a $\mathrm{PC}_{\omega_1,\omega}$-class with exactly $\aleph_1$ non-isomorphic countable models.

**Exercise 3.11 (Kunen)** A linear order is *1-transitive* if for all $a, b$ there is an order preserving automorphism taking $a$ to $b$.

For $\alpha < \omega$, we say that the support of $f : \alpha \to \mathbb{Z}$ is $\{\beta < \alpha : f(\beta) \neq 0\}$ and let $\mathbb{Z}^\alpha$ be the set of finite support functions from $\alpha$ to $\mathbb{Z}$ ordered lexicographically. [Note $\mathbb{Z}^0 = \{\emptyset\}$.]

a) Prove that each $\mathbb{Z}^\alpha$ is 1-transitive.

b) Prove that every countable 1-transitive linear order is isomorphic to $\mathbb{Z}^\alpha$ or $\mathbb{Q} \times \mathbb{Z}_\alpha$ ordered lexicographically.

c) Use b) describe a PC-class with exactly $\aleph_1$ non-isomorphic countable models.

We also know that the isomorphism equivalence relation on $\mathrm{Mod}(\phi)$ is also the orbit equivalence relation of the action of the Polish group $S_\infty$ on the standard Borel space $\mathrm{Mod}(\phi)$. This leads to the Topological Vaught Conjecture which is also still open.

**Topological Vaught Conjecture** Suppose $G$ is a Polish group and there is a Borel-measurable action of $G$ on a standard Borel space $\mathbb{X}$. If $G$ has uncountably many orbits, then there is a perfect set of elements in distinct orbits.

Although Morely's Theorem follow's immediately from Burgess' Theorem, it is instructive to see how to prove it using only Silver's Theorem. For each $\alpha < \omega_1$ let $\sim_\alpha$ be as in Definition 1.16. These are Borel equivalence relations, that is for each $n$ the equivalence relation $(\mathcal{M}, \overline{a}) \sim (\mathcal{N}, \overline{b})$, where $\overline{a}, \overline{b}$ are of length $n$, is a Borel equivalence relation on $\mathbb{X}_\tau \times \omega^n$. This is an easy induction since

- $\sim_0$ is closed since $(\mathcal{M}, \overline{a}) \sim_0 (\mathcal{N}, \overline{b})$ if and only if $\mathcal{M} \models \phi(\overline{a}) \Leftrightarrow \mathcal{N} \models \phi(\overline{b})$ for all quantifier free $\phi$ ;
- for $\alpha$ a limit, $\sim_\alpha = \bigcap_{\beta<\alpha} \sim \beta$;
- $(\mathcal{M}, \overline{a}) \sim_{\alpha+1} (\mathcal{N}, \overline{c}) \Leftrightarrow \forall c \exists d\, (\mathcal{M}, \overline{a}, c) \sim_\alpha (\mathcal{N}, \overline{b}, d)$ and $\forall d \exists c\, (\mathcal{M}, \overline{a}, c) \sim_\alpha (\mathcal{N}, \overline{b}, d)$. Thus $\sim_\alpha$ is a countable intersection of countable unions of Borel sets and, hence, Borel.

For each $\mathcal{M} \models \phi$, if $\alpha$ is the Scott rank of $\mathcal{M}$, then

$$\mathcal{M} \sim_\alpha \mathcal{N} \Leftrightarrow \mathcal{M} \cong \mathcal{N}.$$

Suppose there are $\aleph_2$ non-isomorphic countable models of $\phi$. Then for some $\alpha < \omega_1$ there are $\aleph_2$ models of Scott rank at most $\alpha$. But then, by Silver's Theorem, there must be a perfect set of $\sim_\alpha$-inequivalent elements and hence a perfect set of non-isomorphic elements.

One consequence of Silver's Theorem is that if $\cong$ is a Borel equivalence relation on $\mathrm{Mod}(\phi)$, then Vaught's Conjecture holds for $\phi$. When does this happen? The arguments above show that if there is $\alpha < \omega_1$ such that all models have Scott rank at most $\alpha$, then the isomorphism relation is just $\sim_\alpha$, a Borel equivalence relation. We will argue in Theorem 11.19 that $\cong$ is Borel if and only if there is a countable bound on Scott rank.

## 3.3 Scattered Sentences and Morley's Proof

**Definition 3.12** We say that $\phi$ is *scattered* if $S_n(\mathbb{A}, \phi)$ is countable for all countable fragments $\mathbb{A}$.

Non-scattered theories have many non-isomorphic models.

**Lemma 3.13** *Let $\mathbb{A}$ be a countable fragment. If $S_n(\mathbb{A},\phi)$ is uncountable, then $|S_n(\mathbb{A},\phi)| = 2^{\aleph_0}$.*

**Proof** We know similar results for type spaces in first order logic. Those proofs though always use the compactness of first order logic to prove the compactness of the type spaces. Here we can not rely on compactness.

Let $X$ be the set of $\mathbb{A}$-formulas with $n$-free variables. We identify $S_n(\mathbb{A},\phi)$ as a subset of the Polish space $2^X$. Consider the standard Borel space $\mathrm{Mod}(\phi) \times \omega^n$ and the map $F$ from $\mathrm{Mod}(\phi) \times \omega^n$ onto $S_n(\mathbb{A},\phi)$ given by $(\mathcal{M},\overline{a}) \mapsto \mathrm{tp}^{\mathcal{M}}(\overline{a},\mathbb{A})$. By the arguments above this map is Borel. Thus $S_n(\mathbb{A},\phi)$ is $\mathbf{\Sigma}^1_1$ and, by the Perfect Set Theorem for $\mathbf{\Sigma}^1_1$-sets (see [21] 14.13), must contain a non-empty perfect subset of $2^X$. □

**Corollary 3.14** *If $\phi$ is not scattered there are $2^{\aleph_0}$ non-isomorphic countable models of $\phi$. Indeed, there is a perfect set of non-isomorphic models.*

**Proof** There is a countable fragment $\mathbb{A}$ such that $|S_n(\mathbb{A},\phi)| = 2^{\aleph_0}$. As each model only realizes countably many types, there must be $2^{\aleph_0}$ non-isomorphic countable models.

To find a perfect set of non-isomorphic models, consider the equivalence relation on $\mathrm{Mod}(\phi)$ where $\mathcal{M} \approx \mathcal{N}$ if and only $\{\mathrm{tp}^{\mathcal{M}}(\overline{a},\mathbb{A}) : \overline{a} \in \mathcal{M}\} = \{\mathrm{tp}^{\mathcal{N}}(\overline{a},\mathbb{A}) : \overline{a} \in \mathcal{N}\}$, i.e., $\mathcal{M} \approx \mathcal{N}$ if and only if they realize the same $\mathbb{A}$-types. This is a Borel equivalence relation and if $\mathcal{M} \cong \mathcal{N}$, then $\mathcal{M} \approx \mathcal{N}$. But if $|S_N(\mathbb{A},\phi)|$ is uncountable, $\approx$ has uncountably many equivalence class. Thus, by Silver's Theorem, there is a perfect set of $\approx$-inequivalent, and hence non-isomorphic models. □

The last step of Morely's original proof is to show that if $\phi$ is scattered, then there are at most $\aleph_1$ non-isomorphic models. This proof uses a variant of the type of analysis used in the proof of Scott's Isomorphism Theorem.

We build a sequence of countable fragments $(\mathbb{A}_\alpha : \alpha < \omega_1)$ of $\mathcal{L}_{\omega_1,\omega}$. Let $\mathbb{A}_0$ be a countable fragment of $\mathcal{L}_{\omega_1,\omega}$-containing $\phi$. If $\alpha$ is a limit ordinal, let $\mathbb{A}_\alpha = \bigcup_{\beta<\alpha} \mathbb{A}_\beta$. Given $\mathbb{A}_\alpha$, let $\mathbb{A}_{\alpha+1}$ be the smallest fragment of $\mathcal{L}_{\omega_1,\omega}$ containing all formulas

$$\Theta_p(\overline{v}) = \bigwedge_{\psi\in p} \psi(\overline{v})$$

for all $p \in S_n(\mathbb{A},\phi)$ and $n \in \omega$. Since $\phi$ is scattered, $\mathbb{A}_{\alpha+1}$ is a countable fragment.

**Lemma 3.15** *For each $\mathcal{M} \models \phi$ and $\overline{a} \in \mathcal{M}$ there is $\alpha < \omega_1$ such that if $\mathrm{tp}^{\mathcal{M}}(\overline{a}, \mathbb{A}_\alpha) = \mathrm{tp}^{\mathcal{M}}(\overline{b}, \mathbb{A}_\alpha)$, then $\mathrm{tp}^{\mathcal{M}}(\overline{a}, \mathbb{A}_\beta) = \mathrm{tp}^{\mathcal{M}}(\overline{b}, \mathbb{A}_\beta)$ for all $\beta < \omega_1$.*

*We call the least such $\alpha$ the* Morley height *of $\overline{a} \in M$.*

**Proof** Let $\alpha = \sup\{\beta + 1 : \exists \overline{b} \in \mathcal{M}\ \mathrm{tp}^{\mathcal{M}}(\overline{a}, \mathbb{A}_\beta) = \mathrm{tp}^{\mathcal{M}}(\overline{b}, \mathbb{A}_\beta)$ but $\mathrm{tp}^{\mathcal{M}}(\overline{a}, \mathbb{A}_{\beta+1}) \neq \mathrm{tp}^{\mathcal{M}}(\overline{b}, A_{\beta+1})\}$ . □

We call the least such $\alpha$ the *Morley height* of $\overline{a} \in M$. The Morley height of $\mathcal{M}$ is the sup of all Morley height's of tuples in $\mathcal{M}$.

**Lemma 3.16** *Suppose $\mathcal{M}$ has Morley height $\alpha$ and $\mathcal{N} \equiv_{\mathbb{A}_{\alpha+1}} \mathcal{M}$. If $\overline{a}, \overline{b} \in \mathcal{N}$ and $\mathrm{tp}^{\mathcal{N}}(\overline{a}, \mathbb{A}_\alpha) = \mathrm{tp}^{\mathcal{N}}(\overline{b}, \mathbb{A}_\alpha)$, then $\mathrm{tp}^{\mathcal{N}}(\overline{a}, \mathbb{A}_{\alpha+1}) = \mathrm{tp}^{\mathcal{N}}(\overline{b}, \mathbb{A}_{\alpha+1})$. Thus $\mathcal{N}$ has Morley height $\alpha$ as well.*

**Proof** Let $p = \mathrm{tp}^{\mathcal{N}}(\overline{a}) = \mathrm{tp}^{\mathcal{N}}(\overline{b})$. Then, for any $\psi \in \mathbb{A}_{\alpha+1}$

$$\mathcal{M} \models \forall \overline{v} \forall \overline{w}\ [(\Theta_p(\overline{v}) \leftrightarrow \Theta_p(\overline{w})) \rightarrow (\psi(\overline{v}) \leftrightarrow \psi(\overline{w}))].$$

But this sentence is in $\mathbb{A}_{\alpha+1}$ and, hence, is also true in $\mathcal{N}$. □

**Lemma 3.17** *If $\mathcal{M}$ and $\mathcal{N}$ are countable models of $\phi$, $\mathcal{M}$ has Morley height $\alpha$ and $\mathcal{M} \equiv_{\mathbb{A}_{\alpha+1}} \mathcal{N}$, then $\mathcal{M} \cong \mathcal{N}$.*

**Proof** Via a back and forth argument, we build a sequence of finite functions $f_0 \subseteq f_1 \subseteq \ldots$ from $\mathcal{M}$ to $\mathcal{N}$ so that $f = \bigcup_{n\in\omega} f_n$ is the desired isomorphism. We will always have that if $\overline{a}$ is the domain of $f_i$, then $\mathrm{tp}^{\mathcal{M}}(\overline{a}, \mathbb{A}_\alpha) = \mathrm{tp}^{\mathcal{N}}(f_i(\overline{a}), \mathbb{A}_\alpha)$. Since $\mathcal{M} \equiv_{\mathbb{A}_{\alpha+1}} \mathcal{N}$, we can take $f_0 = \emptyset$. Suppose we have $f_i$ with domain $\overline{a}$ and $f(\overline{a}) = \overline{b}$ and we want to add $c$ to the domain of $f_\alpha$. Let $p$ be the $\mathbb{A}_\alpha$ type of $(\overline{a}, c)$. Then

$$\mathcal{M} \models \exists \overline{v} \exists w \Theta_p(\overline{v}, w).$$

Thus there is $(\overline{s}, t) \in \mathcal{N}$ such that $\mathrm{tp}^{\mathcal{N}}(\overline{s}, t, \mathbb{A}_\alpha) = \mathrm{tp}^{\mathcal{M}}(\overline{a}, c)$. In particular, $\mathrm{tp}^{\mathcal{N}}(\overline{s}, \mathbb{A}_\alpha) = \mathrm{tp}^{\mathcal{N}}(\overline{b}, \mathbb{A}_\alpha)$ and, by the previous lemma $\mathrm{tp}^{\mathcal{N}}(\overline{s}, \mathbb{A}_{\alpha+1}) = \mathrm{tp}^{\mathcal{N}}(\overline{b}, \mathbb{A}_{\alpha+1})$. But the formula $\exists w \Theta_p(\overline{v}, w) \in \mathrm{tp}^{\mathcal{N}}(\overline{s}, \mathbb{A}_{\alpha+1})$. Thus we can find $d$ such that $\mathrm{tp}^{\mathcal{N}}(\overline{b}, d) = p$.

Similarly, we can add elements of $\mathcal{N}$ to the domain of any $f_i$. Thus by the usual back-and-forth bookkeeping, we can build $f_0 \subset f_1 \subset \ldots$ so that $\bigcup f_n$ is the desired isomorphism. □

**Corollary 3.18** *If $\phi$ is scattered, then $\phi$ has at most $\aleph_1$ countable models.*

**Proof** Suppose for contradiction that $T$ has at least $\aleph_2$ models. For some $\alpha < \omega_1$ there are uncountably many models of Morley height at most $\alpha$. But any model of Morley height $\alpha$ is determined up to isomorphism by its $\mathbb{A}_{\alpha+1}$-theory and there are only countably many choices for the theory, a contradiction. $\square$

This finishes Morley's proof. Either $\phi$ is scattered and there are at most $\aleph_1$ countable models or $\phi$ is not scattered and there are continuum many models.

We will examine several equivalents of being scattered.

**Proposition 3.19** *Let $\phi \in \mathcal{L}_{\omega_1,\omega}$ then $\phi$ is scattered if and only if $\sim_\alpha$ has countably many equivalence classes of models of $\phi$ for all $\alpha < \omega_1$;*

**Proof** ($\Leftarrow$) Suppose $\mathbb{A}$ is a countable fragment such that $S_n(\phi, \mathbb{A})$ is uncountable. Let $\alpha$ be a bound on the quantifier rank of formulas. Since

$$(\mathcal{M}, \overline{a}) \sim_\alpha (\mathcal{N}, \overline{b}) \Leftrightarrow (\mathcal{M}, \overline{a}) \equiv_\alpha (\mathcal{N}, \overline{b})$$

there are uncountably many $\sim_\alpha$-classes.

($\Rightarrow$) Suppose $\alpha$ is least such that for some $n$ there are uncountably many $\sim_\alpha$ classes of $(\mathcal{M}, a_1, \ldots, a_n) \models \phi$.

If $\alpha = 0$ then there are uncountably many quantifier free types and $\phi$ is not scattered.

If $\alpha = \beta + 1$, let $\mathbb{A}$ be the smallest fragment containing $\phi$ and all Scott sentences $\Phi^{\mathcal{M}}_{\overline{m},\beta}$ for $(\mathcal{M}, \overline{m}) \models \phi$ for $\overline{m} \in \mathcal{M}^{<\omega}$. Since there are only countably many $\sim_\beta$-classes, $\mathbb{A}$ is countable. Suppose $(\mathcal{M}, \overline{a}) \not\sim_\alpha (\mathcal{N}, \overline{b})$. Then, without loss of generality, there is a $c \in \mathcal{M}$ such $(\mathcal{M}, \overline{a}, c) \not\sim_\beta (\mathcal{N}, \overline{b}, d)$ for any $d \in \mathcal{N}$. Thus the formula

$$\exists w\ \Phi^{\mathcal{M}}_{\overline{a},c,\beta}(\overline{v}, w) \in \mathrm{tp}^{\mathcal{M}}(\overline{a}, \mathbb{A}) \setminus \mathrm{tp}^{\mathcal{N}}(\overline{b}, \mathbb{A}).$$

Since the are uncountably many $\sim_\alpha$ classes there are uncountably many $\mathbb{A}$-types and $\phi$ is not scattered.

If $\alpha$ is a limit ordinal, let $\mathbb{A}$ be the countable fragment generated by all formulas $\Phi^{\mathcal{M}}_{\overline{a},\beta}$ for $\beta < \alpha$. If $(\mathcal{M}, \overline{a}) \not\sim_\alpha (\mathcal{N}, \overline{b})$, then $(\mathcal{M}, \overline{a}) \not\sim_\beta (\mathcal{N}, \overline{b})$ for some $\beta < \alpha$ and we proceed as above. $\square$

We showed that if $\phi$ is not scattered there is a perfect set of non-isomorphic models. We now show that the converse is also true.

**Proposition 3.20** *If there is a perfect set of non-isomorphic models, then $\phi$ is not scattered.*

**Proof** We will give a different proof in Proposition 11.25. We quickly sketch here an argument using forcing and absoluteness.

**claim** "$\phi$ is scattered" and "there is a perfect set of non-isomorphic models of $\phi$" are absolute.

$\phi$ is scattered if and only if for all countable sets of codes $A$ for $\mathcal{L}_{\omega_1,\omega}$ formulas $\mathbb{A}$ there is no perfect set of elements $(\mathcal{M}, \overline{a}) \models \phi$ where all $\mathrm{tp}^{\mathcal{M}}(\overline{a}, A)$ are distinct. This can be shown to be $\Pi^1_2(\phi)$ and, hence, absolute by Shoenfield's Absoluteness Theorem (see [19] 25.20). Note, one ingredient of a full proof would be noting that the set of codes for $\mathcal{L}_{\omega_1,\omega}$-formulas is a $\Pi^1_1$ condition (see [36] 8.2).

There is a perfect set of non-isomorphic models is easily seen to be by $\Sigma^1_2(\phi)$ and hence absolute.

Suppose $\phi$ is scattered and there is a perfect set of non-isomorphic models of $\phi$. By Morley's argument, this is only possible if $2^{\aleph_0} = \aleph_1$. Consider a forcing extension of the universe $\mathbb{V}[G] \models 2^{\aleph_0} > \aleph_1$. By absolutness, $\phi$ is still scattered in $\mathbb{V}[G]$ and there is a perfect set of non-isomorphic elements but this contradicts Morley's results. □

Putting this all together.

**Corollary 3.21** *The following are equivalent*

*i) $\phi$ is scattered;*

*ii) $\sim_\alpha$ has countably many equivalence classes of models of $\phi$ for all $\alpha < \omega_1$;*

*iii) there is no perfect set of non-isomorphic countable models of $\phi$.*

When studying Vaught's Conjecture we often look instead at the *Strong Vaught Conjecture*: If $\phi \in \mathcal{L}_{\omega_1,\omega}$, then either there are countably many models of $\phi$ or there is a perfect set of non-isomorphic models of $\phi$. Clearly the strong version implies the usual one. Moreover, if the original version of Vaught's Conjecture is provable for $\phi$ in ZFC, then so is strong form. Suppose not. If there is no perfect set of non-isomorphic models, then $\phi$ is scattered. Suppose we can find $(\mathcal{M}_\alpha : \alpha < \omega_1)$ a sequence of pairwise non-isomorphic countable models of $\phi$. Pass to a forcing extension $\mathbb{V}[G]$ of the universe making $2^{\aleph_0} > \aleph_1$ without collapsing cardinals (for example, add $\aleph_2$

Cohen reals). By Shoenfield Absoluteness, $\mathbb{V}[G] \models \phi$ is scattered and there are uncountably many non-isomorphic models of $\phi$. But then, by Morely's analysis there are exactly $\aleph_1$ non-isomorphic countable models, contradicting the fact that Vaught's Conjecture is true in all models of ZFC.

For this reason we tend to focus on the strong conjecture. This way it makes sense to study counterexamples without worrying about whether the Continuum Hypothesis is true. This leads to the following definition.

**Definition 3.22** A *Vaught counterexample* is a scattered $\phi$ with uncountably many non-isomorphic countable models. Equivalently, it is a sentence with uncountably many non-isomorphic countable models but no perfect set of non-isomorphic models.

We will examine uncountable models of Vaught counterexamples in §7.

# 4 The Model Existence Theorem and Applications

## 4.1 Consistency Properties and Model Existence

Although we don't have the Compactness Theorem or a useful Completeness Theorem, Henkin-style arguments can still be used in some contexts to build models. In this section we describe the general framework and give several applications.

Throughout this section we will assume that $\tau$ is a countable signature and that $C$ is an infinite set of constant symbols of $\tau$ (though perhaps not all of the constant symbols of $\tau$). The following idea, due to Makkai, is the key. It tells us exactly what we need to do a Henkin argument.

**Definition 4.1** A *consistency property* $\Sigma$ is a collection of countable sets $\sigma$ of $\mathcal{L}_{\omega_1,\omega}$-sentences with the following properties. For $\sigma \in \Sigma$:

C0) if $\mu \subseteq \sigma$, then $\mu \in \Sigma$;

C1) if $\phi \in \sigma$, then $\neg\phi \notin \sigma$;

C2) if $\neg\phi \in \sigma$, then there is $\mu \in \Sigma$ such that $\sigma \cup \{\sim \phi\} \subseteq \mu$;

C3) if $\bigwedge_{\phi\in X} \phi \in \sigma$, then for all $\phi \in X$ there is $\mu \in \Sigma$ such that $\sigma \cup \{\phi\} \subseteq \mu$;

C4) if $\bigvee_{\phi\in X} \phi \in \sigma$, then there is $\mu \in \Sigma$ and $\phi \in X$ such that $\sigma \cup \{\phi\} \subseteq \mu$;

C5) if $\forall v\ \phi(v) \in \sigma$, then for all $c \in C$ there is $\mu \in \Sigma$ such that $\sigma \cup \{\phi(c)\} \subseteq \mu$;

C6) if $\exists v\ \phi(v) \in \sigma$, then there is $\mu \in \Sigma$ and $c \in C$ such that $\sigma \cup \{\phi(c)\} \subseteq \mu$;

C7) let $t$ be a term with no variables and let $c, d \in C$,

a) if $c = d \in \sigma$, then there is $\mu \in \Sigma$ such tht $\sigma \cup \{d = c\} \subseteq \mu$;

b) if $c = t, \phi(t) \in \sigma$, then there is $\mu \in \Sigma$ such that $\sigma \cup \{\phi(c)\} \subseteq \mu$;

c) there is $\mu \in \Sigma$ and $e \in C$ such that $\sigma \cup \{e = t\} \subseteq \mu$.

**Lemma 4.2** *Let $\Sigma$ be a consistency property with $\sigma \in \Sigma$, $c, d, e \in C$.*

*i) There is $\mu \in \Sigma, \sigma \subseteq \mu$ with $c = c \in \mu$.*

*ii) Suppose $c = d, d = e \in \sigma$. Prove that $\sigma \cup \{c = e\} \in \Sigma$.*

*iii) If $\phi, \phi \to \psi \in \sigma$, then there is $\mu \in \Sigma$, $\sigma \subseteq \mu$ with $\psi \in \mu$.*

**Proof** i) By C7c) there is $d \in C$ such that $\sigma \cup \{d = c\} \in \Sigma$. Then by C7a) $\sigma \cup \{c = d, d = c\} \in \Sigma$. Applying C7b) with the "$v = c$" as $\phi(v)$, $\sigma \cup \{c = d, d = c, c = c\} \in \Sigma$. □

**Exercise 4.3** Prove ii) and iii) of the Lemma.

The next Exercise shows we really only need to verify C1)–C7).

**Exercise 4.4** Suppose $\Sigma_0$ satisfies C1)–C7). Let

$$\Sigma = \{\Delta : \exists \mu \in \Sigma_0\ \Delta \subseteq \mu\}.$$

Prove that $\Sigma$ is a consistency property.

**Exercise 4.5** Suppose $\Sigma_0$ is a consistency property.

a) Show that $\Sigma_1 = \{\sigma \in \Sigma_0 : \sigma \text{ is finite}\}$ is a consistency property.

b) Show that $\Sigma_2 = \{\sigma \in \Sigma_0 :$ only finitely many constants from $C$ occur in $\sigma\}$ is a consistency property.

**Theorem 4.6 (Model Existence Theorem)** *If $\Sigma$ is a consistency property and $\sigma \in \Sigma$, there is a countable $\mathcal{M} \models \sigma$.*

**Proof** Let $\Lambda$ be the smallest set of $\mathcal{L}_{\infty,\omega}$-sentences such that:

- $\sigma \subseteq \Lambda$;
- if $\phi$ is a subsentence of a sentence in $\Lambda$, then $\phi \in \Lambda$;
- if $\phi(\overline{v})$ is a subformula of a sentence in $\Lambda$ and $\overline{c} \in C$, then $\phi(\overline{c}) \in \Lambda$;
- If $\neg\phi \in \Lambda$, then $\sim \phi \in \Lambda$;
- if $c, d \in C$, then $c = d \in \Lambda$.

Let $\phi_0, \phi_1, \ldots$ list all sentences in $\Lambda$. We assume that each sentence is listed infinitely often. Let $t_0, t_1, \ldots,$ list all $\tau$-terms with no variables.

Using the fact that $\Sigma$ is a consistency property, we build

$$\sigma = \sigma_0 \subseteq \sigma_1 \subseteq \ldots$$

such that each $\sigma_i \in \Sigma$ and

A) if $\sigma_n \cup \{\phi_n\} \in \Sigma$, then $\phi_n \in \sigma_{n+1}$, in this case:

a) if $\phi_n$ is $\bigvee_{\phi \in X} \phi$, then $\phi \in \sigma_{n+1}$ for some $\phi \in X$;

b) if $\phi_n$ is $\exists v\ \phi(v)$, then $\phi(c) \in \sigma_{n+1}$ for some $c \in C$;

B) $c = t_n \in \sigma_{n+1}$ for some $c \in C$.

Let $\Gamma = \bigcup_{n=1}^{\infty} \sigma_n$. We will build a model of $\Gamma$. For $c, d \in C$ we say $c \sim d$ if $c = d \in \Gamma$.

**claim 1** $\sim$ is an equivalence relation.

Let $c \in C$. The sentence $c = c$ is $\phi_n$ for some $n$. By Lemma 4.2, and A) $c = c \in \sigma_{n+1} \subseteq \Gamma$. Thus $c \sim c$.

If $c = d \in \Gamma$, then we can find $n$ such that $c = d \in \sigma_n$ and $d = c$ is $\phi_n$. Then by C7a) and condition A) $d = c \in \Gamma$.

If $c = d, d = e \in \Gamma$, choose $n$ such that $c = d, d = e \in \sigma_n$ and $c = e$ is $\phi_n$. Then by Lemma 4.2 and A) $c = e \in \Gamma$.

Let $[c]$ denote the $\sim$-class of $c$. Let $M = \{[c] : c \in C\}$.

If $d$ is a constant symbol of $\tau$ (perhaps not in $C$), then by B) there is $c \in C$ such that $c = d \in \Gamma$. We interpret $d^{\mathcal{M}}$ as $[c]$.

Let $f$ be an $n$-ary function symbol of $\tau$ and let $c_1, \ldots, c_n \in C$. By B) there is $d \in C$ such that $d = f(c_1, \ldots, c_n) \in \Gamma$. Suppose $d_1 \in C$ and $d_1 = f(c_1, \ldots, c_n)$ is also in $\Gamma$, using C7b), Lemma 4.2 and A) we see that $d = d_1 \in \Gamma$ and $d \sim d_1$. Also note that if $c_0 = f(c_1, \ldots, c_n) \in \Gamma$, and $d_i \sim c_i$ for $i = 0, \ldots, n$ then, a similar argument shows $d_0 = f(d_1, \ldots, d_n) \in \Gamma$. Thus we can define $f^{\mathcal{M}} : M^n \to M$ by

$$f([c_1], \ldots, [c_n]) = [d] \Leftrightarrow f(c_1, \ldots, c_n) = d \in \Gamma.$$

**claim 2** Let $t(v_1, \ldots, v_n)$ be a term, $c_1, \ldots, c_n, d \in C$. If $d = t(c_1, \ldots, c_n) \in \Gamma$, then $t^{\mathcal{M}}([c_1], \ldots, [c_n]) = [d]$.

We prove this by induction on complexity of terms. Suppose $t = f(t_1, \ldots, t_m)$. For $i \leq m$ there is $d_i \in C$ such that $d_i = t_i(\bar{c}) \in \Gamma$. By induction, $[d_i] = t_i^{\mathcal{M}}([\bar{c}])$. By C7b) and A) $d = f(d_1, \ldots, d_m) \in \Gamma$. Thus

$$t^{\mathcal{M}}([\bar{c}]) = f^{\mathcal{M}}([d_1], \ldots, [d_m]) = [d]$$

as desired.

For $c_1, \ldots, c_n$ and $R$ an $n$-ary relation symbol of $\tau$, we define

$$R^{\mathcal{M}}([c_1], \ldots, [c_n]) \Leftrightarrow R(c_1, \ldots, c_n) \in \Gamma.$$

As above, we can show this does not depend on the choice of $c_i$.

**claim 3** If $\phi \in \Gamma$, then $\mathcal{M} \models \phi$.

We prove this by induction on complexity.[7]

[7] We need to be slightly careful how we define "complexity". For example, $\exists v\ \psi(v)$ is more complex than any $\psi(c)$ and $\neg \bigvee_{\psi \in X} \psi$ is more complicated than $\neg\psi$ for any $\psi \in X$.

- $\phi$ is $t_1 = t_2$

  There are $c_1, c_2 \in C$ such that $c_1 = t_1, c_2 = t_2 \in \Gamma$. Then $c_1 = c_2 \in \Gamma$ and by claim 2, $t_1^{\mathcal{M}} = t_2^{\mathcal{M}}$, so $\mathcal{M} \models \phi$.

- $\phi$ is $R(t_1, \ldots, t_n)$

**Exercise 4.7** Prove $\mathcal{M} \models R(t_1^{\mathcal{M}}, \ldots, t_n^{\mathcal{M}})$.

- $\phi$ is $\bigwedge_{\psi \in X} \psi$

  By C3) and A), $\psi \in \Gamma$ for each $\psi \in X$. By induction $\mathcal{M} \models \psi$ for all $\psi \in X$. Thus $\mathcal{M} \models \phi$.

- $\phi$ is $\bigvee_{\psi \in X} \psi$

  By C4) and A), there is $\psi \in X$ such that $\psi \in \Gamma$. By induction $\mathcal{M} \models \psi$. Thus $\mathcal{M} \models \phi$.

- $\phi$ is $\forall v \ \psi(v)$

  By C5) and A), $\psi(c) \in \Gamma$ for all $c \in C$. By induction $\mathcal{M} \models \psi(c)$ for all $c \in C$. Since every element of $M$ is named by a constant in $C$, $\mathcal{M} \models \phi$.

- $\phi$ is $\exists v \ \psi(v)$

  By A) there is $c \in C$ such that $\psi(c) \in \Gamma$. By induction, $\mathcal{M} \models \psi(c)$. Thus $\mathcal{M} \models \phi$.

- $\phi$ is $\neg\psi$

  By C2) and A), $\sim \psi \in \Gamma$. This now breaks into cases depending on $\psi$.

  - $\psi$ is $t_1 = t_2$

    There are constants $c_1, c_2 \in C$ such that $c_i = t_i \in \Gamma$. Since $\sim \psi$ is $t_1 \neq t_2$, by C7) and A), $c_1 \neq c_2 \in \Gamma$. Suppose, for contradiction, that $\mathcal{M} \models \psi$. By claim 2 $[c_i] = t_i^{\mathcal{M}}$. Thus $\mathcal{M} \models c_1 = c_2$. Hence $c_1 \sim c_2$ and $c_1 = c_2 \in \Gamma$. This contradicts C1).

  - $\psi$ is $R(t_1, \ldots, t_m)$

    Then $\neg R(t_1, \ldots, t_m) \in \Gamma$. There are $c_1, \ldots, c_m \in C$ such that $c_i = t_i \in \Gamma$. Then $\neg R(c_1, \ldots, c_m) \in \Gamma$. By C1), $R(c_1, \ldots, c_m) \notin \Gamma$. Thus

    $$\mathcal{M} \models \neg R(c_1, \ldots, c_m).$$

By claim 2,

$$\mathcal{M} \models \neg R(t_1, \ldots, t_m).$$

Thus $\mathcal{M} \models \phi$.

– $\psi$ is $\neg\theta$

  Then $\theta \in \Gamma$. By induction $\mathcal{M} \models \theta$ and $\mathcal{M} \models \phi$.

– $\psi$ is $\bigwedge_{\theta \in X} \theta$

  Then $\bigvee_{\theta \in X} \sim \theta \in \Gamma$ and $\sim \theta \in \Gamma$ for some $\theta \in X$. By induction, $\mathcal{M} \models \neg\theta$. Thus $\mathcal{M} \models \phi$.

– $\psi$ is $\bigvee_{\theta \in X} \theta$

  Then $\bigwedge_{\theta \in X} \sim \theta \in \Gamma$ and $\sim \theta \in \Gamma$ for all $\theta \in X$. By induction $\mathcal{M} \models \neg\theta$ for all $\theta \in X$. Thus $\mathcal{M} \models \phi$.

– $\psi$ is $\exists v \theta(v)$

  Then $\forall v \sim \theta(v) \in \Gamma$ and $\sim \theta(c) \in \Gamma$ for all $c \in C$. By induction, $\mathcal{M} \models \theta(c)$ for all $c \in C$. Thus $\mathcal{M} \models \phi$.

– $\psi$ is $\forall v \theta(v)$.

  Then $\exists v \sim \theta(v) \in \Gamma$ and $\sim \theta(c) \in \Gamma$ for some $c \in C$. But then $\mathcal{M} \models \neg\theta(c)$ and $\mathcal{M} \models \phi$.

This completes the induction. Thus $\mathcal{M} \models \Gamma$. □

We can easily modify the Henkin argument above to prove the following useful variant.

**Theorem 4.8 (Extended Model Existence Theorem)** *Let $\tau$ be as above. Let $T$ be a countable set of $\mathcal{L}_{\omega_1,\omega}(\tau)$-sentences. Suppose $\Sigma$ is a consistency property such that for all $\sigma \in \Sigma$ and $\phi \in T$, $\sigma \cup \{\phi\} \in \Sigma$. Then there is $\mathcal{M} \models T$.*

**Proof** Let $\theta_0, \theta_1, \ldots$ enumerate $T$. Modify the construction of $\sigma_0 \subseteq \sigma_1 \subseteq \ldots$ such that $\theta_n \in \sigma_{n+1}$ for all $n$. Then $T \subset \Gamma$ and $\mathcal{M} \models T$. □

## 4.2 Omitting Types and Atomic Models

As a first application of the Model Existence Theorem we will show how to prove a generalization of the Omitting Types Theorem of first order logic.

**Theorem 4.9 (Omitting Types Theorem)** *Let $\mathbb{A}$ be a countable fragment of $\mathcal{L}_{\omega_1,\omega}(\tau)$, let $T \subset \mathbb{A}$ be a satisfiable theory and let $\Theta_n(v_1, \ldots, v_{m_n})$ be a set of $\mathbb{A}$-formulas for each $n \in \omega$. Suppose that for all $\mathbb{A}$-formulas $\psi(v_1, \ldots, v_{m_n})$ such that $T + \exists \overline{v}\ \psi(\overline{v})$ is satisfiable, there is $\theta \in \Theta_n$ such that $T + \exists \overline{v}\ (\psi(\overline{v}) \wedge \theta(\overline{v}))$ is satisfiable.*

*Then*

$$T + \bigwedge_{n \in \omega} \forall \overline{v} \bigvee_{\theta \in \Theta_n} \theta(\overline{v})$$

*is satisfiable.*

**Proof** Let $\tau^* \supset \tau$ be obtained by adding a countable set of new constant symbols $C$ and let $\mathbb{A}^*$ be the smallest fragment containing $\mathbb{A}$ such that if $\phi(\overline{v})$ is a formula in $\mathbb{A}$ and $\overline{c} \in C$ then $\phi(\overline{c}) \in \mathbb{A}^*$.

Let

$$\Delta = \{ \bigvee_{\theta \in \Theta_n} \theta(c_1, \ldots, c_{m_n}) : n \in \omega, \overline{c} \in C\}$$

and $\Sigma$ be the set of all sets $\sigma$ where

$$\sigma = \sigma_0 \cup T \cup \Delta$$

where $\sigma_0$ is a finite set of $\mathbb{A}^*$-sentences such that $T \cup \sigma_0$ is satisfiable. Note that $\sigma_0$ contains only finitely many constants from $C$.

We claim that $\Sigma$ is a consistency property. Once we have shown this, the Model Existence Theorem will give us a model of $T \cup \Delta$ where every element is the interpretation of a constant in $C$. This is the desired model.

We will show that C4) holds. The rest of the verification is routine and left as an exercise. Suppose $\sigma$ is as above and $\chi = \bigvee_{\psi \in X} \psi$.

<u>case 1</u> $\chi \in \sigma_0 \cup T$.

In this case we know there is $\mathcal{M} \models \sigma_0 \cup T$ and $\mathcal{M} \models \psi$ for some $\psi \in X$. Then

$$\sigma_0 \cup \{\psi\} \cup T \cup \{ \bigvee_{\theta \in \Theta_n} \theta(c_1, \ldots, c_{m_n}) : n \in \omega, \overline{c} \in C\} \in \Sigma$$

as desired.

<u>case 2</u> $\chi = \bigvee_{\theta \in \Theta_n} \theta(\overline{c})$ for some $n$ and $\overline{c} \in C$.

Let $\overline{d}$ be the constants from $C$ occurring in $\sigma_0$ but not in $\overline{c}$. Let

$$\Phi(\overline{c}, \overline{d}) = \bigwedge_{\phi \in \sigma_0} \phi.$$

Then

$$T + \exists\overline{x}\exists\overline{y}\ \Phi(\overline{x}, \overline{y})$$

is satisfiable. By our assumptions, there is $\theta \in \Theta_n$ such that

$$T + \exists\overline{x}\ \ (\theta \wedge \exists\overline{y}\ \Phi(\overline{x}, \overline{y}))$$

is satisfiable. Thus $T + \Phi(\overline{c}, \overline{d}) \wedge \theta(\overline{c})$ is satisfiable and

$$\sigma_0 \cup \{\theta(\overline{c})\} \cup T \cup \Delta \in \Sigma.$$

Thus C4) holds. □

**Exercise 4.10** Complete the verification that $\Sigma$ is a consistency property.

**Exercise 4.11** Show that Theorem 4.9 is a generalization of the classical Omitting Types Theorem for first order logic.

### Atomic and Prime Models

Once we have the Omitting Types Theorem, many of Vaught's indexVaught, R. proofs about prime and atomic models for first order logic can easily be adapted in the infinitary setting.

Let $\mathbb{A}$ be a countable fragment of $\mathcal{L}_{\omega_1,\omega}$. Let $T \subset \mathbb{A}$ be a satisfiable theory that is $\mathbb{A}$-complete, i.e., $T \models \phi$ or $T \models \neg\phi$ for any $\mathbb{A}$-sentence $\phi$.

**Definition 4.12** A satisfiable formula $\theta(\overline{v})$ is $\mathbb{A}$-*complete* if for any $\mathbb{A}$-formula $\phi(\overline{v})$ either

$$T \models \theta(\overline{v}) \to \phi(\overline{v}) \text{ or } T \models \theta(\overline{v}) \to \neg\phi(\overline{v}).$$

A formula $\phi(\overline{v})$ is $\mathbb{A}$-*completable* if there is a complete $\theta(\overline{v}) \in \mathbb{A}$ with $T \models \theta(\overline{v}) \to \phi(\overline{v})$. We say that $\phi$ is *incompletable* if it is satisfiable but non completable.

**Definition 4.13** $\mathcal{M} \models T$ is $\mathbb{A}$-*atomic* if for all $\overline{a} \in \mathcal{M}$ there is an $\mathbb{A}$-complete formula $\theta(\overline{v})$ such that $\mathcal{M} \models \theta(\overline{a})$

**Definition 4.14** $\mathcal{M} \models T$ is $\mathbb{A}$-*prime* if for every $\mathcal{N} \models T$ there is an $\mathbb{A}$-elementary embedding of $\mathcal{M}$ into $\mathcal{N}$.

**Theorem 4.15** *$\mathcal{M} \models T$ is $\mathbb{A}$-prime if and only if it is countable and $\mathbb{A}$-atomic.*

**Proof** (⇒) Suppose $\mathcal{M}$ is an $\mathbb{A}$-prime model of $T$. Since $T$ has a countable model, $\mathcal{M}$ must be countable. Suppose $\mathcal{M}$ is not atomic. Let $\overline{a} \in \mathcal{M}$ and suppose that $\overline{a}$ satisfies no $\mathbb{A}$-complete formula. Let $\Delta = \{\phi(\overline{v}) \in \mathbb{A} : \mathcal{M} \models \phi(\overline{a})\}$. We apply the Omitting Types Theorem to $T$ and $\bigvee_{\phi\in\Delta} \neg\phi(\overline{v})$. Suppose $T + \exists v\ \psi(\overline{v})$ is satisfiable.

If $\mathcal{M} \models \neg\psi(\overline{a})$, then there is $\mathcal{N} \models T + \exists\overline{v}\ (\psi(\overline{v}) \wedge \neg\neg\psi(\overline{v}))$, so we satisfy the hypotheses of the Omitting Types Theorem.

If $\mathcal{M} \models \psi(\overline{a})$, then $\psi$ is not $\mathbb{A}$-complete, so we can find $\phi$ such that $\mathcal{M} \models \phi(\overline{a})$ and

$$T \not\models \psi(\overline{v}) \to \phi(\overline{v}) \text{ and } T \not\models \psi(\overline{v}) \to \phi(\overline{v}).$$

Thus $T + \exists\overline{v}\ (\psi(\overline{v}) \wedge \neg\phi(\overline{v}))$ is satisfiable and we again satisfy the hypothesis of the Omitting Types Theorem.

Thus there is a

$$\mathcal{N} \models T + \forall\overline{v}v \bigvee_{\mathcal{M}\models\phi(\overline{a})} \neg\phi(\overline{a})$$

and there is no $\mathbb{A}$-elementary embedding of $\mathcal{M}$ into $\mathcal{N}$.

(⇐) Suppose $\mathcal{M} \models T$ is countable and $\mathbb{A}$-atomic and $\mathcal{N}$ models $T$. We must build an $\mathbb{A}$-elementary embedding of $\mathcal{M}$ into $\mathcal{N}$. Let $a_0, a_1, \ldots$ enumerate the elements of $\mathcal{M}$. We build a family of partial $\mathbb{A}$-elementary maps

$$f_0 \subset f_1 \subset \ldots$$

maps into $\mathcal{N}$ where $\mathrm{dom}(f_i) = \{a_0, \ldots, a_{i-1}\}$. Since $T$ is complete, we can start with $f_0 = \emptyset$.

Given $f_n$, let $b_i = f_n(a_i)$ for $i < n$. There is an $\mathbb{A}$-complete formula $\theta(v_0, \ldots, v_{n-1})$ such that $\mathcal{M} \models \theta(a_0, \ldots, a_{n-1})$ and an $\mathbb{A}$-complete formula $\psi(v_0, \ldots, v_n)$ such that $\mathcal{M} \models \psi(a_0, \ldots, a_n)$. Since $\theta_0$ is $\mathbb{A}$-complete,

$$\mathcal{M} \models \forall\overline{v}(\theta(v_0, \ldots, v_{n-1}) \to \exists v_n\ \psi(v_0, \ldots, v_n)).$$

Since $f_n$ is $\mathbb{A}$-elementary, $\mathcal{N} \models \theta(b_0, \ldots, b_{n-1})$. Thus we can find $b_n \in \mathcal{N}$ such that $\mathcal{N} \models \psi(b_0, \ldots, b_n)$. Extend $f_n$ to $f_{n+1}$ by sending $a_n$ to $b_n$. Since $\theta(\overline{v})$ is $\mathbb{A}$-complete, this is $\mathbb{A}$-elementary. The map $f = \bigcup f_n$ is the desired $\mathbb{A}$-elementary embedding of $\mathcal{M}$ into $\mathcal{N}$. □

**Exercise 4.16** Show that any two $\mathbb{A}$-prime models of $T$ are isomorphic.

**Definition 4.17** $T$ is $\mathbb{A}$-*atomic* if every satisfiable formula in $\mathbb{A}$ is completable.

**Theorem 4.18** *$T$ has a countable $\mathbb{A}$-atomic model if and only if $T$ is $\mathbb{A}$-atomic.*

**Proof** ($\Rightarrow$) Suppose $\mathcal{M}$ is a countable $\mathbb{A}$-atomic model. If $\psi(\overline{v})$ is satisfiable, then, since $T$ is $\mathbb{A}$-complete, there is $\overline{a} \in \mathcal{M}$ such that $\mathcal{M} \models \psi(\overline{a})$. But $\overline{a}$ satisfies an $\mathbb{A}$-complete formula $\theta(\overline{v})$ and we must have $T \models \theta(\overline{v}) \to \phi(\overline{v})$. Thus $\phi$ is $\mathbb{A}$-completable.

($\Leftarrow$) For each $n$ let $\Delta_n$ be the set of all complete formulas $\theta(v_1, \ldots, v_n)$. If $\psi(v_1, \ldots, v_n)$ is satisfiable, then there is $\theta(\overline{v}) \in \Delta_n(\overline{v})$ such that $T \models \theta(\overline{v}) \to \psi(\overline{v})$. In particular,

$$T + \exists \overline{v}\ (\theta(\overline{v}) \wedge \psi(\overline{v}))$$

is satisfiable and, by the Omitting Types Theorem, there is an $\mathbb{A}$-atomic $\mathcal{M} \models T$. □

We next show that if there are only countably many $\mathbb{A}$-types then there is an $\mathbb{A}$-atomic model. One way to do this in the first order case is to assume there is an incompletable formula and then build a perfect tree of inconsistent incompletable formulas. By compactness, each branch gives rise to a different type. So we have that a non-atomic theory has $2^{\aleph_0}$ types. Here we use the Omitting Types Theorem directly to prove that an $\mathbb{A}$-theory with countably many types is atomic. We could combine this with Morley's argument from Lemma 3.13 to conclude that a non-$\mathbb{A}$-atomic theory has $2^{\aleph_0}$ $\mathbb{A}$-types.

**Theorem 4.19** *If for all $n \in \omega$ there are only countably many $\mathbb{A}$-types, then $T$ is $\mathbb{A}$-atomic and has an $\mathbb{A}$-prime model.*

**Proof** Suppose $\phi(\overline{v})$ is satisfiable but not $\mathbb{A}$-completable. Let $\gamma_0(\overline{v}), \gamma_1(\overline{v}), \ldots$ list all $\mathbb{A}$-types containing $\phi$. Since none of those types contains a complete formula for any satisfiable formula $\psi(\overline{v})$ and any $i \in \omega$ we can find $\theta \in \gamma_i$ such that $T + \exists \overline{v}\ (\psi(\overline{v}) \wedge \neg\theta(\overline{v}))$ is satisfiable. By the Omitting Types Theorem we can find

$$\mathcal{M} \models T + \forall \bigwedge_{n \in \omega} \overline{v} \bigvee_{\theta \in \gamma_n} \neg\theta(\overline{v}).$$

Clearly $\mathcal{M}$ omits each $\gamma_i$. But these are all $\mathbb{A}$-types containing $\phi$. But $T$ is complete and $\phi$ is satisfiable, thus $\mathcal{M} \models \exists \overline{v} \phi(\overline{v})$, a contradiction. □

We give one application from Nadel [42].

**Theorem 4.20** *If $\phi \in \mathcal{L}_{\omega_1,\omega}$ is complete and $\mathcal{M} \models \phi$, then the Scott rank of $\phi$ is at most* $\mathrm{qr}(\phi)+\omega$, *indeed* $\mathrm{SR}(\mathcal{M}) \leq \mathrm{qr}(\phi)+\omega$.

**Proof** Let $\mathbb{A}$ be the smallest fragment containing $\phi$. Note that every formula in $\mathbb{A}$ has quantifer rank less than $\mathrm{qr}(\phi)+\omega$. There are only countably many $\mathbb{A}$-types. Thus if $\mathcal{M}$ is the unique countable model of $\phi$, then $\mathcal{M}$ is $\mathbb{A}$-atomic. By the usual back-and-forth arguments, for each $\overline{a}$ there is $\psi(\overline{v}) \in \mathbb{A}$ such that if $\mathcal{M} \models \psi(\bar{b})$ there is an automorphism of $\mathcal{M}$ mapping $\overline{a}$ to $\bar{b}$. It follows that there is $\beta < \mathrm{qr}(\phi)+\omega$ such that if $(\mathcal{M},\overline{a}) \sim_\beta (\mathcal{M},\bar{b})$, then $(\mathcal{M},\overline{a}) \sim_\alpha (\mathcal{M},\bar{b})$ for all $\beta$. Thus $\mathrm{SR}(\mathcal{M}) \leq \mathrm{qr}(\phi)+\omega$. $\square$

## 4.3 The Interpolation Theorem

The next application of the Model Existence Theorem is the $\mathcal{L}_{\omega_1,\omega}$ version of Craig's Interpolation Theorem from first order logic. This result was first proved by Lopez-Escobar by different means.

**Theorem 4.21** *Suppose $\phi_1$ and $\phi_2$ are $\mathcal{L}_{\omega_1,\omega}$-sentences with $\phi_1 \models \phi_2$. There is an $\mathcal{L}_{\omega_1,\omega}$-sentence $\theta$ such that $\phi_1 \models \theta$, $\theta \models \phi_2$ and every relation, function and constant symbol occurring in $\theta$ occurs in both $\phi_1$ and $\phi_2$.*

**Proof** Let $C$ be a countably infinite collection of new constant symbols. Let $\tau_i$ be the smallest vocabulary containing $C$ and all symbols in $\phi_i$ and let $\tau = \tau_1 \cap \tau_2$.

Let $\Sigma$ be the collection of finite $\sigma = \sigma_1 \cup \sigma_2$ where $\sigma_i$ is a set of $\tau_i$-sentences and if $\psi_1, \psi_2$ are $\tau$-sentences such that $\sigma_1 \models \psi_1$ and $\sigma_2 \models \psi_2$, then $\psi_1 \wedge \psi_2$ is satisfiable.

**claim** $\Sigma$ is a consistency property.

We will verify properties C3) and C4) and leave the remainder as an exercise.

C3) Suppose $\bigwedge_{\psi \in X} \psi \in \sigma \in \Sigma$, where $\sigma = \sigma_1 \cup \sigma_2$ as above. Suppose $\bigwedge_{\psi \in X} \psi \in \sigma_1$. Let $\sigma_1' = \sigma_1 \cup \{\psi\}$. We claim that $\sigma_1' \cup \sigma_2 \in \Sigma$. Suppose $\sigma_1' \models \theta_1$ and $\sigma_2 \models \theta_2$. Then $\sigma_1 \models \theta_1$. By assumption, $\theta_1 \wedge \theta_2$ is satisfiable. Thus $\sigma_1' \cup \sigma_2 \in \Sigma$.

This is similar if $\bigwedge_{\psi \in X} \psi \in \sigma_2$.

C4) Suppose $\bigvee_{\psi \in X} \psi \in \sigma_1$. Let $\sigma_{1,\psi} = \sigma_1 \cup \{\psi\}$. We claim that some $\sigma_{1,\psi} \cup \sigma_2 \in \Sigma$. Suppose not. Then for each $\psi$ there are $\mathcal{L}_{\omega_1,\omega}(\tau)$-sentences

$\theta_{1,\psi}$, $\theta_{2,\psi}$ such that $\sigma_{1,\psi} \models \theta_{1,\psi}$, $\sigma_2 \models \theta_{2,\psi}$ and $\theta_{1,\psi} \wedge \theta_{2,\psi}$ is unsatisfiable. Then $\theta_{1,\psi} \models \neg\theta_{2,\psi}$. Since

$$\sigma_1 \models \bigvee_{\psi \in X} \psi,$$

$$\sigma_1 \models \bigvee_{\psi \in X} \theta_{1,\psi}.$$

But

$$\sigma_2 \models \bigwedge_{\psi \in X} \theta_{2,\psi}$$

and

$$\bigvee_{\psi \in X} \theta_{1,\psi} \models \neg \bigwedge_{\psi \in X} \theta_{2,\psi}$$

contradicting that $\sigma \in \Sigma$.

**Exercise 4.22** Finish the proof that $\Sigma$ is a consistency property.

We now finish the proof of the Interpolation Theorem. Since $\phi_1 \models \phi_2$, by the Model Existence Theorem, $\{\phi_1, \neg\phi_2\} \notin \Sigma$. Thus there are $\mathcal{L}_{\omega_1,\omega}(\tau)$-sentences $\theta_1$ and $\theta_2$ such that $\phi_1 \models \theta_1$, $\neg\phi_2 \models \theta_2$ and $\theta_1 \wedge \theta_2$ is unsatisfiable.

Thus

$$\phi_1 \models \theta_1, \theta_1 \models \neg\theta_2, \text{ and } \neg\theta_2 \models \phi_2.$$

It follows that

$$\phi_1 \models \theta_1 \text{ and } \theta_1 \models \phi_2.$$

We would be done except $\theta_1$ may contain constants from $C$. Let $\theta_1 = \psi(\overline{c})$, where $\psi(\overline{v})$ is a $\tau$-formula with no constants from $C$. Then

$$\phi_1 \models \forall\overline{v}\ \psi(\overline{v}) \text{ and } \exists\overline{v}\ \psi(\overline{v}) \models \phi_2.$$

Since

$$\forall\overline{v}\ \phi(\overline{v}) \models \exists\overline{v}\ \phi(\overline{v})$$

we can take

$$\forall\overline{v}\ \psi(\overline{v})$$

as the interpolant. □

The Interpolation Theorem gives rise to a separation theorem for $\mathrm{PC}_{\omega_1,\omega}$-classes, analogous to the Separation Theorem for $\mathbf{\Sigma}^1_1$-sets in Descriptive Set Theory.

**Corollary 4.23** *Suppose $\mathcal{K}_0$ and $\mathcal{K}_1$ are disjoint $\mathrm{PC}_{\omega_1,\omega}$-classes of $\tau$-structures. There is $\phi \in \mathcal{L}_{\omega_1,\omega}$ such that $\mathcal{M} \models \phi$ for $\mathcal{M} \in K_0$ and $\mathcal{M} \models \neg\phi$ for $\mathcal{M} \in K_1$.*

**Proof** Let $\tau_0, \tau_1 \supseteq \tau$ and let $\phi_i \in \mathcal{L}_{\omega_1,\omega}(\tau_i)$ such that $\mathcal{K}_i$ is the class of $\tau$-reducts of models of $\phi_i$. We may assume that $\tau_1 \cap \tau_2 = \tau$. Since $\mathcal{K}_0$ and $\mathcal{K}_1$ are disjoint we have that $\phi_0 \models \neg\phi_1$, i.e., if there is any expansion of $\mathcal{M}$ that makes $\phi_0$ true, then there is no expansion of $\mathcal{M}$ making $\phi_1$ true. By the Interpolation Theorem there is $\phi \in \mathcal{L}_{\omega_1,\omega}(\tau)$ such that $\phi_0 \models \phi$ and $\phi \models \neg\phi_1$. Then every structure in $\mathcal{K}_0$ is a model of $\phi$ and no structure in $\mathcal{K}_1$ is a model of $\phi$. $\square$

Indeed we can use these results to prove an invariant version of the Separation Theorem for $\mathbf{\Sigma}^1_1$-sets. In Exercise 3.5 we noted that the collection of countable models in a $\mathrm{PC}_{\omega_1,\omega}$-class is an invariant $\mathbf{\Sigma}^1_1$-sets. The converse is also true.

**Lemma 4.24** *Every invariant $\mathbf{\Sigma}^1_1$ subset $A$ of $\mathbb{X}_\tau$ is the set of countable models in a $\mathrm{PC}_{\omega_1,\omega}$-class.*

**Proof** For notational simplicity we assume $\tau = \{R\}$, where $R$ is a binary relation symbol. Fix $p : \omega^2 \to \omega$ a bijective pairing function. For $R \subseteq \omega \times \omega$, let $f_R : \omega \to 2$ be defined by $f_R(p(i,j)) = 1$ if and only if $R(i,j)$.

Since $A$ is analytic, there is a tree $T \subseteq \{(\eta,\nu) : \eta \in 2^{<\omega}, \nu \in \omega^{<\omega}, |\eta| = |\nu|\}$ such that $(\omega, R) \in A$ if and only if there is $g$ such that $(f_R, g)$ is a path through $T$.

Let $\tau^* = \{s, c, f, g, S_n : n = 1, 2, \ldots\}$ where $s, f, g$ are unary functions, $c$ is a constant symbol and $S_n$ is a $2n$-ary relation symbol. Let $\Theta$ be a sentence asserting:

- $s$ is one-to-one, every element except $c$ has a preimage, there are no cycles and $\forall x \ \bigvee_{i=0}^{\infty} s^{(i)}(c) = x$;
- $\forall x \ f(x) = c \vee f(x) = s(c)$;
- $\displaystyle\bigwedge_{i,j} [R(s^{(i)}(c), s^{(j)}(c)) \leftrightarrow f(s^{(p(i,j))}(c)) = s(c)]$;
- $$\bigwedge_{n=1}^{\infty} \Big[ \bigwedge_{(\eta,\nu)\in T} S_n(s^{\eta(0)}(c), \ldots, s^{\eta(n-1)}(c), s^{\nu(0)}(c), \ldots, s^{\nu(n-1)}(c)) \wedge$$ $$\bigwedge_{\eta\in 2^n, \nu\in\omega^n, (\eta,\nu)\notin T} \neg S_n(s^{\eta(0)}(c), \ldots, s^{\eta(n-1)}(c), s^{\nu(0)}(c), \ldots, s^{\nu(n-1)}(c)) \Big];$$

- $\bigwedge_{n=1}^{\infty} S_n(f(c), f(s(c)), \ldots, f(s^{n-1}(c)), g(c), g(s(c)), \ldots, g(s^{n-1}(c)))$

Suppose $(\omega, R) \in A$. Let $f_R$ be as above. Interpret $c$ as 0 and $s$ as $x \mapsto x+1$. Choose $g$ such that $(f_R, g)$ is a path thru $T$. Interpret $S_n$ by $S_n(i_0, \ldots, i_{n-1}, j_0, \ldots, j_{n-1})$ if and only if $(\bar{i}, \bar{j}) \in T$. The resulting structure is a model of $\Theta$.

Suppose $\mathcal{M}^* = (\omega, R, s, f, g, S_1, \ldots) \models \Theta$. Define $\eta_0 \subset \eta_1 \subset \ldots$ and $\nu_0 \subset \nu_1 \subset \ldots$ such that $f(s^i(c)) = s^{\eta(i)}(c)$ and $g(s^i(c)) = s^{\nu(i)}(c)$. Let $\widehat{f} = \bigcup \eta_i$ and $\widehat{g} = \bigcup \nu_i$. Then $(\widehat{f}, \widehat{g})$ is a path thru $T$. Define $\widehat{R} \subseteq \omega \times \omega$ such that $f_{\widehat{R}} = \widehat{f}$ and let $\mathcal{N} = (\omega, R^*)$. Then $\mathcal{N} \in A$. But $s$ gives an isomorphism between $\mathcal{N}$ and $\mathcal{M} = (\omega, R)$. Thus, since $A$ is invariant, $\mathcal{M} \in A$. $\square$

**Corollary 4.25** *If $A$ and $B$ are disjoint invariant $\mathbf{\Sigma}^1_1$-sets there is an invariant Borel $C$ with $A \subset C$ and $B \cap C = \emptyset$.*

We next prove the converse to Corollary 3.3

**Theorem 4.26** *Every invariant Borel subset of $\mathbb{X}_\tau$ is of the form* $\mathrm{Mod}(\phi)$ *for some $\mathcal{L}_{\omega_1,\omega}(\tau)$-sentence $\phi$.*

**Proof** If $A$ is invariant Borel, then $A$ and $\mathbb{X}_\tau \setminus A$ are disjoint invariant $\mathbf{\Sigma}^1_1$ sets. We can find $\mathrm{PC}_{\omega_1,\omega}$-classes $\mathcal{K}_0$ and $\mathcal{K}_1$ such that $A = \mathbb{X}_\tau \cap \mathcal{K}_0$ and $\mathbb{X}_\tau \setminus A = \mathbb{X}_\tau \cap \mathcal{K}_1$. Since there are no countable structures in $\mathcal{K}_0 \cap \mathcal{K}_1$, by Löwenheim-Skolem, $\mathcal{K}_0$ and $\mathcal{K}_1$ are disjoint classes. Thus by Corollary 4.23 there is an $\mathcal{L}_{\omega_1,\omega}(\tau)$-sentence $\phi$ such that $A \subset \mathrm{Mod}(\phi)$ and $\mathrm{Mod}(\phi) \cap (\mathbb{X}_\tau \setminus A) = \emptyset$. Clearly $A = \mathrm{Mod}(\phi)$.[8] $\square$

## 4.4 The Undefinability of Well-Ordering

Let $\tau = \{<, \ldots\}$.

**Theorem 4.27** *Suppose $\phi$ is an $\mathcal{L}_{\omega_1,\omega}$-sentence and for all $\alpha < \omega_1$ there is $\mathcal{M} \models \phi$ where $(\alpha, <)$ embeds into $<^{\mathcal{M}}$. Then there is $\mathcal{N} \models \phi$ where $(\mathbb{Q}, <)$ embeds into $<^{\mathcal{N}}$.*

[8] Alternatively, using Exercise 1.14, we could just assume all models in $\mathcal{K}_0$ and $\mathcal{K}_1$ are countable to begin with.

**Corollary 4.28** *If $\phi$ is an $\mathcal{L}_{\omega_1,\omega}$-sentence and $<^{\mathcal{M}}$ is well-ordered for all $\mathcal{M} \models \phi$, then there is $\alpha < \omega_1$ such that $<^{\mathcal{M}}$ has order type at most $\alpha$ for all $\mathcal{M} \models \phi$.*

**Proof** Let $\tau_0$ be our original vocabulary. Form $\tau$ by adding a countable set of new constants $C$ and distinct constants $D = \{d_q : q \in \mathbb{Q}\}$. Let $\Sigma$ be all $\sigma$ of the form

$$\sigma_0 \cup \{\phi\} \cup \{d_q < d_r : q < r\}$$

where $\sigma_0$ is a finite set of $\mathcal{L}_{\omega_1,\omega}$-sentences using only finitely many constants from $C \cup D$ such that if we let

$$\theta(\overline{c}, d_{i_1}, \ldots, d_{i_m}) = \bigwedge_{\psi \in \sigma_0} \psi$$

where $i_1 < \ldots < i_m$ then for all $\alpha < \omega_1$ there is an $\mathcal{M}$ such that the following property (*) holds:

$$\mathcal{M} \models \phi \wedge \exists \overline{x}\ \theta(\overline{x}, b_1 \ldots, b_m)$$

and $A \subset \mathcal{M}$ well ordered by $<^{\mathcal{M}}$, $\overline{b} \in A$ and

$$\alpha \leq b_1, b_1 + \alpha \leq b_2, \ldots, b_{m-1} + \alpha \leq b_m.$$

In particular, taking $\sigma_0 = \emptyset$, we see that, by assumption, $\{\phi\} \cup \{d_q < d_r : q < r\} \in \Sigma$. We claim that $\Sigma$ is a consistency property. Once we have shown this we will know there is a model of $\phi$ containing a densely ordered set.

We do several of the non-routine claims and leave the rest of the verification that $\Sigma$ is a consistency property as an exercise.

C4) Suppose $\bigvee_{\psi \in X} \psi \in \sigma$ where $\sigma \in \Sigma$. Then for each $\alpha$ there is $\psi_\alpha \in X$ and $\mathcal{M}$ such that (*) holds and $\mathcal{M} \models \psi_\alpha$. There is $\psi \in X$ such that $\psi = \psi_\alpha$ for uncountably many $\alpha$. Note that if $\psi$ works for $\alpha$ it works for all $\beta < \alpha$. Thus $\sigma \cup \{\psi\} \in \Sigma$.

C7c) Let $\sigma \in \Sigma$. Suppose $t = d_r$ and $\sigma_0$ uses only $d_{i_1}, \ldots, d_{i_m}$ where $i_0 < \ldots < i_m$. Suppose $i_s < r < i_{s+1}$. Let $c$ be element of $C$ not yet used. We claim that $\sigma \cup \{c = d_r\} \in \Sigma$.

Let $\alpha < \omega_1$. Pick $\beta > \alpha + \alpha$. By (*) there is

$$\mathcal{M} \models \phi \wedge \exists \overline{x}\ \theta(\overline{x}, \overline{b})$$

where

$$\beta \leq b_1, b_1 + \beta \leq b_2, \ldots, b_{m-1} + \beta \leq b_m.$$

Let $b = b_s + \alpha$. Then $b_s + \alpha \leq b$ and $b + \alpha \leq b_{s+1}$ as desired. □

**Exercise 4.29** Complete the proof that $\Sigma$ is a consistency property.

We will give a sharper version of this result in Theorem 11.3 and a useful extension to uncountable models in Corollary 6.9.

# 5 Hanf Numbers and Indiscernibles

In Exercise 1.27 we showed that the Upward Löwenheim–Skolem Theorem fails in $\mathcal{L}_{\omega_1,\omega}$, by giving a sentence with models of size $2^{\aleph_0}$ but no larger models. In this section we will show that there is a cardinal $\kappa$ such that if $\phi$ has models of cardinality $\kappa$, then $\phi$ has arbitrarily large models. We call the least such cardinal the *Hanf number* of $\mathcal{L}_{\omega_1,\omega}$. It is general nonsense that there is an Hanf number.

**Exercise 5.1** Let $I$ be a set. For each $i \in I$, let $\mathcal{K}_i$ be a class of structures. Let $\mathcal{K} = \{\mathcal{K}_i : i \in I\}$. Prove there is a cardinal $\kappa$ such that for all $i$, if $\mathcal{K}_i$ has a structure of size $\kappa$, then $\mathcal{K}_i$ contains arbitrarily large structures. The least such $\kappa$ is the *Hanf number* for $\mathcal{K}$.

Recall that for $\kappa$ an infinite cardinal and $\alpha$ an ordinal, we inductively define $\beth_\alpha(\kappa)$ by $\beth_0(\kappa) = \kappa$ and

$$\beth_\alpha(\kappa) = \sup_{\beta<\alpha} 2^{\beth_\beta(\kappa)}.$$

In particular $\beth_1(\kappa) = 2^\kappa$. We let $\beth_\alpha = \beth_\alpha(\aleph_0)$. Under the Generalized Continuum Hypothesis $\beth_\alpha = \aleph_\alpha$.

The main theorem, due to Morley is that the Hanf number for $\mathcal{L}_{\omega_1,\omega}$ is $\beth_{\omega_1}$.

**Theorem 5.2** *Let $\phi \in \mathcal{L}_{\omega_1,\omega}$. If for all $\alpha < \omega_1$ there is $\mathcal{M} \models \phi$ with $|M| \geq \beth_\alpha$, then $\phi$ has models of all infinite cardinalities.*

The next exercise generalizes Exercise 1.27 and shows that $\beth_{\omega_1}$ is optimal.

**Exercise 5.3** Let $\alpha < \omega_1$. Let $\tau = \{U_\beta : \beta \leq \alpha + 1\} \cup \{E\} \cup \{c_0, c_1, \ldots\}$ where $U_\beta$ is a unary relation symbol, $E$ is a binary relation symbol and $c_0, c_1, \ldots$ are constants. Let $\phi$ assert:

i) $U_0 = \{c_0, c_1, \ldots\}$ and $\forall x\, U_{\alpha+1}(x)$;

ii) $U_\gamma \subseteq U_\beta$ for $\gamma < \beta$;

iii) $\forall x\, (U_\beta(x) \leftrightarrow \bigvee_{\alpha<\beta} U_\alpha(x))$ for $\beta$ a limit ordinal;

iv) $\forall x \forall y\, [(U_{\beta+1}(x) \wedge E(y, x)) \rightarrow U_\beta(y)]$;

v) if $\{x : E(x, y)\} = \{x : E(x, z)\}$, then $y = z$.

a) Show that there is $\mathcal{M} \models \phi$ with $|M| = \beth_{\alpha+1}$.

b) Show that every model of $\phi$ has cardinality at most $\beth_{\alpha+1}$.

Here is the main idea of the proof of Theorem 5.2.

- By expanding the signature we may assume that we have $\phi \in T$ where $T$ is a satisfiable theory in a countable fragment with built-in Skolem functions.

- Under the assumptions of Theorem 5.2 we can find a model of $T$ with an infinite set of indiscernibles.

- Taking Skolem hulls we get models of all infinite cardinalities.

The first and third steps are exactly as in first order model theory. In first order model theory the second step is accomplished using Ramsey's Theorem and a compactness argument. We will need a stronger partition theorem and an application of the Model Existence Theorem.

## 5.1 The Erdös–Rado Partition Theorem

The material in this section is verbatim from §5 of [35].

For $X$ a set and $\kappa, \lambda$ (possibly finite) cardinals, we let $[X]^\kappa$ be the collection of all subsets of $X$ of size $\kappa$. We call $f : [X]^\kappa \to \lambda$ a *partition* of $[X]^\kappa$. We say that $Y \subseteq X$ is *homogeneous* for the partition $f$ if there is $\alpha < \lambda$ such that $f(A) = \alpha$ for all $A \in [Y]^\kappa$ (i.e., $f$ is constant on $[Y]^\kappa$). Finally, for cardinals $\kappa, \eta, \mu$, and $\lambda$, we write $\kappa \to (\eta)^\mu_\lambda$ if whenever $|X| \geq \kappa$ and $f : [X]^\mu \to \lambda$, then there is $Y \subseteq X$ such that $|Y| \geq \eta$ and $Y$ is homogeneous for $f$.

Stated in this notation Ramsey's Theorem becomes $\aleph_0 \to (\aleph_0)^n_m$ for all $n, m \in \omega$.

When we begin partitioning sets into infinitely many pieces it becomes harder to find homogeneous sets.

**Proposition 5.4** $2^{\aleph_0} \not\to (3)^2_{\aleph_0}$.

**Proof** We define $F : [2^\omega]^2 \to \omega$ by $F(\{f, g\})$ is the least $n$ such that $f(n) \neq g(n)$. Clearly, we cannot find $\{f, g, h\}$ such that $f(n) \neq g(n)$, $g(n) \neq h(n)$, and $f(n) \neq h(n)$. $\square$

On the other hand, if $\kappa > 2^{\aleph_0}$, then $\kappa \to (\aleph_1)^2_{\aleph_0}$. This is a special case of an important generalization of Ramsey's Theorem.

**Theorem 5.5 (Erdös–Rado Theorem)** $\beth_n(\kappa)^+ \to (\kappa^+)^{n+1}_\kappa$.

**Proof** We prove this by induction on $n$. For $n = 0$, $\kappa^+ \to (\kappa^+)^1_\kappa$ is just the Pigeonhole Principle.

Suppose that we have proved the theorem for $n-1$. Let $\lambda = \beth_n(\kappa)^+$, and let $f : [\lambda]^{n+1} \to \kappa$. For $\alpha < \lambda$, let $f_\alpha : [\lambda \setminus \{\alpha\}]^n \to \kappa$ by $f_\alpha(A) = f(A \cup \{\alpha\})$.

We build $X_0 \subseteq X_1 \subseteq \ldots \subseteq X_\alpha \subseteq \ldots$ for $\alpha < \beth_{n-1}(\kappa)^+$ such that $X_\alpha \subseteq \beth_n(\kappa)^+$ and each $X_\alpha$ has cardinality $\beth_n(\kappa)$. Let $X_0 = \beth_n(\kappa)$. If $\alpha$ is a limit ordinal, then $X_\alpha = \bigcup_{\beta<\alpha} X_\beta$.

Suppose we have $X_\alpha$ with $|X_\alpha| = \beth_n(\kappa)$. Because

$$\beth_n(\kappa)^{\beth_{n-1}(\kappa)} = (2^{\beth_{n-1}(\kappa)})^{\beth_{n-1}(\kappa)} = 2^{\beth_{n-1}(\kappa)} = \beth_n(\kappa),$$

there are $\beth_n(\kappa)$ subsets of $X_\alpha$ of cardinality at most $\beth_{n-1}(\kappa)$. Also note that if $Y \subset X_\alpha$ and $|Y| \leq \beth_{n-1}(\kappa)$, then there are at most $\beth_n(\kappa)$ functions $g : [Y]^n \to \kappa$ because

$$\kappa^{\beth_{n-1}(\kappa)} = 2^{\beth_{n-1}(\kappa)} = \beth_n(\kappa).$$

Thus, we can find $X_{\alpha+1} \supseteq X_\alpha$ such that $|X_{\alpha+1}| = \beth_n(\kappa)$, and if $Y \subset X_\alpha$ with $|Y| = \beth_{n-1}(\kappa)$ and $\beta \in \lambda \setminus Y$, then there is $\gamma \in X_{\alpha+1} \setminus Y$ such that $f_\beta|[Y]^n = f_\gamma|[Y]^n$.

Let $X = \bigcup_{\alpha<\beth_{n-1}(\kappa)^+} X_\alpha$. If $Y \subset X$ with $|Y| \leq \beth_{n-1}(\kappa)$, then $Y \subset X_\alpha$ for some $\alpha < \beth_{n-1}(\kappa)^+$. If $\beta \in \lambda \setminus Y$, then there is $\gamma \in X \setminus Y$ such that $f_\beta|[Y]^n = f_\gamma|[Y]^n$.

Fix $\delta \in \lambda \setminus X$. Inductively construct $Y = \{y_\alpha : \alpha < \beth_{n-1}(\kappa)^+\} \subseteq X$. Let $y_0 \in X$. Suppose that we have constructed $Y_\alpha = \{y_\beta : \beta < \alpha\}$. Choose $y_\alpha \in X$ such that $f_{y_\alpha}|[Y_\alpha]^n = f_\delta|[Y_\alpha]^n$.

By the induction hypothesis, there is $Z \subseteq Y$ such that $|Z| \geq \kappa^+$ and $Z$ is homogeneous for $f_\delta$. Say $f_\delta(B) = \gamma$ for all $B \in [Z]^n$. We claim that $Z$ is homogeneous for $f$. Let $A \in [Z]^{n+1}$. There are $\alpha_1 < \ldots < \alpha_{n+1}$ such that $A = \{y_{\alpha_1}, \ldots, y_{\alpha_{n+1}}\}$. Then

$$f(A) = f_{y_{\alpha_{n+1}}}(\{y_{\alpha_1}, \ldots, y_{\alpha_n}\}) = f_\delta(\{y_{\alpha_1}, \ldots, y_{\alpha_n}\}) = \gamma.$$

Thus, $Z$ is homogeneous for $f$. □

We will use the following corollary.

**Corollary 5.6** $\beth^+_{\alpha+n} \to (\beth^+_\alpha)^{n+1}_{\beth_\alpha}$.

**Proof** This follows from Erdös–Rado as $\beth_{\alpha+n} = \beth_n(\beth_\alpha)$. □

## 5.2 The Hanf Number of $\mathcal{L}_{\omega_1,\omega}$

We now prove Theorem 5.2. We assume that the reader is used to construction and use of indiscernibles in first order logic as in §5 of [35].

**Exercise 5.7** Suppose $\mathbb{A}$ is a countable fragment of $\mathcal{L}_{\omega_1,\omega}$ and $T$ is an $\mathbb{A}$-theory with built-in Skolem functions. Suppose $\mathcal{M} \models T$ contains an infinite set of indiscernibles for the fragment $\mathbb{A}$. Then $T$ has arbitrarily large models.

We begin by expanding the signature we may assume that:

- there is a countable fragment $\mathbb{A}$ of $\mathcal{L}_{\omega_1,\omega}$ and $T \subset \mathbb{A}$ a theory with built in Skolem functions such that $\phi \in T$ and $T$ has models of cardinality $\beth_\alpha$ for all $\alpha < \omega_1$;
- we have two disjoint countably infinite sets of constant symbols $C = \{c_0, c_1, \ldots\}$ and $D = \{d_0, d_1, \ldots\}$, but formulas in $\mathbb{A}$ use only finitely many constant symbols from $C \cup D$.

Let $\Gamma = \{d_i \neq d_j : i < j\} \cup \{\theta(d_{i_1}, \ldots, d_{i_n}) \leftrightarrow \theta(d_{j_1}, \ldots, d_{j_n}) : \theta(v_1, \ldots, v_n) \in \mathbb{A}, i_1 < \ldots < i_n, j_1 < \ldots < j_n$ and no constants from $C \cup D$ occur in $\theta\}$.

If we can find $\mathcal{M} \models T \cup \Gamma$, then the interpretation of $D$ in $\mathcal{M}$ gives an infinite set of indiscernibles. We can then stretch the indiscernibles to build arbitrarily large models.

If $\sigma$ is a finite set of $\mathbb{A}$-sentences, we let $\Theta_\sigma$ be the $\mathbb{A}$-formula with no constants from $C \cup D$ such that $\Theta_\sigma(\overline{c}, \overline{d})$ is the conjunction of $\sigma$.

Let $\Sigma$ be the set of all finite sets $\sigma$ of $\mathbb{A}$-sentences such that for all $\alpha < \omega_1$ there is $\mathcal{M} \models T$, $A \subseteq \mathcal{M}$ with $|A| = \beth_\alpha$, $<$ a linear order of $A$ and

$$\mathcal{M} \models \exists \overline{v}\ \Theta_\sigma(\overline{v}, a_1, \ldots, a_n)$$

for all $a_1 < \ldots < a_n$ in $A$.

We need to show that $\Sigma$ is a consistency property and if $\sigma \in \Sigma$ and $\psi \in T \cup \Gamma$, then $\sigma \cup \{\psi\} \in \Sigma$. Once we have established these claims we can use the Extended Model Existence Theorem 4.8 to conclude that there is $\mathcal{M} \models T \cup \Gamma$.

**claim** $\Sigma$ is satisfiable.

The only tricky case is C4). Suppose $\bigvee_{\psi \in X} \psi \in \sigma \in \Sigma$. Let $\Theta(\overline{c}, d_1, \ldots, d_n)$ be conjunction of $\sigma$. For all $\alpha < \omega_1$ we can find $\mathcal{M}_\alpha \models T$, $A_\alpha \subset \mathcal{M}_\alpha$ of cardinality at least $\beth_{\alpha+n}$ and $<$ a linear order of $A$ such that

$$\mathcal{M}_\alpha \models \exists \overline{v}\ \Theta(\overline{v}, a_1, \ldots, a_n)$$

for all $a_1 < \ldots < a_n \in A$. Let $f_\alpha : [A_\alpha]^n \to X$ such that if $f_\alpha(\overline{a}) = \psi(\overline{c}, d_1, \ldots, d_n)$, then

$$\mathcal{M}_\alpha \models \exists \overline{v}[\Theta(\overline{v}, \overline{a}) \wedge \psi(\overline{v}, \overline{a})].$$

Since $\beth_{\alpha+n} \geq \beth^+_{\alpha+n-1}$, by Erdös–Rado we can find $A'_\alpha \subseteq A_\alpha$ of cardinality at least $\beth_\alpha$ and $\psi \in X$ such that

$$\mathcal{M}_\alpha \models \exists \overline{v} \; [\Theta(\overline{v}, a_1, \ldots, a_n) \wedge \psi(\overline{v}, a_1, \ldots, a_n)]$$

for all $a_1 < \ldots < a_n$ in $A'_\alpha$. By the Pigeonhole Principle, we can find a $\psi$ that works for cofinally many $\alpha$. Thus $\sigma \cup \{\psi(\overline{c}, d_1, \ldots, d_n)\} \in \Sigma$.

**claim** If $\sigma \in \Sigma$ and $\psi \in T \cup \Gamma$, then $\sigma \cup \{\psi\} \in \Sigma$.

Suppose $\sigma \in \Sigma$. It is clear that if $\chi \in T$, then $\sigma \cup \{\chi\} \in \Sigma$. Suppose $\chi$ is

$$\psi(d_{i_1}, \ldots, d_{i_m}) \leftrightarrow \psi(d_{j_1}, \ldots, d_{j_m})$$

where $i_1 < \ldots < i_m$ and $j_1 < \ldots < j_m$ and no constants from $C \cup D$ occur in $\psi$.

Let $\Theta(\overline{c}, d_1, \ldots, d_n m)$ be the conjunction of $\sigma$. We may assume all $i_k, j_k \leq n$. For all $\alpha < \omega_1$ there is $\mathcal{M}_\alpha \models T$, $A_\alpha \subset M_\alpha$ of cardinality $\beth_{\alpha+m}$ such that

$$\mathcal{M}_\alpha \models \exists \overline{v} \; \Theta(\overline{v}, a_1, \ldots, a_m)$$

for $a_1 < \ldots < a_m$ in $A_\alpha$.

Let $f_\alpha : [A_\alpha]^m \to \{0, 1\}$ with $f_\alpha(\overline{a}) = 1$ if $\mathcal{M}_\alpha \models \psi(\overline{a})$. By Erdös–Rado we can find $A'_\alpha \subset A_\alpha$ of cardinality $\beth_\alpha$ homogeneous for $f_\alpha$. Then

$$\mathcal{M}_\alpha \models \exists v \; [\Theta(\overline{v}, a_1, \ldots, a_n) \wedge (\psi(a_{i_1}, \ldots, a_{i_m}) \leftrightarrow \psi(a_{j_1}, \ldots, a_{j_n})]$$

for increasing sequences from $A'_\alpha$. Thus $\sigma \cup \{\chi\} \in \Sigma$.

This completes the Proof of Theorem 5.2. □

The following Corollary is proven directly, by essentially the same argument, in Theorem 5.2.14 of [35].

**Corollary 5.8** *Suppose $T$ is a first order theory in a countable signature and $p_1, p_2, \ldots$ are partial types. If for all $\alpha < \omega_1$ there are models of $T$ of cardinality $\beth_\alpha$ omitting all of the $p_i$, then there are arbitrarily large models of $T$ omitting all of the $p_i$.*

**Exercise 5.9** Show that Theorem 5.2 follows immediately from Corollary 5.8 and the characterization in Theorem 1.31 of $\mathcal{L}_{\omega_1,\omega}$-elementary classes as reducts of classes of models of a first order theory omitting a set of types.

**Exercise 5.10** Suppose $\phi \in \mathcal{L}_{\omega_1,\omega}$ is $\kappa$-categorical for some $\kappa \geq \beth_{\omega_1}$. Prove that if $\mathcal{M}, \mathcal{N} \models \phi$ have cardinality at least $\beth_{\omega_1}$ then $\mathcal{M} \equiv_{\omega_1,\omega} \mathcal{N}$. (Compare this to Exercise 2.29.)

By carefully choosing the order type of the indiscernibles, we can build large models realizing few types.

**Theorem 5.11** *If $\phi$ has arbitrarily large models, then for all $\kappa$ there is a model of $\phi$ of cardinality $\kappa$ realizing only countably many $\mathcal{L}_{\omega_1,\omega}$-types*

**Proof** We follow the proof of Theorem 5.2. We can find a countable vocabulary $\tau$, a countable fragment $\mathbb{A}$ and $T \subseteq \mathbb{A}$ a theory with built in $\mathbb{A}$-Skolem functions such that $\phi \in T$ and $T$ has a countable model with an infinite set of order indiscernibles.

Suppose $(A, <)$ is a linear order of cardinality $\kappa$ that is $n$-transitive for all $n \in \omega$, i.e., for all $a_1 < \ldots < a_n$ and $b_1 < \ldots < b_n$ there is an automorphism $\sigma$ of $(A, <)$ such that $\sigma(\overline{a}) = \overline{b}$. We see below that such orders exist. Stretch the indiscernibles of $\mathcal{M}$ to build a model $\mathcal{N}$ which is the hull of indiscernibles of order type $(A, <)$. Then any order automorphism of $(A, <)$ extends to an automorphism of extends to an automorphism of $\mathcal{N}$. In particular for any Skolem term $f(v_1, \ldots, v_n)$, $a_1 < \ldots < a_n$ and $b_1 < \ldots < b_n$, $f(\overline{a})$ and $f(\overline{b})$ have the same $\mathcal{L}_{\omega_1,\omega}$-type. Thus $\mathcal{N}$ realizes only countably many $\mathcal{L}_{\omega_1,\omega}$-types. □

**Exercise 5.12** Show that for all $\kappa$ there is a linear order of cardinality $\kappa$ that is $n$-transitive for all $n$. [Hint: Consider the order type of an ordered field of cardinality $\kappa$.]

**Corollary 5.13** *If $\phi$ has arbitrarily large models and is $\kappa$-categorical, there is a complete sentence $\psi$ such that $\psi \models \phi$ and $\psi$ has a model of cardinality $\kappa$.*

**Proof** We saw in Exercise 2.34 that a model realizing countable many $\mathcal{L}_{\omega_1,\omega}$-types is $\mathcal{L}_{\infty,\omega}$-equivalent to a countable model. □

We will return to the question of building uncountable models realizing few types in §6.3.

We can refine the Hanf number question by looking at complete sentences.

**Exercise 5.14** Let $\tau = \{+, 0, G_1, G_2, \ldots\}$ and let $\phi$ be the $\mathcal{L}_{\omega_1,\omega}$ sentence asserting:

i) we have an Abelian group where every element has order 2;

ii) $G_1$ is an index 2 subgroup and $G_{n+1}$ is an index 2 subgroup of $G_n$ for each $n$;

iii) $\bigcap G_n = \{0\}$.

Prove that $\phi$ is complete and every model of $\phi$ has cardinality at most $2^{\aleph_0}$.

Baumgartner [9], building on work of Malitz [33], showed that Hanf number for complete $\mathcal{L}_{\omega_1,\omega}$-sentences is still $\beth_{\omega_1}$. Knight [24] showed there is a complete sentence with models of size $\aleph_1$ but no larger. Hjorth [17] extended this by showing that for all $\alpha < \omega_1$ there is a complete sentence with models of size $\aleph_\alpha$ but no larger. We will give Hjorth's proof for $\aleph_1$ and an application to Vaught's Conjecture in §5.4. Without assumptions on cardinal exponentiation Hjorth's result is as far as we can go as Shelah [49] has shown that it consistent with ZFC that if $\phi$ has a model of size $\aleph_{\omega_1}$, then $\phi$ has a model of cardinality $2^{\aleph_0}$.

## 5.3 Morley's Two Cardinal Theorem

We give one more application of using Erdös–Rado to build useful indiscernibles.

**Definition 5.15** Let $\tau = \{U, \ldots\}$ where $U$ is a unary predicate. We say that a $\tau$-structure $\mathcal{M}$ is a $(\kappa, \lambda)$-model if $|\mathcal{M}| = \kappa$ and $|U(\mathcal{M})| = \lambda$.

**Theorem 5.16 (Morley's Two Cardinal Theorem)** *Let $\phi$ be an $\mathcal{L}_{\omega_1,\omega}$-sentence. Suppose for arbitrarily large $\alpha < \omega_1$ there is an infinite $\kappa$ and $\mathcal{M}_\alpha$ a $(\beth_\alpha(\kappa), \kappa)$-model of $\phi$. Then for all $\kappa$ there is a $(\kappa, \aleph_0)$-model of $\phi$.*

**Proof** We can extend the signature to find a countable fragment $\mathbb{A}$, an $\mathbb{A}$-theory $T$ with built-in Skolem functions with $\phi \in T$ such that for arbitrarily large $\alpha < \omega_1$ there is a $(\beth_\alpha(\kappa), \kappa)$-model of $T$ for some infinite $\kappa$.

Suppose $\mathcal{M} \models T$. Suppose $I \subseteq M$ and $<$ linearly orders $I$. We say that $I$ is *indiscernible over $U$* if

$$\mathcal{M} \models \psi(x_1, \ldots, x_n, \overline{a}) \leftrightarrow \psi(y_1, \ldots, y_n, \overline{a})$$

for all $\psi(v_1, \ldots, v_n, \overline{u}) \in \mathbb{A}$, $\overline{a} \in U(\mathcal{M})$, $x_1 < \ldots < x_n, y_1 < \ldots < y_n$ in $I$.

**Exercise 5.17** Suppose there is $\mathcal{M} \models T$ countable with $I \subseteq \mathcal{M}$ an infinite set of indiscernibles over $U$. Show that $T$ has $(\kappa, \aleph_0)$ models for all infinite $\kappa$. [Hint: Prove that if $x_1 < \ldots < x_n \in I$, $\overline{a} \in U(\mathcal{M})$ and $f(\overline{x}, \overline{a}) \in U(\mathcal{M})$, then $f(\overline{y}, \overline{a}) = f(\overline{x}, \overline{a})$ for all $y_1 < \ldots < y_n$ in $I$.]

Add two new countable infinite sets of constant symbols $C$ and $D$. Let $\Gamma = \{d_i \neq d_j : i < j\} \cup \{\forall \overline{u}\ [U(\overline{u}) \to (\psi(d_{i_1}, \ldots, d_{i_m}, \overline{u}) \leftrightarrow \psi(d_{j_1}, \ldots, d_{j_m}, \overline{u}) :$ for $\phi \in \mathbb{A}$ (with no contants from $C \cup D$), $i_1 < \ldots < i_m$, $j_1 < \ldots < j_m\}$.

We define a consistency property $\Sigma$. If $\sigma$ is a finite set of $\mathbb{A}$-sentences with finitely many constants from $C \cup D$, then $\sigma \in \Sigma$ if and only if for arbitrarily large $\alpha < \omega_1$ there is $\mathcal{M} \models T$ and $A \subseteq M$ with $|A| \geq \beth_\alpha(U(\mathcal{M}))$ and $<$ an ordering of $A$ such that

$$\mathcal{M} \models \exists \overline{v} \Theta(\overline{v}, \overline{a})$$

for all $a_1 < \ldots < a_n$ in $A$, where $\Theta(\overline{c}, d_1, \ldots, d_n)$ is the conjunction of all formulas in $\sigma$.

**Exercise 5.18** Prove that $\Sigma$ is a consistency property.

**Exercise 5.19** Prove that if $\sigma \in \Sigma$ and $\psi \in T \cup \Gamma$, then $\sigma \cup \{\psi\} \in \Sigma$.

Combining the exercises as in the proof of Theorem 5.2 we conclude that for all infinite $\kappa$, there is a $(\kappa, \aleph_0)$-model of $T$. □

The next exercise shows that the assumptions are optimal.

**Exercise 5.20** Modify Exercise 5.3 to show that for all $\alpha < \omega_1$ there is a sentence $\phi \in \mathcal{L}_{\omega_1,\omega}$ with a $(\beth_\alpha, \aleph_0)$-model that does not have $(\kappa, \aleph_0)$-models for all infinite $\kappa$.

By a similar argument Morley proved a sharper result for first order theories. For a proof see Theorem 4.2.15 of [35].

**Theorem 5.21** *Let $T$ be a first order theory in a countable language. Suppose that $T$ has a $(\beth_n, \aleph_0)$-model for all $n \in \omega$. Then for all $\kappa \geq \lambda$, $T$ has a $(\kappa, \lambda)$-model.*

## 5.4 Completely Characterizing $\aleph_1$

**Theorem 5.22 (Knight [24])** *There is a complete $\mathcal{L}_{\omega_1,\omega}$-sentence $\phi$ such that $\phi$ has models of cardinality $\aleph_1$ but no larger models, indeed no model of size $\aleph_1$ has a proper extension that is a model of $\phi$.*

We will give Hjorth's proof of a refinement on Knight's theorem and his application to Vaught counterexamples. Hjorth's construction uses a variation of the Fraïssé construction. We begin with a quick review of the method.

**Definition 5.23** Let $\mathcal{K}$ be a class of finite structures closed under isomorphism. The class $\mathcal{K}$ has the *joint embedding property* if whenever $A_0, A_1 \in \mathcal{K}$ there is $B \in \mathcal{K}$ and an embedding $f_i : A_i \to B$.

The class $\mathcal{K}$ has the *amalgamation property* if whenever $A, B_0, B_1 \in \mathcal{K}$ and $f_i : A \to B_i$ are embeddings, there is $C \in \mathcal{K}$ and embeddings $g_i : B_i \to C$. We say that $\mathcal{K}$ has the *disjoint embedding property* if we can choose $g_0, g_1$, and $C$ such that $g_0(B_0 \setminus f_0(A)) \cap g_1(B_1 \setminus f_1(A)) = \emptyset$.

**Definition 5.24** We say that $\mathcal{M}$ is $\mathcal{K}$-*generic* if

i) for every finite $A \subset \mathcal{M}$ there is $B \in \mathcal{K}$ such that $A \subseteq B \subset \mathcal{M}$;

ii) for every $A \in \mathcal{K}$, there is an embedding of $A$ into $\mathcal{M}$;

iii) if $A, B \in \mathcal{K}$, $A \subset B$ and $A \subset \mathcal{M}$ there is $A \subset B_0 \subset \mathcal{M}$ such that there is an isomorphism $f : B \to B_0$ such that $f|A$ is the identity.

**Exercise 5.25** Suppose $(\mathcal{M}_\alpha : \alpha < \beta)$ is a chain $\mathcal{M}_0 \subset \ldots \subset \mathcal{M}_\alpha \subset \ldots$ of models of $\Phi_\mathcal{K}$. Show that $\bigcup_{\alpha<\beta} \mathcal{M}_\alpha \models \Phi_\mathcal{K}$.

**Proposition 5.26** *Suppose there are $\aleph_0$ isomorphism types of structures in $\mathcal{K}$ and $\mathcal{K}$ has the joint embedding property and the amalgamation property. Then there is a $\mathcal{K}$-generic model of cardinality $\aleph_0$ and any two such models are isomorphic.*

**Proof** We build a countable $\mathcal{K}$-generic model $\mathcal{M}$ as a union of $A_0 \subset A_1 \subset \ldots$ of elements of $\mathcal{K}$. We can take $A_0 \in \mathcal{K}$ arbitrary. At stage $s$ of our construction we are trying to insure an instance of ii) or iii).

Suppose we are given $A_s \in \mathcal{K}$ and $B \in \mathcal{K}$. Since $\mathcal{K}$ has the joint embedding property we can find $A_{s+1} \in \mathcal{K}$ such that $A_s$ and $B$ both embed in $A_{s+1}$.

Suppose $A \subset A_s$, $A \subset B$ where $A_s, A$ and $B \in \mathcal{K}$. Since $\mathcal{K}$ has the amalgamation property we can find $A_{s+1} \supset A_s$ and $B' \subset A_{s+1}$ such that $B$ and $B'$ are isomorphic over $A$.

Let $\mathcal{M} = \bigcup_{s\in\omega} A_s$. Clearly $\mathcal{M}$ satisfies i) and by carefully organizing our construction we can ensure $\mathcal{M}$ satisfies ii) and iii).

Suppose $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{K}$-generic models of cardinality $\aleph_0$. We build a sequence of partial embeddings

$$f_0 \subset f_1 \subset \ldots$$

such that $A_s = \mathrm{dom}(f_s) \in \mathcal{K}$.

Let $A_0 \subset \mathcal{M}$ be an arbitrary element of $\mathcal{K}$. By ii) there is an embedding of $A_0$ into $\mathcal{N}$. Suppose we have $f_s : A_s \to \mathcal{N}$ where $A_s \in \mathcal{K}$ and $A_s \subset \mathcal{M}$. Let $a \in \mathcal{M} \setminus A_s$. By i) there is $A_{s+1} \subset \mathcal{M}_s$ such that $A_s \cup \{b\} \subseteq A_{s+1}$ and $A_{s+1} \in \mathcal{K}$. By iii) we can find $f_{s+1} \supset f_s$ such that $f_{s+1} : A_{s+1} \to \mathcal{N}$. Similarly, we can extend $f_s$ to add any element of $\mathcal{N}$ to the image. Thus we can construct the sequence of partial embedding such that $f = \bigcup f_s$ is an isomorphism between $\mathcal{M}$ and $\mathcal{N}$. □

In our examples we will have $\emptyset \in \mathcal{K}$. In this case the amalgamation property implies the joint embedding property, so we can ignore the latter.

**Exercise 5.27** Show that conditions i), ii) and iii) give a simple description of $\Phi_{\mathcal{K}}$ a complete sentence satisfied by the countable $\mathcal{K}$-generic model.

**Proposition 5.28** *Suppose in addition that $\mathcal{K}$ has the disjoint amalgamation property and $\emptyset \in \mathcal{K}$. If $\mathcal{M}$ is the countable $\mathcal{K}$-generic model there is an embedding $f : \mathcal{M} \to \mathcal{M}$ such that* $\mathrm{img}(f)$ *is a proper subset of $\mathcal{M}$.*

*If $\Phi_{\mathcal{K}}$ is the Scott sentence of $\mathcal{M}$, then $\Phi_{\mathcal{K}}$ has a model of size $\aleph_1$.*

**Proof** Let $\tau' = \tau \cup \{P\}$ where $P$ is a unary function symbol. Let $\mathcal{K}'$ be the class of all expansions of structures in $\mathcal{K}$ to $\tau'$.

**Exercise 5.29** Show that since $\mathcal{K}$ has the disjoint amalgamation property and $\emptyset \in \mathcal{K}$, $\mathcal{K}'$ has joint embedding property and the amalgamation property.

Let $\mathcal{M}'$ be the unique $\mathcal{K}'$-generic model and let $\mathcal{M}$ be it's reduct to $\tau$. We claim that there is a $\tau$-embedding $f : \mathcal{M} \to P(\mathcal{M}')$. Then $\mathcal{M}$ is a proper embedding of $\mathcal{M}$ into itself.

Suppose $A \in \mathcal{K}$, $A \subset \mathcal{M}$ and $f : A \to P(\mathcal{M}')$ is a $\tau$-embedding. Let $a \in \mathcal{M} \setminus A$ and let $B_1 \in \mathcal{K}', B_1 \subset \mathcal{M}'$ such that $A \cup \{a\} \subseteq B_1$. Let $B$ be the $\tau$-reduct of $B_1$. Let $B_2$ be a $\tau'$-structure such that the $\tau$-reduct is isomorphic to $B$ and $P(B_2) = B_2$. Since $\mathcal{M}'$ is $\mathcal{K}'$-generic, we can extend $f$ to $\widehat{f} : B \to P(\mathcal{M}')$. Carefully iterating this construction we can build a $\tau$-embedding $f : \mathcal{M} \to P(\mathcal{M})$.

If $\mathcal{M}$ is the countable $\mathcal{K}$-generic model, we can build a chain $\mathcal{M}_0 \subset \mathcal{M}_1 \subset \ldots \subset \mathcal{M}_\alpha \subset \ldots$ for $\alpha < \omega_1$ of countable models of $\Phi_K$. At successor stages we use the construction above to build $\mathcal{M}_\alpha \subset \mathcal{M}_{\alpha+1}$ and at limit stages we use Exercise 5.25. Then $\mathcal{N} = \bigcup_{\alpha<\omega_1} \mathcal{M}_\alpha$ is the desired model of size $\aleph_1$. $\square$

For Hjorth's example we let $\tau_0 = \{S_i, R_i : i \in \omega\}$ where $S_i$ is a binary relation symbol and $R_i$ is an $i+2$-ary relation symbol. We will consider the class $\mathcal{K}_0$ of finite $\tau$-structures with the following properties:

i $\forall a, b \ \bigvee_{i\in\omega} S_i(a, b)$;

ii $\forall a, b \ \bigwedge_{i\neq j}(S_i(a, b) \rightarrow \neg S_j(a, b))$;

iii $\forall a_0, a_1, b_0, \ldots, b_{k-1} \ [R_k(a_0, a_1, b_0, \ldots, b_{k-1}) \rightarrow (a_0 \neq a_1 \wedge b_i \neq b_j)]$, for all $i < j < k$;

iv $\forall a_0, a_1, b_0, \ldots, b_{k-1} \ [R_k(a_0, a_1, b_0, \ldots, b_{k-1}) \leftrightarrow R_k(a_0, a_1, b_{\sigma(0)}, \ldots, b_{\sigma(k-1)})]$ for all $k$ and $\sigma$ a permutation of $\{0, \ldots, k-1\}$;

v $\forall a_0, a_1, b_0, \ldots, b_{k-1} \ [R_k(a_0, a_1, b_0, \ldots, b_{k-1}) \rightarrow (S_i(a_0, b_j) \leftrightarrow S_i(a_1, b_j))]$ for all $i, k$ and $j < k$;

vi $\forall a_0, a_1, b_0, \ldots, b_{k-1}, c \ [(R_k(a_0, a_1, b_0, \ldots, b_{k-1}) \wedge \bigwedge_{i<k} c \neq b_i) \rightarrow \bigwedge_{i\in\omega}(S_i(a_0, c) \rightarrow \neg S_i(a_1, c))]$ for all $k$;

vii $\forall a_0, a_1[a_0 \neq a_1 \rightarrow \bigvee_{k\in\omega} \exists b_0, \ldots, b_{k-1} \ R_k(a_0, a_1, b_0, \ldots, b_{k-1})]$.

We can think of structures in $\mathcal{K}_0$ as being complete graphs where we color the directed edge $(a, b)$ with the unique color $i$ such that $S_i(a, b)$. For each $a_0$ and $a_1$ there is a maximal set $\{b_0, \ldots, b_{k-1}\}$ such that each edge $(a_0, b_i)$ has the same color as the edge $(a_1, b_i)$ for $i = 1 \ldots k$ and $R_k(a_0, a_1, b_0, \ldots, b_k)$.

**Exercise 5.30** a) Show that there are only countably many isomorphism types in $\mathcal{K}_0$.

b) Let $A$ be a set of cardinality $n$. Color the complete directed graph on $A$ such that each edge gets a distinct color. Let $R_0$ hold of all pairs $(a, b)$ where $a \neq b$ and no other $R_i$ relation holds. Show that $A \in \mathcal{K}_0$.

c) Conclude that $\mathcal{K}_0$ is a countably infinite set of isomorphism types and $\emptyset \in \mathcal{K}_0$.

**Lemma 5.31** *$\mathcal{K}_0$ has the disjoint amalgamation property. Thus there is a countable $\mathcal{K}_0$-generic model $\mathcal{M}$. If $\Phi_{\mathcal{K}_0}$ is the Scott sentence of $\mathcal{M}$, then $\Phi_{\mathcal{K}_0}$ has models of size $\aleph_1$.*

**Proof** Without loss of generality, we may suppose we have $A, B_0, B_1 \in \mathcal{K}_0$ with $A = B_0 \cap B_1$. We make $B_0 \cup B_1$ into a $\tau_0$-structure as follows.

- If $a_0 \in B_l$ and $a_1 \in B_{1-l}$ let $b_0, \ldots, b_{k-1}$ be the elements of $A$ such that $S_i(a_0, b_j)$ and $S_i(a_1, b_j)$ for some $i$. Make $R_k(a_0, a_1, b_0, \ldots, b_{k-1})$ hold.
- For each pair $a_0 \in B_l, a_1 \in B_{1-l}$, choose a new $i$ and make $S_i(a_0, a_1)$ hold.

It is easy to check that this makes $B_0 \cup B_1$ into a structure in $\mathcal{K}_0$. $\square$

**Lemma 5.32** *If $\mathcal{M} \models \Phi_{\mathcal{K}_0}$ is uncountable, then there is no proper extension $\mathcal{M} \subset \mathcal{N}$ where $\mathcal{N} \models \Phi_{\mathcal{K}_0}$. In particular, $\Phi_{K_0}$ has no models of size $\aleph_2$.*

**Proof** Suppose $\mathcal{M} \subset \mathcal{N}$ and $a_0, a_1 \in \mathcal{M}$. For some $k$ there are $b_0, \ldots, b_{k-1} \in \mathcal{M}$ such that $\mathcal{M} \models R_k(a_0, a_1, b_0, \ldots, b_{k-1})$. If $b \in \mathcal{N} \setminus \mathcal{M}$, by vi) if $S_i(a_0, b)$, then $\neg S_i(a_1, b)$. Thus the function that sends $a \in \mathcal{M}$ to the unique $i$ such that $S_i(a, b)$ is injective and $\mathcal{M}$ must be countable. $\square$

Thus $\Phi_{\mathcal{K}_0}$ is the desired complete $\mathcal{L}_{\omega_1,\omega}$-sentence with models of size $\aleph_1$ but no models of size $\aleph_2$.

### Vaught counterexamples with no models of size $\aleph_2$

Let $\tau_0$ and $\mathcal{K}_0$ be as above. Let $\tau_1 = \tau_0 \cup \{P, Q, F\}$ where $P$ and $Q$ are unary relation symbols and $F$ is a binary relation symbol. Let $\mathcal{K}_1$ be the class of $\tau_1$ structures $A$ such that:

- $A$ is the disjoint union of $P(A)$ and $Q(A)$;
- $F$ defines a function from $P(A)$ into $Q(A)$;
- the relation symbols from $\tau_0$ only hold of elements from $P(A)$;
- $P(A) \in \mathcal{K}_0$.

**Exercise 5.33** a) Show that $\mathcal{K}_1$ has the amalgamation and joint embedding properties (this argument should only need the fact that $\mathcal{K}_0$ has the disjoint embedding property and $\emptyset \in \mathcal{K}_0$).

b) Let $\mathcal{M}$ be the countable $\mathcal{K}_1$-generic model and let $\Phi_{\mathcal{K}_1}$ be it's Scott sentence. Show that $P(\mathcal{M}) \models \Phi_{K_0}$.

c) Show that in $\mathcal{M}$ the map from $P(\mathcal{M})$ to $Q(\mathcal{M})$ is surjective.

d) Show $\Phi_{K_1}$ has models of size $\aleph_1$, but no uncountable model has a proper extension that is a model of $\Phi_{K_1}$.

**Definition 5.34** Let $\mathcal{M}$ be a model and let $X \subset \mathcal{M}$ be $\mathcal{L}_{\omega_1,\omega}$-definable. We say that $X$ is a set of *absolute indiscernibles* in $\mathcal{M}$ if for every permutation $\sigma$ of $X$ there is an automorphism $f$ of $\mathcal{M}$ such that $f \supset \sigma$.

**Lemma 5.35** *Let $\mathcal{M}$ be the countable $\mathcal{K}_1$-generic model. Then $Q(\mathcal{M})$ is a set of absolute indiscernibles.*

**Proof** Let $\sigma$ be a permutation of $Q(\mathcal{M})$. Suppose we have $A \subset \mathcal{M}$ such that $A \in \mathcal{K}_1$ and $f : A \to \mathcal{M}$ such $f|Q(A) = \sigma|Q(A)$. Suppose $A \subset B \subset \mathcal{M}$ and $B \in \mathcal{K}_1$. We need to show that there is $g : A \to \mathcal{M}$ extending $f$ such that $f|Q(B) = \sigma|Q(B)$.

Let $c_1, \ldots, c_m$ be the elements of $Q(B) \setminus Q(A)$. Note that $A' = A \cup \{c_1, \ldots, c_m\} \in \mathcal{K}_1$. We can extend $f$ to $f' : A' \to \mathcal{M}$ by defining $f'(c_i) = \sigma(c_i)$. We now use the fact that $\mathcal{M}$ is $\mathcal{K}_1$-generic to extend $f'$ to $g : B \to \mathcal{M}$. Since $B \setminus A' \subset P(\mathcal{M})$, $g|Q(B) = \sigma|Q(B)$ as desired.

Similarly, we can add elements of $\mathcal{M} \setminus f(A)$ to the image of $f$. This allows us to build an automorphism of $\mathcal{M}$ that extends $\sigma$. □

**Corollary 5.36 (Hjorth[18])** *If there is a counterexample $\psi \in \mathcal{L}_{\omega_1,\omega}$ to Vaught's Conjecture, then there is a counterexample with no model of size $\aleph_2$.*

**Proof** Suppose $\psi$ is a Vaught counterexample in $\mathcal{L}_{\omega_1,\omega}(\tau)$ where we may assume $\tau \cap \tau_1 = \emptyset$ and $\tau$ is relational. Consider the $\tau \cup \tau_1$-sentence $\Psi$ with models $\mathcal{N}$ such that: that

- $\mathcal{N} \models \Phi_{\mathcal{K}_1}$;
- all of the $\tau$-structure is on $Q(\mathcal{N})$;
- $Q(\mathcal{N}) \models \psi$.

Suppose $\mathcal{N}_1$ and $\mathcal{N}_2$ are countable models of $\Psi$. We claim that if $Q(\mathcal{N}_1) \cong Q(\mathcal{N}_2)$, then $\mathcal{N}_1 \cong \mathcal{N}_2$. Suppose $f_0 : Q(\mathcal{N}_1) \to Q(\mathcal{N}_2)$ is a $\tau$-isomorphism. Since the $\tau_1$-reducts of $\mathcal{N}_1$ and $\mathcal{N}_2$ are $\mathcal{K}_1$-generic, there is $f : \mathcal{N}_1 \to \mathcal{N}_2$ a $\tau_1$-isomorphism. Then $\sigma_0 = f^{-1} \circ f_0$ a permutation of $Q(\mathcal{N}_1)$. Since $Q(\mathcal{N}_1)$

is a set of absolute indiscernibles, there is $\sigma \supset \sigma_0$ an automorphism of the $\tau_1$-reduct of $\mathcal{N}_1$. But then $f \circ \sigma$ is a $\tau_1$-isomorphism between $\mathcal{N}_1$ and $\mathcal{N}_2$. But $f \circ \sigma \supset \sigma_0$, so this is the desired isomorphism between $\mathcal{N}_1$ and $\mathcal{N}_2$

For any countable $\mathcal{A} \models \psi$, there is a countable $\mathcal{N} \models \Psi$ with $Q(\mathcal{N}) \cong \mathcal{A}$. Since $\psi$ has uncountably many non-isomorphic countable models but no perfect set of non-isomorphic models the same is true of $\Psi$. But if $\mathcal{M} \models \Psi$, $|Q(\mathcal{M})| \leq |P(\mathcal{M})| \leq \aleph_1$. Thus $\Psi$ is a Vaught counterexample with no models of size $\aleph_2$. $\square$

Hjorth's original proof was somewhat different. He used a slight variant of $\mathcal{K}_1$ and showed that the automorphism group of the $\mathcal{K}_1$-generic countable model had a closed subgroup with a surjective homomorphism onto $S_\infty$. He was then able to construct the counterexample with no model of size $\aleph_2$ using some basic facts about the dynamics of Polish group actions. In [4] the authors notice that Hjorth's construction can be easily adapted to get a model with a set of absolute indiscernibles and from this the main result quickly follows.

**Exercise 5.37** Show that the example $\Psi$ has models of size $\aleph_1$.

We will see in §7 that every Vaught counterexample has models of size $\aleph_1$.

# Part II
# Building Uncountable Models

## 6 Elementary Chains

### 6.1 Elementary End Extensions

The failure of compactness leaves us with no general tool for building elementary extensions and elementary chains. In this section we will give a criteria which will allow us to use the Model Existence Theorem to build elementary extensions in some special circumstances. Iterating this construction will be a useful tool for building models of cardinality $\aleph_1$.

**Definition 6.1** Let $\tau = (<, \ldots)$. Let $\mathcal{M}$ be linearly ordered by $<$. We say that $\mathcal{M} \subset \mathcal{N}$ is an *end extension* if $\mathcal{M} \neq \mathcal{N}$ and $\mathcal{N} \models\ a < b$ for all $a \in M$ and $b \in \mathcal{N} \setminus \mathcal{M}$.

The following theorem of Keisler gives necessary and sufficient conditions for constructing elementary end extensions. For the purposes of the next few sections we will need to assume our fragments have a slightly strong closure property, namely

$$(\dagger) \quad \text{If } \exists \overline{v} \bigvee_{\psi \in X} \psi(\overline{v}) \in \mathbb{A},\ \text{then } \bigvee_{\psi \in X} \exists \overline{v}\ \psi(\overline{v}) \in \mathbb{A}.$$

For the next sections we let $\exists^* x\ \phi$ abbreviate $\forall y \exists x > y\ \phi$ and let $\forall^* x \phi$ denote the dual quantifier, $\exists y \forall x > y\ \phi$.

**Theorem 6.2** *Let $\tau = \{<, \ldots\}$ be a countable vocabulary, let $\mathcal{M}$ be a countable $\tau$-structure such that $<$ is a linear order, and let $\mathbb{A}$ be a countable fragment of $\mathcal{L}_{\omega_1,\omega}(\tau)$. The following are equivalent:*

1. *$\mathcal{M}$ has a countable $\mathbb{A}$-elementary end extension;*

2. *Each of the following is true in $\mathcal{M}$.*

   (a) $\forall x \exists y\ x < y$

   (b) $\forall \overline{w} \left[\exists^* x\ \bigvee_{n\in\omega} \phi_n(x, \overline{w}) \rightarrow \bigvee_{n\in\omega} \exists^* x\ \phi_n(x, \overline{w})\right]$, *for* $\bigvee_{n\in\omega} \phi_n \in \mathbb{A}$.

*(c)* $\forall \overline{w}\,[\exists^* x \exists y\ \phi(x,y,\overline{w}) \rightarrow (\exists y \exists^* x\ \phi(x,y,\overline{w}) \vee \exists^* y \exists x\ \phi(x,y,\overline{w}))]$, *for* $\phi \in \mathbb{A}$.

**Proof** ($1 \Rightarrow 2$) Suppose $\mathcal{M} \prec_{\mathbb{A}} \mathcal{N}$ and $\mathcal{N}$ is an end extension of $\mathcal{M}$. The main idea is the following *overspill principle*: if $b \in \mathcal{N} \setminus \mathcal{M}$, $\overline{a}, c \in \mathcal{M}$ and $\mathcal{N} \models \phi(b,\overline{a})$, then $\mathcal{M} \models \exists x > c\ \phi(x,\overline{a})$. Thus $\mathcal{M} \models \exists^* x\ \phi(x,\overline{a})$.

(a) If $a \in \mathcal{M}$ and $b \in \mathcal{N} \setminus \mathcal{M}$, $a < b$. Since $\mathcal{M} \prec \mathcal{N}$, there is $c \in \mathcal{M}$ such that $a < c$.

(b) If $\mathcal{M} \models \exists^* x \bigvee \phi_n(x,\overline{a})$, then the same is true in $\mathcal{N}$ and we can find $b \in \mathcal{N}\setminus\mathcal{M}$ and $n \in \omega$ such that $\mathcal{N} \models \phi_n(b,\overline{a})$. By overspill $\mathcal{M} \models \exists^* x \phi_n(x,\overline{a})$.

(c) If $\mathcal{M} \models \exists^* x \exists y\ \phi(x,y)$, then $\mathcal{N} \models \exists^* x \exists y\ \phi(x,y)$. Thus there is $b \in \mathcal{N} \setminus \mathcal{M}$ such that $\mathcal{N} \models \exists y\ \phi(b,y)$. Suppose $\mathcal{N} \models \phi(b,c)$. If $c \in \mathcal{M}$, then, by overspill, $\mathcal{M} \models \exists^* x\ \phi(x,c)$. If $c \notin \mathcal{M}$, then, by overspill, $\mathcal{M} \models \exists^* y \exists x\ \phi(x,y)$.

($2 \Rightarrow 1$) We expand the vocabulary $\tau$ to $\tau'$ by adding a constant symbol for each element of $\mathcal{M}$ and a new constant symbol $d$. Let $\mathbb{A}'$ be the smallest fragment containing $\mathbb{A}$ and the new constant symbols. Let

$$T = \{\chi(d,\overline{m}) : \mathcal{M} \models \forall^* x \chi(x,\overline{m}) \text{ for } \chi \in \mathbb{A}, \overline{m} \in \mathcal{M}\}.$$

Note that if $\mathcal{N} \models T$ then $\mathcal{M} \prec_{\mathbb{A}} \mathcal{N}$. For $a \in \mathcal{M}$ let $\Theta_a$ be the sentence

$$\forall y \bigvee_{b \in M} (y = b \vee a < y).$$

If $\mathcal{N} \models T + \bigwedge_{a \in \mathcal{M}} \Theta_a$, then $\mathcal{M} \prec_{\mathbb{A}} \mathcal{N}$ and $\mathcal{N}$ is an end extension of $\mathcal{M}$. We will build $\mathcal{N}$ using the Omitting Types Theorem 4.9. We must show that $T$ is satisfiable and that if $\exists y\ \phi(y) \in \mathbb{A}'$, $T \cup \exists y\ \phi(y)$ is satisfiable and $a \in \mathcal{M}$, then there is $b \in M$ such that $T + \exists y\ [\phi(y) \wedge (y = b \vee a < y)]$ is satisfiable.

**claim** $T$ is satisfiable.

Let $\tau''$ be obtained by adding $C$ a countable set of new constant symbols and let $\mathbb{A}''$ be the smallest fragment containing $\mathbb{A}'$ and the new constants. Let $\Sigma$ be all finite sets $\sigma$ of $\mathbb{A}''$-sentences such that if $\theta_\sigma(c_1,\ldots,c_m,d)$ is the conjunction of all sentences in $\sigma$, then $\mathcal{M} \models \exists^* x \exists y_1 \ldots \exists y_m \theta_\sigma(y_1,\ldots,y_m,x)$.

We claim that $\Sigma$ is a consistency property. Note that, by condition (a), $\exists^* x\ x = x$. Thus $\Sigma$ is nonempty. As usual we will only check C4). Suppose $\bigvee_{\psi \in X} \psi(c_1,\ldots,c_m,d) \in \sigma \in \Sigma$. Then

$$\mathcal{M} \models \exists^* x \exists v_1 \ldots \exists v_m\ \theta_\sigma(v_1,\ldots,v_m,x).$$

But then

$$\mathcal{M} \models \exists^* x \bigvee_{\psi \in X} \exists v_1 \ldots \exists v_m \; (\psi(\overline{v}, x) \wedge \theta_\sigma(\overline{v}, x))$$

and, by condition (b), there is $\psi \in X$ such that

$$\mathcal{M} \models \exists^* x \exists v_1 \ldots \exists v_m \; (\psi(\overline{v}, x) \wedge \theta_\sigma(\overline{v}, x)).$$

Thus $\sigma \cup \{\psi(\overline{c}, d)\} \in \Sigma$.

To show that $T$ is satisfiable, we will show that if $\psi \in T$ and $\sigma \in \Sigma$, then $\sigma \cup \{\psi\} \in \Sigma$. The Extended Model Existence Theorem 4.8 will then guarantee the existence of $\mathcal{M} \models T$.

Suppose $\mathcal{M} \models \exists y \forall x > y \; \chi(x)$ and $\sigma \in \Sigma$. Then

$$\mathcal{M} \models \exists^* x (\chi(x) \wedge \exists \overline{v} \; \theta_\sigma(\overline{v}, x)).$$

Hence $\sigma \cup \{\chi(d)\} \in \Sigma$ as desired.

Thus $T$ is satisfiable.

**claim** For $\overline{m} \in \mathcal{M}$, $\phi(x, \overline{y}) \in \mathbb{A}$, $T + \phi(d, \overline{m})$ is satisfiable if and only if $\mathcal{M} \models \exists^* x \; \phi(x, \overline{m})$.

($\Rightarrow$) Since $\neg\phi(d, \overline{m}) \notin T$,

$$\mathcal{M} \not\models \exists x \forall y > x \neg\phi(y, \overline{m}).$$

Thus $M \models \exists^* y \; \phi(y, \overline{m})$.

($\Leftarrow$) We use the consistency property $\Sigma$. If $\mathcal{M} \models \exists^* x \; \phi(x, \overline{m})$, then $\{\phi(d, \overline{m})\} \in \Sigma$. By the arguments above and the Extended Model Existence Theorem 4.8, $T + \psi(d, \overline{m})$ is satisfiable.

**claim** If $\exists y \; \phi(y) \in \mathbb{A}'$, $T \cup \exists y \; \phi(y)$ is satisfiable and $a \in \mathcal{M}$, then there is $b \in M$ such that $T + \exists y \; [\phi(y) \wedge (y = b \vee a < y)]$ is satisfiable.

Suppose $T + \exists y \; \psi(y, d, \overline{m})$ is satisfiable and $a \in \mathcal{M}$. By the claim above, $\mathcal{M} \models \exists^* x \exists y \psi(y, x, \overline{m})$. Thus

$$\mathcal{M} \models \exists^* x \; \left[(\exists y \; (\psi(y, x, \overline{m}) \wedge a < y)) \vee (\exists y \; (\psi(y, x, \overline{m}) \wedge y \leq a))\right].$$

By condition b) we are in at least one of two cases.

<u>case 1</u> $\mathcal{M} \models \exists^* x \exists y (\psi(y, x, \overline{m}) \wedge a < y)$

By the previous claim $T + \exists y (\psi(y, d, \overline{m}) \wedge a < y)$ is satisfiable, so certainly

$$T + \exists y (\psi(y, d, \overline{m}) \wedge (y = b) \vee a < y)$$

is satisfiable for any $b \in \mathcal{M}$.

<u>case 2</u> $\mathcal{M} \models \exists^* x \exists y\ (\psi(y, x, \overline{m}) \wedge y \leq a)$

By now apply condition c). Clearly

$$\mathcal{M} \not\models \exists^* y \exists x\ ((\psi(y, x, \overline{m}) \wedge y \leq a).$$

Thus by c) there is $b \in \mathcal{M}$ such that

$$\mathcal{M} \models \exists^* x\ (\psi(b, x, \overline{m}) \wedge b \leq a).$$

But the $T + \exists y(\psi(y, d, \overline{m} \wedge y = b)$ is satisfiable, so

$$T + \exists y(\psi(y, d, \overline{m}) \wedge (y = b) \vee a < y)$$

is satisfiable for some $b \in \mathcal{M}$.

We have verified the conditions of the Omitting Types Theorem. Thus there is

$$\mathcal{N} \models T \cup \bigwedge_{a \in \mathcal{M}} \forall y \bigvee_{b \in \mathcal{M}} (y = b \vee a < y).$$

Then $\mathcal{N}$ is an $\mathbb{A}$-elementary end extension of $\mathcal{M}$. □

With mild extra assumptions on $\mathbb{A}$ we can iterated this method to build elementary extensions of size $\aleph_1$. In general if $\exists^* x \bigvee_{\theta \in X} \in \mathbb{A}$, we don't know that $\bigvee_{\theta \in X} \exists^* x \theta(x) \in \mathbb{A}$. We suppose that $\mathbb{A}$ has this additional closure property.

**Corollary 6.3** *Let $\mathcal{M}$ be a countable $\tau$-structure and let $\mathbb{A}$ be a countable fragment of $\mathcal{L}_{\omega_1,\omega}(\tau)$ with the above additional closure property. If $\mathcal{M}$ has an $\mathbb{A}$-elementary end extension, then $\mathcal{M}$ has an $\mathbb{A}$-elementary end extension of cardinality $\aleph_1$.*

**Proof** We can build an $\mathbb{A}$-elementary chain of countable elementary end extensions $(\mathcal{M}_\alpha : \alpha < \omega_1)$. Because $\mathbb{A}$ satisfies condition (†), the conditions (a)–(c) of Theorem 6.2 are $\mathbb{A}$-elementary, so given $\mathcal{M} \prec_{\mathbb{A}} \mathcal{M}_\alpha$ we can build an $\mathbb{A}$-elementary end extension of $\mathcal{M}_\alpha$. $\mathcal{N} = \bigcup_{\alpha < \omega_1} \mathcal{M}_\alpha$ is the desired $\mathbb{A}$-elementary end extension of cardinality $\aleph_1$. □

Keisler's Theorem works in a slightly more general setting than linear orders. Let $\tau = \{R, \ldots\}$ be a countable vocabulary, where $R$ is a binary relation symbol.

**Definition 6.4** We say that $\mathcal{N}$ is an end extension of $\mathcal{M}$ if whenever $a \in \mathcal{N}$, $b \in \mathcal{M}$ and $aRb$, then $a \in \mathcal{M}$.

We say that $\mathcal{N}$ is a *strong end extension* if $\mathcal{M} \neq \mathcal{N}$ and $\mathcal{N} \models aRb \wedge \neg bRa$ for all $a \in \mathcal{M}$ and $b \in \mathcal{N} \setminus \mathcal{M}$.

If $R$ is a linear ordering, then any end extension is strong.

**Exercise 6.5** Modify the proof of Theorem 6.2 to show that if $\mathcal{M}$ is a countable $\tau$-structure such that $R$ is transitive and irreflexive, then $\mathcal{M}$ has a strong $\mathbb{A}$-elementary end extension if and only if a′),b) c) hold where

(a′) is $\forall x \forall y \exists z\ (xRz \wedge yRz)$

(b) and (c) are as above but we interpret $\exists^* x\ \phi$ as $\forall y \exists x\ (yRx \wedge \phi)$.

**Exercise 6.6** Suppose $\mathcal{M} \models$ ZFC. Recall that we have a definable rank function $\rho : \mathcal{M} \to On^{\mathcal{M}}$ where $\rho(x)$ is the least ordinal $\alpha$ of $\mathcal{M}$ such that $x \in \mathbb{V}_{\alpha+1} \setminus \mathbb{V}_\alpha$. Suppose $\mathcal{M} \subset \mathcal{N}$. We say that $\mathcal{N}$ is a *top extension* if $\rho(m) < \rho(n)$ for all $m \in \mathcal{M}$ and $n \in \mathcal{N} \setminus \mathcal{M}$. Prove that every countable $\mathcal{M} \models$ ZFC has a top extension of cardinality $\aleph_1$. [Hint: Let $R(a, b)$ be the relation $\rho(a) < \rho(b)$ and show the version of Keisler's Theorem in the Exercise above applies.]

### Vaught's two cardinal property

Let $\tau = \{U, \ldots\}$ be countable where $U$ is a unary predicate.

**Definition 6.7** We say that $\mathcal{M}$ is a $(\kappa, \lambda)$-model if $|\mathcal{M}| = \kappa$ and $|U(\mathcal{M})| = \lambda$.

In first order logic, Vaught's Two Cardinal Theorem says that if a countable theory $T$ has a $(\kappa, \lambda)$-model where $\kappa > \lambda \geq \aleph_0$. Then $T$ has an $(\aleph_1, \aleph_0)$-model. The usual proofs break down as they all require some remnant of compactness. Keisler found the following proof using Theorem 6.2.

**Theorem 6.8** *If $\phi \in \mathcal{L}_{\omega_1,\omega}$ has a $(\kappa, \lambda)$-model where $\kappa > \lambda \geq \aleph_0$, then $\phi$ has an $(\aleph_1, \aleph_0)$-model.*

**Proof** Suppose $\mathcal{M}_0$ is a $(\kappa, \lambda)$-model of $\phi$. Let $\mathbb{A}_0$ be a countable fragment of $\mathcal{L}_{\omega_1,\omega}$ with $\phi \in \mathbb{A}$. By Löwenheim–Skolem we can find $\mathcal{M}_1 \prec_{\mathbb{A}_0} \mathcal{M}_0$ a $(\lambda^+, \lambda)$-model. Let $<$ be a well ordering of $\mathcal{M}_1$ of order type $\lambda^+$ and let $\tau^* = \tau \cup \{<\}$. Note that, by the reguarlity of $\lambda^+$ all of the following are true in $\mathcal{M}_1$:

(a) $\neg\exists^* x\ U(x)$;

(b) $\forall\overline{v}\ [\exists^* x\ \bigvee_{\psi\in X}\psi(x,\overline{v})\to\bigvee_{\psi\in X}\exists^* x\ \psi(x,\overline{v})]$, for all $\bigvee_{\psi\in X}\psi\in\mathcal{L}_{\omega_1,\omega}$;

(c) $\forall\overline{v}\ [\exists^* x\exists y\ \psi(x,y,\overline{v})\to(\exists^* y\exists x\ \psi(x,y,\overline{v})\vee\exists y\exists^* x\ \psi(x,y,\overline{v}))]$, for all $\psi\in\mathcal{L}_{\omega_1,\omega}$.

Let $\mathbb{A}$ be a countable fragment of $\mathcal{L}_{\omega_1,\omega}(\tau^*)$ such that $\phi\in\mathbb{A}$ and if $\bigvee_{\psi\in X}\psi\in\mathbb{A}$, then $\bigvee_{\psi\in X}\exists^* x\psi\in\mathbb{A}$.

Let $\mathcal{M}\prec_{\mathbb{A}}\mathcal{M}_1$ be countable. Then the three conditions above are true in $\mathcal{M}$ for $\mathbb{A}$-formulas. In particular, there is $b\in\mathcal{M}$ such that $\mathcal{M}\models\forall x\ (U(x)\to x<b)$. By Corollary 6.3 there is $\mathcal{M}\prec_{\mathbb{A}}\mathcal{N}$ such that $\mathcal{N}$ is an end extension and $|\mathcal{N}|=\aleph_1$. Since $\mathcal{N}\models\forall x\ (U(x)\to x<b)$, $U(\mathcal{N})=U(\mathcal{M})$ is countable. □

### Undefinability of well order revisited

We can use end extensions to prove an uncountable version of the undefinability of well order (Theorem 4.27).

**Corollary 6.9** *Let $\tau$ be a countable vocabulary containing a binary relation symbol $<$, let $\phi\in\mathcal{L}_{\omega_1,\omega}(\tau)$, and suppose $\mathcal{M}=(M,<,\ldots)\models\phi$ where $<$ has order type $\omega_1$. Then there is $\mathcal{N}\models\phi$ of cardinality $\aleph_1$ and an order preserving embedding of $(\mathbb{Q},<)$ into $\mathcal{N}$.*

**Proof** Let $\mathbb{A}$ be a countable fragment of $\mathcal{L}_{\omega_1,\omega}(\tau)$ containing $\phi$ with the closure property above. We expand the signature by adding a new unary predicate $P$. Let $\Psi$ assert that $\mathcal{M}$ is an $\mathbb{A}$-elementary end extension of $P(\mathcal{M})$. For any $\alpha<\omega_1$ we can find $P_\alpha$ such that

$$(\mathcal{M},P_\alpha)\models\phi\wedge\Psi$$

and the order type of $P_\alpha$ is at least $\alpha$.

By the undefinability of well order (Theorem 4.27), there is a countable model $(\mathcal{N},P(\mathcal{N}))\models\phi\wedge\Psi$ where $(\mathbb{Q},<)$ can be embedded into $(\mathcal{N},<)$. Since $P(\mathcal{N})\prec_{\mathbb{A}}\mathcal{N}$ and $P(\mathcal{N})$ has an $\mathbb{A}$-elementary end extension, conditions (a)–(c) of Theorem 6.2 hold in $P(\mathcal{N})$ and hence in $\mathcal{N}$. Thus $\mathcal{N}$ has an $\mathbb{A}$-elementary end extension, which, by Corollary 6.3 we may assume has cardinality $\aleph_1$. □

## 6.2 Omitting Types in End Extensions

We continue the study of end extensions begun in §6.1. Keisler also characterized when a countable model has an uncountable elementary end extension omitting a type.

Fix $\tau = \{<, \ldots\}$ be a countable vocabulary and let $\mathbb{A}$ be a countable fragment of $\mathcal{L}_{\omega_1,\omega}(\tau)$. Suppose for $n \in \omega$ we have $\Theta_n(v_1, \ldots, v_{m_n})$ be a set of $\mathbb{A}$-formulas. We will be looking for models of

$$\chi = \bigwedge_{n\in\omega} \forall x_1 \ldots \forall x_{m_n} \bigvee_{\theta\in\Theta_n} \theta(\overline{x}).$$

Note that the formulas $\bigvee_{\theta\in\Theta_n} \theta$ and $\chi$ need not be in $\mathbb{A}$.

Recall that an ordering $<$ is $\omega_1$-*like* if it is uncountable but $\{x : x < a\}$ is countable for all $a$.

**Theorem 6.10** *Suppose $\mathcal{M}$ is a countable $\tau$-structure. The following are equivalent*

1. *$\mathcal{M}$ has an $\omega_1$-like $\mathbb{A}$-elementary end extension $\mathcal{N}$ where $\chi$ holds.*

2. *$\mathcal{M}$ satisfies the conditions from Theorem 6.2 and, in addition, for every $n \in \omega$, $\overline{a} \in \mathcal{M}$ if*

$$\mathcal{M} \models S\overline{y}\exists x_1 \ldots \exists x_{m_n}\psi(\overline{x}, \overline{y}, \overline{a})$$

*then there is $\theta \in \Theta_n$ such that*

$$\mathcal{M} \models S\overline{y}\exists x_1 \ldots \exists x_{m_n}\psi(\overline{x}, \overline{y}, \overline{a}) \wedge \theta(\overline{x})$$

*where $S\overline{y}$ is a string of quantifiers of the form $\exists y_i$ and $\exists^* y_j$.*

**Proof** (1. $\Rightarrow$ 2.) Since $\mathcal{N}$ is an $\mathbb{A}$-elementary end extension, the conditions from Theorem 6.2 are satisfied in $\mathcal{M}$. Fix $n$ and suppose

$$\mathcal{M} \models S\overline{y}\exists x_1 \ldots \exists x_{m_n}\ \psi(\overline{x}, \overline{y})$$

(we suppress the parameters from $\mathcal{M}$ as they play no role). Since $\mathcal{N}$ is $\mathbb{A}$-elementary and $\mathcal{N} \models \chi$,

$$\mathcal{N} \models S\overline{y}\exists x_1 \ldots \exists x_{m_n}\ \psi(\overline{x}, \overline{x}y) \wedge \bigvee_{\theta\in\Theta_n} \theta(\overline{x}).$$

Note that for any countable disjunction $\bigvee_{i=1}^{\infty} \phi_i(\overline{v}, w)$

$$\text{(i) } \mathcal{N} \models \forall \overline{v} \ \left[ \exists w \ \bigvee_{i=1}^{\infty} \phi_i(\overline{v}, w) \leftrightarrow \bigvee_{i=1}^{\infty} \exists w \ \phi_i(\overline{v}, w) \right]$$

and

$$\text{(ii) } \mathcal{N} \models \forall \overline{v} \ \left[ \exists^* w \ \bigvee_{i=1}^{\infty} \phi_i(\overline{v}, w) \leftrightarrow \bigvee_{i=1}^{\infty} \exists^* w \ \phi_i(\overline{v}, w) \right].$$

(i) is true in any model. (ii) is true in any $\omega_1$-like model because $\exists^* x \ \phi(x)$ is equivalent to "there are uncountably many $x$ such that $\phi(x)$". It now follows by induction that

$$\mathcal{N} \models \bigvee_{\theta \in \Theta_n} S\overline{y} \exists \overline{x} \ \psi(\overline{x}, \overline{y}) \wedge \theta(\overline{x})$$

Thus

$$\mathcal{N} \models S\overline{y} \exists \overline{x} \ \psi(\overline{y}, \overline{x}) \wedge \theta(\overline{x})$$

for some $\theta \in \Theta_n$ and, since $\mathcal{M} \prec_{\mathbb{A}} \mathcal{N}$ so does $\mathcal{M}$.

(2. $\Rightarrow$ 1.) Our basic strategy will be to show that if $\mathcal{M}$ is a countable and satisfies Condition 2, then it has a countable end extension $\mathcal{M} \prec_{\mathbb{A}} \mathcal{N}$ that also satisfies condition 2. This will allow us to start building an $\mathbb{A}$-elementary chain of end extensions. It is easy to see that the union of an elementary chain of models satisfying Condition 2 will also satisfy Condition 2. Thus we can interate this process to build an $\mathbb{A}$-elementary chain of length $\omega_1$. The union of this chain will be an $\omega_1$-like model satisfying Condition 2.

If $\mathcal{N}$ is any model satisfying Condition 2, $n \in \omega$ and $a_1, \ldots, a_{m_n} \in \mathcal{N}$, let $\psi(\overline{x})$ be

$$x_1 = a_1 \wedge \ldots \wedge x_{m_n} = a_{m_n}.$$

Then for some $\theta \in \Theta_n$,

$$\mathcal{N} \models \exists \overline{x} \ (\psi(\overline{x}) \wedge \theta(\overline{x})),$$

i.e., $\mathcal{N} \models \theta(\overline{a})$. Thus $\mathcal{N} \models \chi$.

Thus it suffices to show that a countable model satisfying Condition 2 has an $\mathbb{A}$-elementary end extension satisfying Condition 2. We will do this by weaving an additional omitting types argument into the proof of Theorem 6.2.

Recall that in the proof of Theorem 6.2 we expanded the language by constants for the elements of $\mathcal{M}$ and an additional constant $d$. We introduced the theory

$$T = \{\phi(\overline{a}, d) : \phi \in \mathbb{A}, \overline{a} \in \mathcal{M} \text{ and } \mathcal{M} \models \exists^* x\ \phi(\overline{a}, x)\},$$

showed $T$ is satisfiable and used the Omitting Types Theorem to find a countable model of

$$T + \bigwedge_{a \in M} \forall y \bigvee_{b \in M} (y = b \vee a \leq y).$$

We will recast Condition 2 as an omitting types problem and show how to find a model of $T$ solving it. Finding an end extension satisfying Condition 2, will be accomplished by solving these two omitting types problems simultaneously. But once we know we can solve each of them, the Omitting Types Theorem tells us we can solve them both at once.

Fix $n$. We let $\overline{x} = x_1, \ldots, x_{m_n}$. Let $\psi(\overline{x}, \overline{y}, \overline{v})$ be an $\mathbb{A}$-formula with parameters from $\mathcal{M}$. We will show how to use the Omitting Types Theorem to build a model of

$$T + \forall \overline{v} \left[ S\overline{y}\exists\overline{x}\ \psi(\overline{x}, \overline{y}, \overline{v}) \rightarrow \bigvee_{\theta \in \Theta_n} S\overline{y}\exists\overline{x}\ (\psi(\overline{x}, \overline{y}, \overline{v}) \wedge \theta(\overline{x})) \right].$$

Doing this for all $n$ and all $\psi$ will give the desired model of Condition 2.

To apply the Omitting Type Theorem we need to know that if we have $\phi(\overline{v}, d)$ an $\mathbb{A}$-formula with parameters from $\mathcal{M}$ and

$$T + \exists \overline{v}\ \phi(\overline{v}, d)$$

is satisfiable, then

$$T + \exists \overline{v}\ [\phi(\overline{v}, d) \wedge (S\overline{y}\exists\overline{x}\ \psi(\overline{x}, \overline{y}, \overline{v}) \rightarrow S\overline{y}\exists\overline{x}\ (\psi(\overline{x}, \overline{y}, \overline{v}) \wedge \theta(\overline{x})))]$$

is satisfiable for some $\theta \in \Theta_n$.

Suppose $T + \exists \overline{v}\ \phi(\overline{v}, d)$ is satisfiable. We will show that either

(a) $T + \exists \overline{v}\ [\phi(\overline{v}, d) \wedge \neg S\overline{y}\exists\overline{x}\ \psi(\overline{x}, \overline{y}, \overline{v})]$ is satisfiable

or

(b) $T + \exists \overline{v}\ [\phi(\overline{v}, d) \wedge S\overline{y}\exists\overline{x}\ (\psi(\overline{x}, \overline{y}, \overline{v}) \wedge \theta(\overline{x}))]$ is satisfiable for some $\theta \in \Theta_n$.

Recall from the proof of Theorem 6.2, that

$$T + \pi(d) \text{ is satisfiable } \Leftrightarrow \mathcal{M} \models \exists^* x\ \pi(x),$$

for any $\mathbb{A}$-formula with parameters from $\mathcal{M}$.

Suppose $T + \exists \overline{v}\ \phi(\overline{v}, d)$ is satisfiable, but (a) fails. Then

$$\mathcal{M} \models \exists^* w \exists \overline{v}\ \phi(\overline{v}, w)$$

and

$$T \models \forall \overline{v}\ [\phi(\overline{v}, d) \to S\overline{y}\overline{x}\psi(\overline{x}, \overline{y}, \overline{v})].$$

So

$$\mathcal{M} \models \exists^* w \exists \overline{v} S\overline{y} \exists \overline{x}\ (\phi(\overline{v}, w) \wedge \psi(\overline{x}, \overline{y}, \overline{v}))$$

and, by Condition 2, there is $\theta \in \Theta_n$ such that

$$\mathcal{M} \models \exists^* w \exists \overline{v} S\overline{y} \exists \overline{x}\ (\phi(\overline{v}, w) \wedge \psi(\overline{x}, \overline{y}, \overline{v}) \wedge \theta(\overline{x})).$$

But then

$$T + \exists \overline{v} S\overline{y} \exists \overline{x}\ [\phi(\overline{v}, d) \wedge \psi(\overline{x}, \overline{y}, \overline{v}) \wedge \theta(\overline{x})]$$

is satisfiable and (b) holds.

The Omitting Types Theorem now allows us to construct $\mathcal{N}$ a countable $\mathbb{A}$-elementary end extension of $\mathcal{M}$ where Condition 2 holds. $\square$

## 6.3 Uncountable Models Realizing Few Types

In this section we will collect a number of results where build uncountable models realizing countably few types. Results of this type can be very useful when studying both categoricity phenomena and Vaught counterexamples. We have already seen one example of such a theorem in Theorem 5.11 where we showed that if $\phi$ has arbitrarily large models, then it has arbitrarily large models realizing only countably many $\mathcal{L}_{\omega_1,\omega}$-types.

**Definition 6.11** If $\mathbb{A}$ is a fragment of $\mathcal{L}_{\omega_1,\omega}$ we say that $\mathcal{M}$ is $\mathbb{A}$-*small* if $\mathcal{M}$ realizes only countably many $\mathbb{A}$-types and we say that $\mathcal{M}$ is *small* if it is $\mathcal{L}_{\omega_1,\omega}$-small.

Recall that Exercise 2.34 shows that $\mathcal{M}$ is small if and only if there is a countable $\mathcal{N}$ with $\mathcal{M} \equiv_{\infty,\omega} \mathcal{N}$.

**Theorem 6.12** *If $\phi$ is an $\mathcal{L}_{\omega_1,\omega}$-sentence which has an uncountable model that is $\mathbb{A}$-small for every countable fragment $\mathbb{A}$, then $\psi$ has a small model of cardinality $\aleph_1$.*

**Proof** Let $\mathcal{M} \models \phi$ of cardinality $\aleph_1$ be $\mathbb{A}$-small for all countable fragments of $\mathcal{L}_{\omega_1,\omega}$. Add a binary relation symbol to the signature and let $<$ be a well ordering of $\mathcal{M}$ of order type $\omega_1$. For each $n \geq 1$ we also add $2n+1$-ary relation symbols $E_n(x, \overline{y}, \overline{z})$ and $n+1$-ary functions $f_n(x, \overline{y})$ and a constant 0. We let $\Psi$ be an $\mathcal{L}_{\omega_1,\omega}$-sentence asserting:

i) 0 is the $<$ least element and $E_n(0, \overline{a}, \overline{b})$ if and only if $\overline{a}$ and $\overline{b}$ satisfy the same quantifier free formulas in the original signature;

ii) if $\alpha < \beta$ and $E_n(\beta, \overline{a}, \overline{b})$, then $E_n(\alpha, \overline{a}, \overline{b})$;

iii) every $\alpha$ has a $<$-successor $\beta$ and $E_n(\beta, \overline{a}, \overline{b})$ if and only if for all $c$ there is $d$ such that $E_{n+1}(\alpha, \overline{a}, c, \overline{b}, d)$ and for all $d$ there is $c$ such that $E_{n+1}(\alpha, \overline{a}, c, \overline{b}, d)$;

iv) if $\beta$ is a limit, then $E(\beta, \overline{a}, \overline{b})$ if and only if $E(\alpha, \overline{a}, \overline{b})$ for all $\alpha < \beta$;

v) $<$ has an initial segment of order type $\omega$ and each $f_n$ maps $M^{n+1}$ into this segment;

vi) $E_n(\beta, \overline{a}, \overline{b})$ if and only if $f_n(\beta, \overline{a}) = f_n(\beta, \overline{b})$.

i)–iv) assert that $E_n$ gives the usual Scott analysis in $\mathcal{M}$ and v) and vi) assert that each $E_n(\beta, \cdot, \cdot)$ has only countably many equivalence classes.

By Corollary 6.9 there is an uncountable $\mathcal{N} \models \phi \wedge \Psi$ embedding $(\mathbb{Q}, <)$. In particular there is an infinite descending chain $d_0 > d_1 > \ldots$. Define $\overline{a} \sim^* \overline{b}$ if and only if $E_n(d_i, \overline{a}, \overline{b})$ for some $i$. Suppose $\theta(\overline{v})$ is an $\mathcal{L}_{\infty,\omega}$-formula. We prove by induction on subformulas that if $\overline{a} \sim^* \overline{b}$, then $\mathcal{N} \models \theta(\overline{a}) \leftrightarrow \theta(\overline{b})$. For quantifier free formulas this is true by i) and ii). By induction this is clear for finitary and infintiary Boolean connectives. Suppose $\theta(\overline{v})$ is $\exists y\ \chi(\overline{v}, y)$. If $\mathcal{N} \models \chi(\overline{a}, c)$, then by iii) there is $d$ such that $\mathcal{N} \models \chi(\overline{b}, d)$. Thus any $\sim^*$-equivalent elements realize the same type. By iv) and v) there are only countably many $E_n(d_0, \cdot, \cdot)$-classes and, hence, only countably many $\sim^*$-classes. Thus $\mathcal{N}$ is small. □

The existence of a reasonably large uncountable model is enough to imply that there is an uncountable model realizing few $\mathbb{A}$-types for a countable fragment $\mathbb{A}$.

**Theorem 6.13** *Let $\mathbb{A}$ be a countable fragment of $\mathcal{L}_{\omega_1,\omega}$. If $\phi$ has a model of cardinality $(2^{\aleph_0})^+$, then $\mathbb{A}$ has a model of cardinality $\aleph_1$ realizing only countably many $\mathbb{A}$-types.*

**Proof** Let $\mathcal{M}$ be model of $\phi$ of cardinality $\kappa > 2^{\aleph_0}$. There is $U \subset \mathcal{M}$ such that $|U| = \lambda \leq 2^{\aleph_0}$ and for all $\overline{a} \in \mathcal{M}$ there is $\overline{u} \in U$ such that $\mathrm{tp}^{\mathcal{M}}(\overline{a}, \mathbb{A}) = \mathrm{tp}^{\mathcal{M}}(\overline{u}, \mathbb{A})$. Let $\Theta$ be the sentence

$$\bigwedge_{n\in\omega} \forall a_1 \ldots \forall a_n \exists u_1 \ldots \exists u_n \left[\bigwedge_{i=1}^{n} U(u_i) \wedge \bigwedge_{\psi\in\mathbb{A}} \psi(\overline{a}) \leftrightarrow \psi(\overline{u})\right].$$

Then $(\mathcal{M}, U)$ is a $(\kappa, \lambda)$ model of $\phi \wedge \psi$ where $\kappa > \lambda$. By Theorem 6.8, there is $(\mathcal{N}, U^{\mathcal{N}})$ an $(\aleph_1, \aleph_0)$-model of $\phi \wedge \psi$. Then $\mathcal{N} \models \phi$ has cardinality $\aleph_1$ and $\mathcal{N}$ realizes countably many $\mathbb{A}$-types. $\square$

The next result is the analog of the classical theorem that an $\omega$-stable theory realizes few types over any model.

**Theorem 6.14** *Let $\mathbb{A}$ be a countable fragment. Suppose for all countable $\mathcal{M} \models \phi$ there are only countably many $\mathbb{A}$-types over $\mathcal{M}$. Then for every model $\mathcal{N}$ there are only $|\mathcal{N}|$ $\mathbb{A}$-types over $\mathcal{N}$.*

**Proof** Suppose not. Let $\mathcal{N} \models \phi$ have cardinality $\kappa$ where there are at least $\kappa^+$ $\mathbb{A}$-types over $\mathcal{N}$. For simplicity assume that there are $\kappa^+$ 1-types over $\mathcal{N}$. We can find $(\mathcal{N}_\alpha : \alpha < \kappa^+)$ such that $\mathcal{N} \prec_{\mathbb{A}} \mathcal{N}_\alpha$ and there is $a_\alpha \in \mathcal{N}_\alpha$ such that $a_\alpha$ and $a_\beta$ realize different types over $\mathcal{N}$ for $\alpha \neq \beta$. We may also assume that $\mathcal{N}_\alpha \cap \mathcal{N}_\beta = \mathcal{N}$ for $\alpha \neq \beta$. Let $X = \bigcup_{\alpha<\kappa^+} \mathcal{N}_\alpha$. Consider a structure where we have:

- two sorts, one for $X$ and one for $(\kappa^+, <)$;
- a binary relation $R \subset \kappa^+ \times X$ such that

  $$R(\alpha, x) \text{ if and only if } x \in \mathcal{N}_\alpha;$$

- a predicate for $\mathcal{N}$ and $\mathcal{N} \models \phi$;
- all of the $\tau$-structure on each $\mathcal{N}_\alpha$ and $\mathcal{N} \prec_{\mathbb{A}} \mathcal{N}_\alpha$;
- an injection function $f : \kappa^+ \to X$ such that $f(\alpha) = a_\alpha$;
- a bijection $g : \kappa^+ \to X$;

We can write down a sentence $\Theta$ such that if $(\mathcal{M}, \mathcal{N}^{\mathcal{M}}) \models \Theta$, then $\mathcal{N}^{\mathcal{M}} \models \phi$ and there are at least $|\mathcal{M}|$-many $\mathbb{A}$-types over $\mathcal{N}^{\mathcal{M}}$. Since $\Theta$ has a $(\kappa^+, \kappa)$-model, by Theorem 6.8, $\Theta$ has an $(\aleph_1, \aleph_0)$-model. But then there is a countable model $\mathcal{N}$ with uncountably many $\mathbb{A}$-types over $\mathcal{N}$. $\square$

**Exercise 6.15** Let $\mathbb{A}$ be a countable fragment. Suppose for all $\mathcal{M} \models \phi$ and $A \subseteq \mathcal{M}$ countable there are only countably many $\mathbb{A}$-types over $A$. Then for all $\mathcal{M} \models \phi$ and all infinite $B \subseteq \mathcal{M}$ there are only $|B|$ $\mathbb{A}$-types over $B$.

**Theorem 6.16** *Let $\mathbb{A}$ be a countable fragment and suppose $\phi \in \mathbb{A}$ has at least one uncountable model. Then the set of $\mathbb{A}$-types realized in all uncountable models of $\phi$ is countable. Indeed, if $\phi$ has $\omega_1$-like models, then the set of $\mathbb{A}$-types realized in all $\omega_1$-like models is countable.*

**Proof** Suppose $\mathcal{N}$ is a model of cardinality $\aleph_1$. Let $<$ be a linear order of $\mathcal{M}$ of order type $\omega_1$ and let $\widehat{\mathbb{A}}$ be a countable fragment of $\mathcal{L}_{\omega_1,\omega}(\tau \cup \{<\}$ containing $\mathbb{A}$. Let $\mathcal{M}$ be a countable $\widehat{\mathbb{A}}$-elementary initial segment of $\mathcal{N}$.

Let $X$ be the set of all $\mathbb{A}$-types realized in every uncountable model of $\phi$. Then for $p \in X$ we can not find an $\omega_1$-like end extension of $\mathcal{M}$ omitting $p$. By Theorem 6.10 there is a formula $\psi_p$ with parameters from $\mathcal{M}$ such that

$$\mathcal{M} \models S\overline{y}\exists\overline{x}\ \psi_p(\overline{x}, \overline{y})$$

but for all $\gamma(\overline{x}) \in p$,

$$\mathcal{M} \models \neg S\overline{y}\exists\overline{x}\ (\psi_p(\overline{x}, \overline{y}) \wedge \gamma(\overline{x})).$$

We claim that the map $p \mapsto \psi_p$ is injective.

Suppose for contradiction that $p, q \in X$, $p \neq q$ and $\psi_p = \psi_q$. Let $\psi$ denote $\psi_p = \psi_q$. Suppose $\gamma(\overline{x}) \in p$ and $\neg\gamma(\overline{x}) \in q$. Then

$$\mathcal{M} \models S\overline{y}\exists\overline{x}\ \psi(\overline{x}, \overline{y})$$

but

$$\mathcal{M} \models \neg S\overline{y}\exists\overline{x}\ (\psi(\overline{x}, \overline{y}) \wedge \neg\gamma(\overline{x})) \text{ and } \mathcal{M} \models \neg S\overline{y}\exists\overline{x}\ (\psi(\overline{x}, \overline{y}) \wedge \gamma(\overline{x})).$$

Since

$$\mathcal{M} \models S\overline{y}\exists\overline{x}\ \psi(\overline{x}, \overline{y}),$$

$$\mathcal{M} \models S\overline{y}\exists\overline{x}\ [(\psi(\overline{x}, \overline{y}) \wedge \gamma(\overline{x})) \vee (\psi(\overline{y}, \overline{x}) \wedge \neg\gamma(\overline{x}))],$$

But $\exists$ and $\exists^*$ commute with finite disjunctions, Thus

$$\mathcal{M} \models S\overline{y}\overline{x}\ (\psi(\overline{x},\overline{y}) \wedge \gamma(\overline{x})) \text{ or } \mathcal{M} \models S\overline{y}\overline{x}\ (\psi(\overline{x},\overline{y}) \wedge \neg\gamma(\overline{x}))$$

a contradiction. □

We will make use of a special case.

**Corollary 6.17** *If $\mathcal{M}$ is a countable model with an uncountable $\mathbb{A}$-elementary end extension, then the set of $\mathbb{A}$-types realized in every uncountable $\omega_1$-like $\mathbb{A}$-elementary end extensions is countable.*

**Proof** Add constants for the elements of $\mathcal{M}$ and consider the $\mathbb{A}$-theory of $\mathcal{M}$ in this language together with sentences

$$\forall v\ (v < a \rightarrow \bigvee_{b<a} b = v)$$

for all $a \in \mathcal{M}$. Now apply the theorem. □

**Theorem 6.18** *If $\mathbb{A}$ is a countable fragment $\mathcal{L}_{\omega_1,\omega}$ and $\mathcal{M} \models \phi$ realizes uncountably many $\mathbb{A}$-types, then $\phi$ has $2^{\aleph_1}$ non-isomorphic models of cardinality $\aleph_1$.*

**Proof** For notational simplicity we will assume there are uncountably many 1-types. Fix $\mathcal{M} \models \phi$ of cardinality $\aleph_1$ such that $\phi$ realizes uncountably many types. Let $\widehat{\tau} = \tau \cup \{<, U, G\}$ and expand $\mathcal{M}$ to a $\widehat{\tau}$-structure where $<$ is a linear order of $\mathcal{M}$ of order type $\omega_1$, $U$ is a subset where any two elements realize distinct $\mathbb{A}$-types, $G : U \rightarrow \mathcal{M}$ is a bijection. Let $\widehat{\mathbb{A}}$ be a countable fragment of $\mathcal{L}_{\omega_1,\omega}(\widehat{\tau})$ where we can express that any two elements of $U$ realize distinct $\mathbb{A}$-types.

We will build a tree of countable models $(\mathcal{M}_f : f \in 2^{<\omega_1}\}$ and $\mathbb{A}$-types $(p_f : f \in 2^{<\omega_1})$ such that:

i if $f \subset g$, then $\mathcal{M}_g$ is a $\widehat{\mathbb{A}}$-elementary end extension of $\mathcal{M}_f$;

ii if $f \subseteq g$, then $p_f$ is realized in $\mathcal{M}_g$;

iii $\mathcal{M}_f$ satisfies Condition 2 of Theorem 6.10 for types in $F_f = \{p_{f|\alpha^\frown i} :$ where $\alpha \in \text{dom}(f)$ and $f(\alpha) \neq i\}$, the *forbidden types* for $f$.

Choose $\mathcal{M}_\emptyset$ a countable model such that $\mathcal{M}$ is an $\widehat{\mathbb{A}}$-elementary end extension. Note that any uncountable $\widehat{\mathbb{A}}$-elementary extension of $\mathcal{M}_\emptyset$ will realize uncountably many $\mathbb{A}$-types.

Let $p_\emptyset$ be an $\mathbb{A}$-type realized in $\mathcal{M}$.

If $\mathrm{dom}(f) = \alpha$ a limit ordinal, let $\mathcal{M}_f = \bigcup_{\beta<\alpha} \mathcal{M}_\beta$. Since Condition 2) is preserved under unions of chains and $F_g \subseteq F_h$ for $g \subset h$, $\mathcal{M}_f$ satisfies *i–iii*.

Since $\mathcal{M}_f$ satisfies Condition 2), by Theorem 6.10, we can find an $\omega_1$-like $\widehat{\mathbb{A}}$-elementary end extension $\mathcal{N}_0$ of cardinality $\aleph_1$ omitting all of the types forbidden in $F_f$. Since $\mathcal{N}_0$ realizes uncountably many $\mathbb{A}$-types, by Corollary 6.17 it realizes an $\mathbb{A}$-type $p_{f^\frown 0}$ that is not realized in every $\omega_1$-like $\widehat{\mathbb{A}}$-elementary end extension of $\mathcal{M}_f$. Let $\mathcal{M}_{f^\frown 0}$ be a $\widehat{\mathbb{A}}$-elementary initial segment of $\mathcal{N}_0$ containing $\mathcal{M}_f$ and realizing $p_{f^\frown 0}$.

Let $\mathcal{N}_1$ be an $\omega_1$-like $\widehat{\mathbb{A}}$-elementary end extension of $\mathcal{M}_f$ of cardinality $\aleph_1$ omitting $p_{f^\frown 0}$. By Corollary 6.17, there is an $\mathbb{A}$-type $p_{f^\frown 1}$ realized in $\mathcal{N}_1$ that can be omitted in an $\omega_1$-like $\widehat{\mathbb{A}}$-elementary end extension of $\mathcal{M}_{f^\frown 0}$. Let $\mathcal{M}_{f^\frown 1}$ be an $\widehat{\mathbb{A}}$-elementary initial segment of $\mathcal{N}_1$ containing $\mathcal{M}_f$ and a realization of $p_{f^\frown 1}$. By construction, *i–iii* hold.

For $h \in 2^{\omega_1}$, let $\mathcal{M}_h = \bigcup_{\alpha<\omega_1} \mathcal{M}_{h|\alpha}$. If $g \neq h \in 2^{\omega_1}$, then $\mathcal{M}_g$ and $\mathcal{M}_h$ realize different $\mathbb{A}$-types and hence are non-isomorphic as $\tau$-structures. $\square$

**Corollary 6.19** *If $T$ is a first order theory in a countable language and there are uncountably many types over $\emptyset$, then $T$ has $2^{\aleph_1}$ non-isomorphic models of cardinality $\aleph_1$.*

This is a precursor to Shelah's result [48] that that the same is true for an non $\omega$-stable. theory.

**Corollary 6.20** *If $\phi$ has uncountable models and fewer than $2^{\aleph_1}$ models of cardinaility $\aleph_1$, then there is a complete $\mathcal{L}_{\omega_1,\omega}$-sentence $\psi$ such that $\psi \models \phi$ and $\psi$ has a model of cardinality $\aleph_1$. In particular, if $\phi$ is $\aleph_1$-categorical, there is a complete $\aleph_1$-categorical sentence $\psi \models \phi$.*

**Proof** If $\phi$ has uncountable models and $I(\phi, \aleph_1) < 2^{\aleph_1}$, then there are only countably many $\mathbb{A}$-types for each countable fragment $\mathbb{A}$. By Theorem 6.12 and Exercise 2.34 the unique model of cardinality $\aleph_1$ is $\mathcal{L}_{\infty,\omega}$-equivalent to a countable model. $\square$

## 6.4 Extending Models of Set Theory

The main result of this section is due to Keisler and Morley [23]. The proof we give is due to Paul Larson (see [5] for expansions of this idea) and his proof that "$\phi$ has an uncountable model" is absolute.

Start with a countable $\omega$-model $\mathcal{M}$ of ZFC. Our goal is to build a countable elementary extension where we add elements exactly to the sets that $\mathcal{M}$ thinks are uncountable. Originally this was proved by an omitting types argument. Here we use a variant of a Skolem ultrapower argument where the ultrafilter is built by forcing.

Let $\mathcal{P}(\omega_1)^{\mathcal{M}} = \{A \in \mathcal{M} : \mathcal{M} \models A \subseteq \omega_1\}$. Let $P = \{A \in M : \mathcal{M} \models A \subset \omega_1$ contains a stationary set$\}$ ordered by $A \leq B$ if and only if $A \subseteq B$. Suppose $G \subset P$ is an $\mathcal{M}$-generic filter.

**Lemma 6.21** *Let $B \in \mathcal{P}(\omega_1)^{\mathcal{M}}$. The set $D_B = \{A \in P : A \subseteq B$ or $A \subseteq \omega_1^{\mathcal{M}} \setminus B\}$ is dense. Thus $G$ is a non-principle ultrafilter on $\omega_1^{\mathcal{M}}$.*

**Proof** i) Let $A_0 \in P$. Suppose $\mathcal{M} \models S \subseteq A_0$ is stationary. If neither $S \cap B$ nor $S \setminus B$ is stationary, then we can find closed unbounded sets $C_0, C_1 \in \mathcal{M}$ such that $C_0 \cap S \cap B = \emptyset$ and $(C_1 \cap S) \setminus B = \emptyset$. But there is $\alpha \in C_0 \cap C_1 \cap S$, a contradiction. □

We use $G$ to build a restricted ultrapower of $\mathcal{M}$. Let $\mathcal{F} = \{f \in \mathcal{M} : \text{dom}(f) = \omega_1\}$. Define $\sim$ on $\mathcal{F}$ by

$$f \sim g \Leftrightarrow \{\alpha : f(\alpha) = g(\alpha)\} \in G.$$

It is easy to see that $f$ is an equivalence relation. Let $[f]$ denote the equivalence class of $f$ and let $\mathcal{N} = \{[f] : f \in \mathcal{F}\}$. Define $\in^{\mathcal{N}}$ by

$$[f] \in^{\mathcal{N}} [g] \Leftrightarrow \{\alpha : f(\alpha) \in g(\alpha)\} \in G.$$

**Exercise 6.22** Show that if $f \sim f_1$ and $g \sim g_1$ then

$$\{\alpha : f(\alpha) \in g(\alpha)\} \in G \Leftrightarrow \{\alpha : f_1(\alpha) \in g_1(\alpha)\} \in G.$$

Thus $\in^{\mathcal{N}}$ is well definied.

Let $j : \mathcal{M} \to \mathcal{N}$ by $j(a) = [f_a]$ where $f_a(\alpha) = a$ for all $\alpha$.

**Exercise 6.23** Prove that $j$ is an embedding.

**Exercise 6.24** Adapt the proof of the Fundamental Theorem of Ultraproducts to show that $j$ is an elementary embedding. [Hint: Dealing with existential quantifiers will require that $\mathcal{M}$ is a model of the axiom of choice.]

**Lemma 6.25** *If $\mathcal{M} \models a$ is countable and $\mathcal{N} \models [f] \in a$, then $\mathcal{N} \models [f] = j(b)$ for some $b \in \mathcal{M}$.*

**Proof** There is a $g : \omega \to a$ a bijection such that $g \in \mathcal{M}$. Since $[f] \in^{\mathcal{N}} a$. There is $A \in G$ such that $f(\alpha) \in a$ for $\alpha \in A$. We need only show that

$$\{S \in P : f \text{ is constant on } S\}$$

is dense below $A$. Let $B \leq A$. There is $h \in \mathcal{M}$ such that $h : B \to \omega$ and $f(\alpha) = g(h(\alpha))$ for $\alpha \in B$. But $B$ contains a stationary set, and thus there is stationary $S \subseteq B$ such that $h$ is constant on $B$. $\square$

**Exercise 6.26** Let $i : \omega_1^{\mathcal{M}} \to \omega_1^{\mathcal{M}}$ be the identity map. Show that $[i] \in j(\omega_1^{\mathcal{M}})$ but $[i] \neq j(b)$ for any $b \in \mathcal{M}$.

**Lemma 6.27** *If $a \in \mathcal{M}$ and $\mathcal{M} \models a$ is uncountable, then there is $[f] \in \mathcal{N} \setminus \mathrm{img}(j)$ with $f \in j(a)$.*

**Proof** Since $\mathcal{M} \models a$ is uncountable, there is $f \in \mathcal{M}$ such $f : \omega_1^{\mathcal{M}} \to a$ is injective. Clearly $[f] \in j(a)$, but $f \not\sim f_b$ for any $b \in \mathcal{M}$. $\square$

We have now proved the main theorem.

**Theorem 6.28 (Keisler-Morley)** *Let $\mathcal{M}$ be a countable $\omega$-model of ZFC. Then $\mathcal{M}$ has a countable elementary extension $\mathcal{N}$ such that for $a \in \mathcal{M}$ there is $b \in \mathcal{N} \setminus \mathcal{M}$ such that $\mathcal{N} \models b \in a$ if and only if*

$$\mathcal{M} \models a \text{ is uncountable.}$$

We can iterate the construction to build an elementary chain of length $\omega_1$.

**Corollary 6.29** *If $\mathcal{M}$ is countable there is $\mathcal{M} \prec \mathcal{N}$ of cardinality $\aleph_1$ such that*

$$\mathcal{M} \models a \text{ is countable } \Leftrightarrow \{b \in \mathcal{N} : \mathcal{N} \models b \in a\} \text{ is countable.}$$

**The absoluteness of uncountable satisfiability**

Fix $\phi \in \mathcal{L}_{\omega_1,\omega}$. The following result was proved by Larson and also by Friedman, Hyttinen and Koerwien [12].

**Corollary 6.30** *It is absolute whether $\phi$ has an uncountable model.*

**Proof** (sketch) Fix ZFC$^*$ be a fragment of ZFC that is strong enough to prove all of the facts we needed in the above argument but weak enough that, say, $H(\aleph_2) \models$ ZFC$^*$. We claim that $\phi$ has an uncountable model if and only if there is an $\omega$-model $\mathcal{M}$ of ZFC$^*$ with $\phi \in \mathcal{M}$ and $\mathcal{M} \models$ there is an uncountable model of $\phi$.

The ($\Rightarrow$) direction follows because we can take a suitable elementary submodel of $H(\aleph_2)$. For the ($\Leftarrow$) direction start with such an $\mathcal{M}$. By the above construction, we can build an uncountable elementary extension $\mathcal{M} \prec \mathcal{N}$ such that every set that $\mathcal{N}$ believes in uncountable is uncountable. In particular, $\mathcal{N}$ will contain $\mathbb{A}$ uncountable such that $\mathcal{N}$ believe $\mathcal{A} \models \phi$. The tree of subformulas of $\phi$ is a well founded tree which will be in the model $\mathcal{M}$ and hence in $\mathcal{N}$. We can now do an induction to show that indeed

$$\mathcal{A} \models \psi(\overline{a}) \Leftrightarrow \mathcal{N} \models \text{“}\mathcal{A} \models \psi(\overline{a})\text{”}$$

for all subformulas of $\phi$. In particular, $\mathcal{A} \models \phi$.

Thus the existence of an uncountable model is equivalent to a $\Sigma^1_1(\phi)$-formula, and hence absolute. □

The same argument can be used to show that it is absolute whether an $\mathcal{L}_{\omega_1,\omega}(Q)$-sentence has a model where $Q$ is the quantifer “there exists uncountably many”.

Friedman, Hyttinen and Koerwien [12] go on to show it is not absolute whether a sentence has a model in $\aleph_2$ (even assuming GCH). The absoluteness of $\aleph_1$-categoricity is one of the many interesting open questions in the subject.

Baldwin and Larson [5] have expanded the contructions of this style to give a different proof of Theorem 6.18 and to generalize it to more expressive logics.

## 6.5 $\aleph_1$-categorical Sentences Have Models in $\aleph_2$

In this section we will prove the following theorem of Shelah. Our treatment in this section closely follows [1].

**Theorem 6.31 (Shelah [51])** *If $\phi \in \mathcal{L}_{\omega_1,\omega}$ is $\aleph_1$-categorical, then $\phi$ has a model of cardinality $\aleph_2$.*

For the remainder of this section fix $\mathbb{A}$ a countable fragment of $\mathcal{L}_{\omega_1,\omega}$ and $\phi \in \mathbb{A}$.

**Definition 6.32** We say that $(\mathcal{M}, \mathcal{N},)$ is a *proper pair* witnessed by $a$ if $\mathcal{M} \prec_{\mathbb{A}} \mathcal{N}$ and $a \in \mathcal{N} \setminus \mathcal{M}$.

**Lemma 6.33** *Suppose $\phi$ is $\kappa$-categorical, $\mathbb{A}$ is a fragment containing $\phi$ and $\mathcal{M} \prec_{\mathbb{A}} \mathcal{N}$ is a proper pair of models of cardinality $\kappa$, then $\phi$ has a model of cardinality $\kappa^+$.*

**Proof** Build a continuous elementary chain

$$\mathcal{M}_0 \prec_{\mathbb{A}} \mathcal{M}_1 \prec_{\mathbb{A}} \ldots \prec_{\mathbb{A}} \mathcal{M}_\alpha \prec_{\mathbb{A}} \ldots$$

for $\alpha < \kappa^+$, where each $\mathcal{M}_\alpha \models \phi$ has cardinality $\kappa$ and $(\mathcal{M}_\alpha, \mathcal{M}_{\alpha+1}) \cong (\mathcal{M}, \mathcal{N})$. This is possible since $\phi$ is $\kappa$-categorical so all of the $\mathcal{M}_\alpha \cong \mathcal{M}$. The union is the desired model. □

**Exercise 6.34** Show that if $\phi$ is $\kappa$-categorical and $\phi$ has a model of cardinality greater than $\kappa$, then there is a proper pair $\mathcal{M} \prec_{\mathbb{A}} \mathcal{N}$ of cardinality $\kappa$.

**Definition 6.35** We say that a proper pair $(\mathcal{M}, \mathcal{N}, a)$ is a *maximal triple* if there is no proper pair $(\mathcal{M}_1, \mathcal{N}_1)$ witnessed by $a$ with $\mathcal{M} \prec_{\mathbb{A}} \mathcal{M}_1$ and $\mathcal{N} \preceq_{\mathbb{A}} \mathcal{N}_1$

**Lemma 6.36** *If there is a proper pair of cardinality $\kappa$ but no maximal triple of cardinality $\kappa$, then $\phi$ has a proper pair of cardinality $\kappa^+$.*

**Proof** We build continuous elementary chains

$$\mathcal{M}_0 \prec_{\mathbb{A}} \mathcal{M}_1 \prec_{\mathbb{A}} \ldots \prec_{\mathbb{A}} \mathcal{M}_\alpha \prec_{\mathbb{A}} \ldots \text{ and } \mathcal{N}_0 \preceq_{\mathbb{A}} \mathcal{N}_1 \preceq_{\mathbb{A}} \ldots \preceq_{\mathbb{A}} \mathcal{N}_\alpha \preceq_{\mathbb{A}} \ldots$$

of for $\alpha < \kappa^+$ where $\mathcal{M}_\alpha \prec_{\mathbb{A}} \mathcal{N}_\alpha$ is a proper pair of models of cardinality $\kappa$ witnessed by $a$. At successor stages we use the fact that $(\mathcal{M}_\alpha, \mathcal{N}_\alpha, a)$ is not maximal. At limit stages we note that the union is still a proper pair witnessed by $a$. The union of the $\mathcal{M}_\alpha$ is the desired model of cardinality $\kappa^+$. □

**Definition 6.37** We say that $\mathcal{M} \prec_{\mathbb{A}} \mathcal{N}$ is a *cut pair* if there is a descending sequence

$$\mathcal{M} \prec_{\mathbb{A}} \ldots \mathcal{N}_{i+1} \prec_{\mathbb{A}} \mathcal{N}_i \prec_{\mathbb{A}} \ldots \prec_{\mathbb{A}} \mathcal{N}_1 \prec_{\mathbb{A}} \mathcal{N}$$

such that $\bigcap_{i\in\omega} \mathcal{N}_i = \mathcal{M}$.

**Lemma 6.38** *If $\phi$ has a uncountable model, then there is a cut pair in $\aleph_0$.*

**Proof** Let $\mathcal{M} \models \phi$ have cardinality $\aleph_1$. We fix a well ordering of $\mathcal{M}$ of order type $\omega_1$ and identify the universe of $\mathcal{M}$ with $\omega_1$. We can find a continuous $\omega_1$-chain of countable models

$$\mathcal{M}_0 \prec_{\mathbb{A}} \ldots \prec_{\mathbb{A}} \mathcal{M}_\alpha \prec_{\mathbb{A}} \ldots$$

such that $\mathcal{M} = \bigcup \mathcal{M}_\alpha$. Let $R$ be a binary relation on $\mathcal{M}$ such that $R(a, \alpha)$ if and only if $a \in \mathcal{M}_\alpha$. Let $\tau^* = \tau \cup \{<, R\}$. We can write down an $\mathcal{L}_{\omega_1,\omega}(\tau)$-sentence asserting that

- $<$ is a linear order with least element;
- $\forall a \exists b\ (R(a, b) \wedge \forall c < b \neg R(a, c))$
- $\{a : R(a, b)\}$ is a $\tau$-structure that is a model of $\phi$ for all $b$ and an $\mathbb{A}$-elementary submodel of $\mathcal{M}$;
- if $b < c$, then $\{a : R(a, b)\}$ is a proper $\mathbb{A}$-elementary submodel of $\{a : R(a, c)\}$.

By the Undefinability of Well Order Theorem 4.27, there is a countable $\mathcal{N} \models \Theta$ where $\mathbb{Q}$ embeds into $<$. Let $a_0 > a_1 > \ldots$ be an infinite descending sequence in $\mathcal{N}$. Let $\mathcal{N}_i = \{x \in \mathcal{N} : \mathcal{N} \models R(x, a_i)\}$, let $I = \{a \in \mathcal{N} : a < a_i$ for all $i\}$. and let $\mathcal{N}^* = \bigcup_{a\in I}\{x : R(x, a)\}$. Then

$$\mathcal{N}^* \prec_{\mathbb{A}} \ldots \prec_A \mathcal{N}_n \prec_A \ldots \prec_A \mathcal{N}_1 \prec_{\mathbb{A}} \ldots \mathcal{N}$$

and $\bigcap_{i\in\omega} \mathcal{N}_i = \mathcal{N}^*$. Thus $(\mathcal{N}^*, \mathcal{N})$ is a cut pair. □

We will show that if a sentence has a cut pair and a maximal triple then we can construct many models. We will need the following standard set theoretic fact that is often useful in proving many models theorems. See, for example, [35] 5.3.10 for a proof. Other examples of this method are given in Appendix A.

**Lemma 6.39** *Suppose $\kappa$ is regular. There is a family $(X_\alpha : \alpha < \kappa)$ of disjoint stationary subsets of $\kappa$. For $A \subseteq \kappa$, let $X_A = \bigcup_{\alpha \in A} X_\alpha$. Then $(X_A : A \subset \kappa)$ is a family of $2^\kappa$ stationary subsets of $\kappa$ such that $X_A \triangle X_B$ is stationary for $A \neq B$.*

**Lemma 6.40** *Suppose $\phi$ is $\kappa$-categorical and $\phi$ has both a cut pair and a maximal triple of cardinality $\kappa$, then there are $2^{\kappa^+}$ non-isomorphic models of $\phi$ of cardinality $\kappa^+$.*

**Proof** Let $(\mathcal{M}, \mathcal{N}, a)$ be a maximal triple and let $(\mathcal{M}', \mathcal{N}')$ be a cut pair. For any $X \subseteq \kappa^+$ we build $\mathcal{M}(X) \models \phi$ as follows: We build a continuous $\kappa^+$-chain

$$\mathcal{M}_0(X) \prec_{\mathbb{A}} \ldots \prec_{\mathbb{A}} \mathcal{M}_\alpha(X) \prec_{\mathbb{A}} \ldots$$

such that if $\alpha \in X$, then

$$(\mathcal{M}_\alpha(X), \mathcal{M}_{\alpha+1}(X)) \cong (\mathcal{M}, \mathcal{N}),$$

while if $\alpha \notin X$, then

$$(\mathcal{M}_\alpha(X), \mathcal{M}_{\alpha+1}(X)) \cong (\mathcal{M}', \mathcal{N}').$$

Let $\mathcal{M}(X) = \bigcup_{\alpha < \kappa^+} \mathcal{M}_\alpha(X)$.

Suppose $X$ and $Y$ are stationary subsets of $\kappa^+$ such that $X \setminus Y$ is stationary. We claim that $\mathcal{M}(X) \not\cong \mathcal{M}(Y)$. For contradiction, suppose $f : \mathcal{M}(X) \to \mathcal{M}(Y)$ is an isomorphism. Then we can find $\alpha \in X \setminus Y$ such that $f|\mathcal{M}_\alpha(X)$ is an isomorphism between $\mathcal{M}_\alpha(X)$ and $\mathcal{M}_\alpha(Y)$. There is $a$ such that $(\mathcal{M}_\alpha(X), \mathcal{M}_{\alpha+1}(X), a)$ is a maximal triple. Since $(\mathcal{M}_\alpha(Y), \mathcal{M}_{\alpha+1}(Y))$ is a cut pair, there is $\mathcal{M}^*$ such that $\mathcal{M}_\alpha(Y) \prec_{\mathbb{A}} \mathcal{M}^* \preceq_{\mathbb{A}} \mathcal{M}_{\alpha+1}(Y)$ such that $f(a) \notin \mathcal{M}^*$. Choose $\beta \geq \alpha + 1$ such that $f^{-1}(\mathcal{M}^*) \prec_{\mathbb{A}} \mathcal{M}_{\beta(X)}$. Since

- $\mathcal{M}_\alpha(X) \prec_{\mathbb{A}} f^{-1}(\mathcal{M}^*) \prec_{\mathbb{A}} \mathcal{M}_\beta(X)$
- $\mathcal{M}_{\alpha+1}(X) \preceq_{\mathbb{A}} \mathcal{M}_\beta(X)$ and
- $a \in \mathcal{M}_{\beta(X)} \setminus f^{-1}(\mathcal{M}^*)$

this contradicts the fact that $(\mathcal{M}_\alpha(X), \mathcal{M}_{\alpha+1}(X), a)$ is a maximal pair.

Using Lemma 6.39 we can construct $2^{\kappa^+}$ non-isomorphic models of cardinality $\kappa^+$. □

**Proof of Theorem 6.31**

Suppose $\phi$ is $\aleph_1$-categorical. By Corollary 6.20 we may assume that $\phi$ is complete. By Lemma 6.38, there is a cut pair in $\aleph_0$. Since there is a cut pair, by Lemma 6.40 there is no maximal triple of cardinality $\aleph_0$. But then, by Lemma 6.36. There is a proper pair of cardinality $\aleph_1$ and by Lemma 6.33 a model of cardinality $\aleph_2$. □

In fact Shelah [51] proves that the same is true for $\mathcal{L}_{\omega_1,\omega}(Q)$ where $Q$ is the quantifier "there exists uncountably many." See §7 of [1] for a proof.

How many models will $\phi$ have in $\aleph_2$? Possibly, the maximal number.

**Theorem 6.41 (Shelah)** *Suppose $2^\kappa < 2^{\kappa^+}$, $\phi$ is $\kappa$-categorical and amalgamation fails in $\kappa$. Then there are $2^{\kappa^+}$ non-isomorphic models of $\phi$ of cardinality $\kappa^+$.*

See §17 of [1] for a proof.

# 7 Uncountable Models of Vaught Counterexamples

Suppose $\phi \in \mathcal{L}_{\omega_1,\omega}$ is a Vaught counterexample, i.e., $\phi$ is scattered with uncountably many countable models. What can we say about uncountable models of $\phi$? The first results due to Harnik and Makkai show that there are always models of $\phi$ of cardinality $\aleph_1$. This was refined in an unpublished result of Harrington who proved that a Vaught counterexample has models of arbitrarily large Scott rank below $\omega_2$. While most model theorists believe Vaught's Conjecture fails, Harrington's result combined with Hjorth's Theorem from §5.4 raise the seductive possibility of proving Vaught's Conjecture by strengthening Harrington's result to build a model of size $\aleph_2$ contradicting Hjorth's Theorem. Of course, all of the constructions we know of models of size $\aleph_1$ are built up from countable approximations and we have few good techniques for building a model of size $\aleph_2$.

Harrington's result shows that if $\phi$ is a Vaught counterexample then $I(\phi, \aleph_1) \geq \aleph_2$. An interesting open problem is if this can be extended to show $I(\phi, \aleph_1) = 2^{\aleph_1}$. Using two fundamental results of Shelah, Baldwin [2] noticed that this is true for first order theories.

**Proposition 7.1** *Suppose $T$ is a complete first order theory in a countable language which is a Vaught counterexample. Then $I(T, \aleph_1) = 2^{\aleph_1}$.*

**Proof** Shelah [47] proved Vaught's Conjecture for $\omega$-stable theories. Thus $T$ is not $\omega$-stable. But Shelah [48] also proved that a non $\omega$-stable theory has $2^{\aleph_1}$ non-isomorphic models of cardinality $\aleph_1$. □

In the remainder of this section we will prove the results of Harnik, Makkai and Harrington.

## 7.1 Minimal Counterexamples

We will prove the following theorem of Harnik and Makkai.

**Theorem 7.2** *Let $\phi$ be a Vaught counterexample. There is $\mathcal{M} \models \phi$ of cardinality $\aleph_1$ that is not $\mathcal{L}_{\infty,\omega}$-equivalent to a countable model.*

**Definition 7.3** Suppose $\phi \in \mathcal{L}_{\omega_1,\omega}$ is a Vaught counterexample. We say that $\phi$ is a *minimal counterexample* if for every sentence $\psi \in \mathcal{L}_{\omega_1,\omega}$ either $\phi \wedge \psi$ or $\phi \wedge \neg\psi$ has at most countably many countable models.

Minimal counterexamples were introduced by Harnik and Makkai for this proof, but they are used in a number of results around Vaught's Conjecture.

**Lemma 7.4** *If $\phi$ is a Vaught counterexample, there is a minimal counterexample $\theta$ such that $\theta \models \phi$.*

**Proof** For purposes of contradiction, suppose there is no minimal counterexample $\theta$ with $\theta \models \phi$. We will build a tree of counterexamples $(\phi_\eta : \eta \in 2^{<\omega})$ such that:

i) $\phi_\emptyset = \phi$;

ii) $\phi_\eta \models \phi_\nu$ for $\nu \subseteq \eta$;

iii) $\phi_{\eta,0} \wedge \phi_{\eta,1}$ is unsatisfiable, for all $\eta$.

This is easy to do. Given $\phi_\eta$, because $\phi_\sigma$ is not a minimal counterexample, there is $\psi$ such that $\phi_\eta \wedge \psi$ and $\phi_\eta \wedge \neg\psi$ are Vaught counterexamples. Let $\phi_{\eta,0} = \phi_\sigma \wedge \psi$ and let $\phi_{\eta,1} = \phi_\eta \wedge \neg\psi$.

For $f \in 2^\omega$, let

$$T_f = \{\phi_\eta : \eta \subset f\}.$$

Suppose we additionally have

iv) each $T_f$ is satisfiable.

If each $T_f$ is satisfiable, then $I(\phi, \aleph_0) = 2^{\aleph_0}$ so $\phi$ is not a Vaught counterexample. Indeed, if $\alpha < \omega_1$ is a bound on the quantifier ranks of the formulas $(\phi_\eta : \eta \in 2^{<\omega})$, then there are uncountably many models that are not $\equiv_\alpha$, and hence a perfect set of non-isomorphic models.

Unfortunately, condition iv) is not automatic so we will need to build the tree with more care. Add $C$ a countable set of new constant symbols to the language. Let $\Sigma = \{\sigma : \sigma$ is a finite set of $\mathcal{L}_{\omega_1,\omega}$-sentences using only finitely many constant symbols from $C$ such that $\sigma \cup \{\phi\}$ has uncountably many countable models$\}$.

**claim** $\Sigma$ is a consistency property.

We'll only check (C4). Suppose $\bigvee_{\psi \in X} \psi \in \sigma \in \Sigma$. Since there are uncountably many models of $\sigma$ and $X$ is countable, for some $\psi \in X$ there are uncountably many models of $\sigma \cup \{\phi\}$.

If $\sigma \in \Sigma$, and $\theta(\overline{v})$ is an $\mathcal{L}_{\omega_1,\omega}$-formula in the original vocabulary such that $\theta(\overline{c})$ is the conjunction of $\sigma \cup \{\phi\}$. Since there are uncountably many models of $\sigma \cup \{\phi\}$, there are uncountably many models of $\exists\overline{v}\ \theta(\overline{v})$. Thus $\exists\overline{v}\ \theta(\overline{v})$ is a Vaught counterexample.

We will build a sequence of countable fragments $\mathbb{A}_0 \subseteq \mathbb{A}_1 \subseteq \mathbb{A}_2 \subseteq \dots$ and $\mathbb{A} = \bigcup \mathbb{A}_n$. We will let $\chi_0, \chi_1, \dots$ list all $\mathbb{A}$-sentences and $t_0, t_1, \dots$ list all $\mathbb{A}$-terms, both lists with infinite repetition. We will also build a tree $(\sigma_\eta : \eta \in 2^{<\omega})$ such that:

i) $\sigma_\emptyset = \{\phi\}$;

ii) $\sigma_\eta \subseteq \sigma_\nu$ if $\eta \subseteq \nu$;

iii) if $|\eta| = n$ and $\sigma_\eta \cup \{\chi_n\} \in \Sigma$, then $\chi_n \in \sigma_{\eta,i}$ for $i = 0, 1$, in addition

- if $\chi_n = \bigvee_{\psi \in X} \psi$, then there is $\psi \in X$ such that $\psi \in \sigma_{\eta,i}$ for $i = 0, 1$ and
- if $\chi_n = \exists v\ \psi(v)$, then for some constant $c \in C$, $\psi(c) \in \sigma_{\eta,i}$ for $i = 0, 1$;

iv) if $|\eta| = n$, then there is $c \in C$ such that $t_n = c \in \sigma_{\eta,i}$ for $i = 1, 2$;

v) for each $\eta$ there is an $\mathcal{L}_{\omega_1,\omega}$-sentence $\psi$ with no constant symbols from $C$ such that $\psi \in \sigma_{\eta,0}$ and $\neg\psi \in \sigma_{\eta,1}$;

Because $\Sigma$ is a consistency property, given $\sigma_\eta \in \Sigma$ we can find $\sigma_\eta \subset \sigma'_\eta \in \Sigma$ satisfying iii) and iv). Let $\theta(\overline{v})$ be the formula in the original vocabulary such that $\theta(\overline{c})$ is the conjunction of $\sigma'_\eta$. We argued above that $\exists\overline{v}\ \theta(\overline{v})$ is a Vaught counterexample implying $\phi$. Since we are assuming there are no minimal counterexamples, we can find a sentence $\psi$ such that there are uncountably many countable models of both $\psi \wedge \exists\overline{v}\ \theta(\overline{v})$ and $\neg\psi \wedge \exists\overline{v}\ \theta(\overline{v})$. Let $\sigma_{\eta,0} = \sigma'_\eta \cup \{\psi\}$ and $\sigma_{\eta,1} = \sigma'_\eta \cup \{\neg\psi\}$.

The final bookkeeping detail is that we let $\mathbb{A}_n$ be the smallest fragment containing all formulas in all $\sigma_\eta$ for all $|\eta| \leq n$. Although our fragment is expanding during the construction it is easy to build the necessary lists $\chi_0, \chi_1, \dots$ and $t_0, t_1, \dots$.

For $f \in 2^\omega$, let $T_f = \bigcup_{\eta \subset f} \sigma_\eta$. As in the proof of the Model Existence Theorem 4.6, iii) and iv) guarantee that there is a canonical countable model of $T_f$ and condition v) insures that these countable models are $\not\equiv_{\mathbb{A}}$ in the original vocabulary. Thus there are perfectly many non-isomorphic models of $\phi$, a contradiction. $\square$

We will need a refinement of this Lemma.

**Definition 7.5** Let $\phi \in \mathcal{L}_{\omega_1,\omega}(\tau)$ be a Vaught counterexample. Suppose $\psi(\overline{v})$ is an $\mathcal{L}_{\omega_1,\omega}(\tau)$-formula. We say that $\psi$ is *$\phi$-large* if there are uncountably many models of $\phi \wedge \exists\overline{v}\ \psi(\overline{v})$. We say that a $\phi$-large formula $\psi(\overline{v})$ is *minimal $\phi$-large* if for all $\mathcal{L}_{\infty,\omega}$-formulas $\chi(\overline{v})$ exactly one of $\psi(\overline{v}) \wedge \chi(\overline{v})$ and $\psi(\overline{v}) \wedge \neg\chi(\overline{v})$ is $\phi$-large.

**Corollary 7.6** *If $\psi(\overline{v})$ is $\phi$-large, then there is a minimal $\phi$-large $\theta(\overline{v})$ such that $\theta(\overline{v}) \models \psi(\overline{v})$.*

**Proof** If $\psi$ has free variables $v_1, \ldots, v_n$ add new constants $d_1, \ldots, d_n$ and apply Theorem 7.4 to $\phi \wedge \psi(\overline{d})$. $\square$

**The Proof of Theorem 7.2** Let $\phi$ be a Vaught counterexample. By Theorem 7.4 we may, without loss of generality, assume that $\phi$ is a minimal counterexample.

Our proof will need fragments with extra closure properties. We say that a fragment $\mathbb{A}$ is *rich* if for any $\phi$-large $\psi(\overline{v}) \in \mathbb{A}$, there is a minimal $\phi$-large $\theta(\overline{v}) \in \mathbb{A}$ with $\theta \models \phi$. By Corollary 7.6, for any countable fragment $\mathbb{A}$ with $\phi \in \mathbb{A}$, we can find a countable $\mathbb{A}' \supseteq \mathbb{A}$ where $\mathbb{A}'$ is rich.

If $\mathbb{A}$ is a countable rich fragment of $\mathcal{L}_{\omega_1,\omega}$, let $T_{\mathbb{A}} = \{\psi \in \mathbb{A} : \psi$ is a sentence such that $\phi \wedge \psi$ has uncountably many countable models$\}$. Since $\phi$ is a minimal counterexample, for every $\psi \in \mathbb{A}$ exactly one of $\psi$ and $\neg\psi$ is in $T_{\mathbb{A}}$. Moreover, if $\psi$ is an $\mathbb{A}$-sentence not in $T_{\mathbb{A}}$, then $\psi$ has only countably many countable models. Since $\mathbb{A}$ is countable, there are uncountably many countable models of $T_{\mathbb{A}}$.

If $\psi(\overline{v})$ is an $\mathbb{A}$-formula and $T_{\mathbb{A}} \models \exists v\ \psi(\overline{v})$, then $\psi(\overline{v})$ is $\phi$-large. Thus all $\mathbb{A}$-formulas consistent with $T_{\mathbb{A}}$ are $\phi$-large.

If $\theta(\overline{v})$ is a minimal $\phi$-large formula, then for any $\chi(\overline{v}) \in \mathcal{L}_{\omega_1,\omega}$,

$$T_{\mathbb{A}} \models \theta(\overline{v}) \rightarrow \chi(\overline{v}) \text{ or } T_{\mathbb{A}} \models \theta(\overline{v}) \rightarrow \neg\chi(\overline{v}).$$

Thus the minimal $\phi$-large formulas are complete and consistent $\mathbb{A}$-formula consistent with $T_{\mathbb{A}}$ is implied by a minimal $\phi$-large formula in $\mathbb{A}$. Thus $T_{\mathbb{A}}$ is $\mathbb{A}$-atomic. By Theorem 4.19 there is an $\mathbb{A}$-prime model of $T_{\mathbb{A}}$.

We now describe the basic construction. For $\alpha < \omega_1$ we build countable rich fragments

$$\mathbb{A}_0 \subseteq \mathbb{A}_1 \subseteq \ldots \subseteq \mathbb{A}_\alpha \subseteq \ldots$$

.

Let $\mathbb{A}_0$ be a countable rich fragment containing $\phi$.

Given $\mathbb{A}_\alpha$ and $\mathcal{M}_\alpha$ the $\mathbb{A}_\alpha$-prime model of $T_\alpha$. Let $\mathbb{A}_{\alpha+1}$ be a countable rich fragment containing $\Phi_\alpha$ the Scott sentence of $\mathcal{M}_\alpha$. Let $\mathcal{M}_{\alpha+1}$ be the prime model of $T_{\mathbb{A}_{\alpha+1}}$. Since $T_{\mathbb{A}_\alpha} \subset T_{\mathbb{A}_{\alpha+1}}$ we may assume $\mathcal{M}_\alpha \prec_{\mathbb{A}_\alpha} \mathcal{M}_{\alpha+1}$. Of course $\Phi_\alpha \not\in T_{\mathbb{A}_{\alpha+1}}$.

If $\beta$ is a limit ordinal, let $\mathcal{M}_\beta = \bigcup_{\alpha<\beta} \mathcal{M}_\alpha$ and $\mathbb{A}_\beta = \bigcup_{\alpha<\beta} \mathbb{A}_\alpha$. For all $\overline{a} \in M_\beta$, there is a minimal $\phi$-large formula $\theta(\overline{v}) \in A_\beta$ such that $\mathcal{M}_\beta \models \theta(\overline{a})$

and every $\phi$-large formula in $\mathbb{A}_\beta$ is implied by a minimal $\phi$-large formula in $\mathbb{A}_\beta$. Thus $\mathcal{M}_\beta$ is the prime model of $T_{\mathbb{A}_\beta}$.

Note that $\Phi_\alpha$ the Scott sentence of $\mathcal{M}_\alpha$ is not it $T_{\mathbb{A}_{\alpha+1}}$. Thus $\neg\Phi_\alpha \in T_{\mathbb{A}_{\alpha+1}}$. Hence $\mathcal{M}_\alpha \not\cong \mathcal{M}_\beta$ if $\alpha \neq \beta$.

Let $\mathcal{M} = \bigcup_{\alpha<\omega_1} \mathcal{M}_\alpha$. Suppose $\Phi$ is the Scott sentence of a countable $\mathcal{N} \models \phi$. We claim that $\mathcal{M} \not\models \Phi$. Suppose for contradiction that $\mathcal{M} \models \Phi$. By Exercise 1.30, $\{\alpha < \omega_1 : \mathcal{M}_\alpha \models \Phi\}$ is closed unbounded. But if $\mathcal{M}_\alpha, \mathcal{M}_\beta \models \Phi$ then $\mathcal{M}_\alpha \cong \mathcal{M}_\beta$, a contradiction. Thus $\mathcal{M} \not\equiv_{\infty,\omega} \mathcal{N}$ for any countable model $\mathcal{N}$. □

Once we know that Vaught counterexamples have uncountable models we can shows there is a small model of size $\aleph_1$.

**Corollary 7.7 (Makkai)** *Every Vaught counterexample has a small uncountable model.*

**Proof** Let $\phi$ be a counterexample. We saw in Theorem 7.2 that every counterexample has an uncountable model. Since $\phi$ is scattered, that model is $\mathbb{A}$-small for every countable fragment $\mathbb{A}$. Thus, by Theorem 6.12, $\phi$ has an uncountable small model. □

Makkai's orginal proof in [32] used admissible model theory and $\Sigma_{\mathbb{A}}$-saturated models. The proof we give here is due to Baldwin [2].

**Exercise 7.8** If $\phi$ is Vaught counterexample. There are $\aleph_1$-countable models, that are $\equiv_{\infty,\omega}$ to models of cardinality $\aleph_1$. [Hint: Suppose $\Phi_1, \Phi_2, \ldots$ are Scott sentences for countably many models of $\phi$. Consider $\phi \wedge \bigwedge_{n=1}^{\infty} \neg\Phi_n$.]

## 7.2 Harrington's Theorem

Harrington showed that a counterexample to Vaught's Conjecture has models of arbitrarily large Scott rank below $\omega_2$. His orginal proof used admissibility. This proof was never published. I learned of it from lecture notes from a course Victor Harnik gave at Dartmouth in 1974. Recently new proofs have been given in [30] and [4] which avoid admissibility. The proof we will follow is from [4].

### The Morley tree

Fix $\phi \in \mathcal{L}_{\omega_1,\omega}$ scattered with uncountably many countable models, i.e., models of arbitrarily high countable Scott rank. We continue the analysis from §3.3. Recall that we constructed a series of countable fragments

$$\mathbb{A}_0 \subset \mathbb{A}_1 \subset \ldots \subset \mathbb{A}_\alpha \subset \ldots$$

for $\alpha < \omega_1$ such that $\phi \in \mathbb{A}_0$, $\mathbb{A}_{\alpha+1}$ is obtained by adding formulas $\bigwedge_{\psi\in p} \psi(\overline{v})$ for each $\mathbb{A}_\alpha$-type $p$ and $\mathbb{A}_\alpha = \bigcup_{\beta<\alpha} \mathbb{A}_\beta$ for each limit ordinal.

We build a tree of theories $\mathcal{T}$. For $\alpha < \omega_1$, we let $\mathcal{T}_\alpha$ be the elements of $\mathcal{T}$ of height $\alpha$. An $\mathbb{A}_\alpha$-theory $T$ is in $\mathcal{T}_\alpha$ if and only if

i) $\phi \in T$;

ii) $T$ is satisfiable;

iii) $T$ is $\mathbb{A}_\alpha$-complete, ie., $\psi \in T$ or $\neg\psi \in T$ for any $\psi \in \mathbb{A}_\alpha$;

iv) for all $\beta < \alpha$, $T \cap \mathbb{A}_\beta$ is not complete, i.e., not $\aleph_0$-categorical.

Thus the terminal nodes on $\mathcal{T}$ are the $\aleph_0$-categorical theories. We let $\mathcal{T}_{<\alpha}$ denote $\bigcup_{\beta<\alpha} \mathcal{T}_\beta$. We let $\mathcal{T} = \bigcup_{\alpha<\omega_1} \mathcal{T}_\alpha$.

**Exercise 7.9** Show that if $T \in \mathcal{T}_\alpha$ and $\beta < \alpha$, then $T \cap \mathbb{A}_\beta \in \mathcal{T}_\beta$.

Note that since $\phi$ is scattered, each $T \in \mathcal{T}_\alpha$ is $\mathbb{A}_\alpha$-atomic.

**Exercise 7.10** a) Suppose $\alpha < \beta$, $T_0 \in \mathcal{T}_\alpha$, $T_1 \in \mathcal{T}_\beta$ and $T_0 \subset T_1$. Let $\mathcal{M}_0$ be the $\mathbb{A}_\alpha$-prime model of $T_0$ and $\mathcal{M}_1$ be the $\mathbb{A}_\beta$-prime model of $T_1$, then there is an $\mathbb{A}_\alpha$-elementary embedding of $\mathcal{M}_0$ into $\mathcal{M}_1$.

b) Suppose $\alpha$ is a limit ordinal and $(T_\beta : \beta < \alpha)$ is a chain of theories such that $T_\beta \in \mathcal{T}_\beta$ and $T_\gamma \subset T_\beta$ for all $\gamma < \beta$. Then $\bigcup_{\beta<\alpha} T_\beta \in \mathcal{T}_\alpha$. [Hint: the only difficulty should be proving that this theory is satisfiable, and a) should be helpful in showing this.]

**Exercise 7.11** If $\mathcal{M} \models \phi$ is countable, there is a terminal node $T$ in the Morley tree such that $\mathcal{M} \models T$.

**Exercise 7.12** Show $\mathcal{T}_\alpha \neq \emptyset$ for all $\alpha < \omega_1$.

**Exercise 7.13** a) Suppose $\alpha$ is a limit ordinal. Prove that there is $\beta < \alpha$ and $T \in T_\beta$ with a unique extension in $\mathcal{T}_\alpha$. [Hint: if not build a perfect tree of theories in $\mathcal{T}_{<\alpha}$.]

b) There is $T \in \mathcal{T}$ and a closed unbounded set $C \subset \omega_1$ such that $T$ has a unique extension in $\mathcal{T}_\alpha$ for each $\alpha \in C$. [Hint: Fodor's Theorem]

Let $(T_\alpha : \alpha \in C)$ be the sequence of unique extensions of $T$ to $\mathcal{T}_\alpha$.

c) Show that $\bigcup_{\alpha\in C} T_\alpha$ is satisfiable. [Hint: Consider the chain of models $(\mathcal{M}_\alpha : \alpha \in C)$ where $\mathcal{M}_\alpha$ is $\mathbb{A}_\alpha$-prime.]

**Lemma 7.14** *If $\alpha < \omega_1$ is a limit ordinal and $\mathcal{M} \models T \in \mathcal{T}_\alpha$ then the Scott rank of $\mathcal{M}$ is at least $\alpha$.*

**Proof** Let $\mathcal{M} \models \phi$. We prove that for all $\overline{a} \in \mathcal{M}$ and $\alpha < \omega_1$ there is $\Psi^{\mathcal{M}}_{\overline{a},\alpha}(\overline{v}) \in \mathbb{A}_{\alpha+1}$ such that $\mathcal{M} \models \Psi^{\mathcal{M}}_{\overline{a},\alpha}(\overline{a})$ and

$$\models \Psi^{\mathcal{M}}_{\overline{a},\alpha}(\overline{v}) \to \Phi^{\mathcal{M}}_{\overline{a},\alpha}(\overline{v}).$$

Clearly $\text{tp}^{\mathcal{M}}(\overline{a}/\mathbb{A}_0)$ implies $\Phi^{\mathcal{M}}_{\overline{a},0}$ which is the conjunction of atomic and negated atomic formulas satisfied by $\overline{a}$. We let $\Psi^{\mathcal{M}}_{\overline{a},0}(\overline{v})$ be the conjunction of $\text{tp}^{\mathcal{M}}(\overline{a}, \mathbb{A}_0)$.

Let $X$ be the set of $\mathbb{A}_\alpha$-types $q(\overline{v}, w)$ with $\Psi^{\mathcal{M}}_{\overline{a},\alpha}(\overline{v}) \in q(\overline{v}, w)$ such that $q(\overline{a}, w)$ is not realized in $\mathcal{M}$. Since $\phi$ is scattered, $X$ is countable. For $q \in X$ let $\theta_q(\overline{v}, w) \in \mathbb{A}_{\alpha+1}$ be the conjunction of $q$. Then $\text{tp}^{\mathcal{M}}(\overline{a}, \mathbb{A}_{\alpha+1})$ implies

$$\forall w \left( \bigwedge_{b\in\mathcal{M}} \Psi_{\overline{a},b,\alpha}(\overline{v}, w) \wedge \bigwedge_{q\in X} \neg\theta_q(\overline{v}, w) \right).$$

Let $\Psi^{\mathcal{M}}_{\overline{a},\alpha+1}$ be the conjunction of $\text{tp}^{\mathcal{M}}(\overline{a}, \mathbb{A}_{\alpha+1})$. Note that

$$\models \Psi^{\mathcal{M}}_{\overline{a},\alpha+1}(\overline{v}) \to \forall w \bigvee_{b\in\mathcal{M}} \Psi_{\overline{a},b}(\overline{v}, \overline{w})$$

since we have insured that no other types are realized. Thus $\models \Psi^{\mathcal{M}}_{\overline{a},\alpha+1} \Rightarrow \Phi^{\mathcal{M}}_{\overline{a},\alpha+1}$.

If $\alpha$ is a limit ordinal then, by induction $\text{tp}^{\mathcal{M}}(\overline{a}, \mathbb{A}_\alpha)$ implies $\Phi^{\mathcal{M}}_{\overline{a},\alpha}$, so we can take $\Psi^{\mathcal{M}}_{\overline{a},\alpha}$ to be the conjunction of $\text{tp}^{\mathcal{M}}(\overline{a}, \mathbb{A}_\alpha)$.

Suppose now that $\alpha < \omega_1$ is a limit ordinal and $\mathcal{M} \models \phi$ has Scott rank $\beta < \alpha$. If $\overline{a} \in \mathcal{M}$, then

$$\models \Psi^{\mathcal{M}}_{\overline{a},\beta}(\overline{v}) \to \Phi^{\mathcal{M}}_{\overline{a},\beta}(\overline{v}),$$

but

$$\models \Phi^{\mathcal{M}}_{\overline{a},\beta}(\overline{v}) \to \theta(\overline{v})$$

for any formula $\theta \in \mathcal{L}_{\omega_1,\omega}$ such that $\mathcal{M} \models \theta(\overline{a})$. In particular, this is true if $\theta$ is $\Psi^{\mathcal{M}}_{\overline{a},\beta+1}$. Thus

$$\mathcal{M} \models \forall \overline{v} \left( \Psi^{\mathcal{M}}_{\overline{a},\beta}(\overline{v}) \to \Psi^{\mathcal{M}}_{\overline{a},\beta+1}(\overline{v}) \right)$$

for any $\overline{a} \in \mathcal{M}$. We can do a back-and-forth argument to show that the $\mathbb{A}_{\beta+2}$-theory of $\mathcal{M}$ is $\aleph_0$-categorical. Thus $\mathcal{M}$ is not a model of any $T \in \mathcal{T}_\alpha$.
□

In §13 we will need the next result from [39].

**Corollary 7.15** *Suppose $\phi$ is a minimal Vaught counterexample. There is a closed unbounded $C \subset \omega_1$ such that if $\mathcal{M}, \mathcal{N} \models \phi$ and $\mathcal{M}$ and $\mathcal{N}$ have Scott rank at least $\alpha$, then $\mathcal{M} \equiv_\alpha \mathcal{N}$.*

**Proof** For each $\alpha$ let

$$T_\alpha = \{\psi \in \mathbb{A}_\alpha : \phi \wedge \psi \text{ has } \aleph_1 \text{ countable models }\}.$$

Since $\phi$ is a minimal counterexample, $T_\alpha$ is $\mathbb{A}_\alpha$-complete, satisfiable, and all but countably many countable models of $\phi$ are models of $T_\alpha$. Thus there is $\gamma$ such that if $\mathrm{SR}(\mathcal{M}) \geq \gamma$ then $\mathcal{M} \models T_\alpha$.

Let $g : \omega_1 \to \omega_1$ such that if $\mathrm{SR}(\mathcal{M}) \geq g(\alpha)$, then $\mathcal{M} \models T_\alpha$. Let

$$C = \{\alpha < \omega_1 : \alpha \text{ is a limit and } g(\beta) < \alpha \text{ for all } \beta < \alpha\}.$$

Suppose $\alpha \in C$, $\mathcal{M}, \mathcal{N} \models \phi$ and $\mathrm{SR}(\mathcal{M})$, $\mathrm{SR}(\mathcal{N}) \geq \alpha$. Then $\mathcal{M}$ and $\mathcal{N}$ are models of $T_\alpha$. Thus $\mathcal{M} \equiv_{\mathbb{A}_\alpha} \mathcal{N}$. By the arguments above $\mathcal{M} \equiv_\alpha \mathcal{N}$. □

### Twisted direct limits

We make a short digression to discuss a construction we will use in the main proof.

Let $\mu$ be a limit ordinal.

- Let $(\tau_\alpha : \alpha < \mu)$ be sequence of vocabularies with
$$\tau_0 \subseteq \ldots \subseteq \tau_\alpha \subset \ldots$$
and $\tau_\alpha = \bigcup_{\beta<\alpha} \tau_\beta$ for $\alpha$ a limit ordinal.
- Suppose for each $\alpha < \mu$ we have $\mathbb{A}_\alpha$ a fragment of $\mathcal{L}_{\infty,\omega}(\tau_\alpha)$.

**Definition 7.16** If $\tau \subseteq \tau'$, $\mathbb{A}$ and $\mathbb{A}'$ are fragments of $\mathcal{L}_{\infty,\omega}(\tau)$ and $\mathcal{L}_{\infty,\omega}(\tau')$ respectively and $\pi : \mathbb{A} \to \mathbb{A}'$ is an injection, then we say that $\pi$ is a *fragment embedding* if and only if

- $\pi$ is the identity on atomic $\tau$-formulas;

- the free variables of $\pi(\psi)$ are the same as the free variables of $\psi$ for $\psi \in \mathbb{A}$;

- $\pi$ commutes with first order Boolean connectives $\wedge, \vee, \neg$ and quantification $\exists v$ and $\forall v$;

- $\psi$ is an infinite disjunction (conjunction) if and only if $\pi(\psi)$ is an infinite disjunction (conjunction); and if $\psi$ is an infinite disjunction (conjunction) then $\theta$ is a disjunct (conjunct) of $\psi$ if and only if $\pi(\theta)$ is a disjunct (conjunct) of $\pi(\psi)$, for $\psi, \theta \in \mathbb{A}$.

Suppose that we have fragment embeddings $\pi_{\alpha,\beta} : \mathbb{A}_\alpha \to \mathbb{A}_\beta$ for all $\alpha < \beta$. We say that $(\pi_{\alpha,\beta} : \alpha, \beta)$ is a *directed system of fragment embeddings* if and only if $\pi_{\alpha,\gamma} = \pi_{\beta,\gamma} \circ \pi_{\alpha,\beta}$ for $\alpha < \beta < \gamma < \mu$.

**Definition 7.17** Suppose $(\mathbb{A}_\alpha, \pi_{\alpha,\beta} : \alpha < \beta < \mu)$ is a directed system of fragment embeddings. Let $\tau^* = \bigcup_{\alpha<\mu} \tau_\alpha$. Suppose we also have a fragment $\mathbb{A}^*$ of $\mathcal{L}_{\infty,\omega}(\tau^*)$ and $\tau_\alpha$ embeddings $\pi_{\alpha,*} : \mathbb{A}_\alpha \to \mathbb{A}^*$ such that $\pi_{\alpha,*} = \pi_{\beta,*} \circ \pi_{\alpha,\beta}$ for all $\alpha < \beta$ and every element of $\mathbb{A}^*$ is of the form $\pi_{\alpha,*}(\psi)$ for some $\alpha < \mu$ and $\psi \in \mathbb{A}_\alpha$.

We call $(\mathbb{A}^*, \pi_{\alpha,*})$ the *direct limit* of the system $(\pi_{\alpha,\beta} : \alpha, \beta)$.

- Fix a directed system of fragment embeddings $(\pi_{\alpha,\beta} : \alpha < \beta < \mu)$.

- Fix $\mathbb{A}^*$ and $(\pi_{\alpha,*} : \alpha < \mu)$ a direct limit.

- Suppose that $\mathcal{M}_\alpha$ is a $\tau_\alpha$-structure for all $\alpha < \mu$.

- Suppose $\sigma_{\alpha,\beta} : \mathcal{M}_\alpha \to \mathcal{M}_\beta$ is a commuting family of $\tau_\alpha$-embedding for $\alpha < \beta < \mu$.

**Definition 7.18** We say that $(\mathcal{M}_\alpha, \sigma_{\alpha,\beta} : \alpha < \beta < \mu)$ is a *twisted elementary system* if

$$\mathcal{M}_\alpha \models \psi(\overline{a}) \Leftrightarrow \mathcal{M}_\beta \models \pi_{\alpha,\beta}(\psi)(\sigma_{\alpha,\beta}(\overline{a}))$$

for all $\psi \in \mathbb{A}_\alpha$ and $\alpha < \beta < \mu$.

Note that since $\pi_{\alpha,\beta}$ is the identity on atomic $\tau_\alpha$-formulas this is consistent with being a $\tau_\alpha$-embedding.

The following lemma is a twisted version of the standard theorems on elementary chains.

**Lemma 7.19** *Suppose $(\mathcal{M}_\alpha, \sigma_{\alpha,\beta} : \alpha < \beta < \mu)$ is a twisted elementary system. There is a $\tau^*$-structure $\mathcal{M}^*$ and $\tau_\alpha$-embeddings $\sigma_{\alpha,*} : \mathcal{M}_\alpha \to \mathcal{M}^*$ such that*

- *$\sigma_{\alpha,*} = \sigma_{\beta,*} \circ \sigma_{\alpha,\beta}$ for all $\alpha < \beta < \mu$ ;*
- *every element of $\mathcal{M}^*$ is in the image of $\sigma_{\alpha,*}$ for all sufficiently large $\alpha < \mu$;*
- *for $\psi \in \mathbb{A}_\alpha$, $\overline{a} \in \mathcal{M}_\alpha$*

$$\mathcal{M}_\alpha \models \psi(\overline{a}) \Leftrightarrow \mathcal{M}^* \models \pi_{\alpha,*}(\psi)(\sigma_{\alpha,*}(\overline{a}))$$

**Proof** Let $\mathcal{F}$ be all functions with domain $[\alpha, \mu)$ such that $f(\alpha) \in \mathcal{M}_\alpha$ and $f(\beta) = \sigma_{\alpha,\beta}(f(\alpha))$ for all $\alpha < \beta < \mu$.

**Exercise 7.20** Show that if $f, g \in \mathcal{F}$ and $f(\gamma) = g(\gamma)$, then $f$ and $g$ agree on their common domanin.

Thus we have an equivalence relation $f \sim g$ if and only if $\exists \gamma\ f(\gamma) = g(\gamma)$. Let $[f]$ denote the equivalence class of $f$ and let the universe of $\mathcal{M}^*$ be the set of all equivalence classes. Define $\sigma_{\alpha,*}(a) = [f]$ where $f(\alpha) = a$ and $f(\beta) = \sigma_{\alpha,\beta}(a)$ for all $\alpha < \beta < \mu$.

We can define a $\tau^*$-structure on $\mathcal{M}^*$ such that

$$\mathcal{M}_\alpha \models \theta(\overline{a}) \Leftrightarrow \mathcal{M}^* \models \theta(\sigma_{\alpha,*}(\overline{a}))$$

for all atomic formulas $\theta \in \mathbb{A}_\alpha$ and $\overline{a} \in \mathcal{M}_\alpha$.

**Exercise 7.21** Argue that $\mathcal{M}^*$ is well-defined.

We now prove by induction on complexity of formulas that

$$\mathcal{M}_\alpha \models \psi(\overline{a}) \Leftrightarrow \mathcal{M}^* \models \pi_{\alpha,*}(\psi)(\sigma_{\alpha,*}(\overline{a}))$$

for $\psi \in \mathbb{A}_\alpha$, $\overline{a} \in \mathcal{M}_\alpha$.

Consider the case where $\psi \in \mathbb{A}_\alpha$ is an infinite disjunction. Then $\pi_{\alpha,*}(\psi)$ is also an infinite disjuction. Suppose $\mathcal{M}_\alpha \models \psi(\overline{a})$. Then there is a disjunct $\theta(\overline{v})$ such that $\mathcal{M}_\alpha \models \theta(\overline{a})$. Since we have a fragment embedding $\pi_{\alpha,*}(\theta)$ is a disjunct of $\pi_{\alpha,*}(\phi)$. By induction, $\mathcal{M}^* \models \pi_{\alpha,*}(\theta)(\sigma_{\alpha,*}(\overline{a}))$. Thus $\mathcal{M}^* \models \pi_{\alpha,*}(\phi)(\sigma_{\alpha,*}(\overline{a}))$.

Suppose $\mathcal{M}^* \models \pi_{\alpha,*}(\psi)(\sigma_{\alpha,*}(\overline{a}))$. Then $\mathcal{M}^* \models \chi(\sigma_{\alpha,*}(\overline{a}))$ for some disjunct $\chi$ of $\pi_{\alpha,*}(\psi)$. There is $\alpha \leq \beta < \mu$ and $\theta \in \mathbb{A}_\beta$ such that $\pi_{\beta,*}(\theta) = \chi$.

Then by induction $\mathcal{M}_\beta \models \theta(\sigma_{\alpha,\beta}(\overline{a}))$ and $\mathcal{M}_\beta \models \pi_{\alpha,\beta}(\phi)(\sigma_{\alpha,\beta}(\overline{a}))$. Since we have a twisted elementary system, $\mathcal{M}_\alpha \models \phi(\overline{a})$.

**Exercise 7.22** Complete the induction on complexity. □

**Definition 7.23** We say that a twisted elementary system $(\mathcal{M}_\alpha, \sigma_{\alpha,\beta} : \alpha < \beta)$ is *atomic* if each $\mathcal{M}_\alpha$ is $\mathbb{A}_\alpha$ atomic and a formula $\theta(\overline{v}) \in \mathbb{A}_\alpha$ is $\mathbb{A}_\alpha$-complete if and only if $\pi_{\alpha,\beta}(\theta)$ is $\mathbb{A}_\beta$-complete for all $\alpha < \beta < \mu$.

**Lemma 7.24** *Suppose $(\mathcal{M}_\alpha, \sigma_{\alpha,\beta}, \alpha < \beta)$ is an atomic twisted elementary system. Then the direct limit $\mathcal{M}^*$ is atomic and $\theta(\overline{v}) \in \mathbb{A}_\alpha$ is $\mathbb{A}_\alpha$-complete if and only if $\pi_{\alpha,*}(\theta)$ is $\mathbb{A}^*$-complete.*

**Proof** The following exercise is the ($\Leftarrow$) direction.

**Exercise 7.25** Show that if $\theta(\overline{v}) \in \mathbb{A}_\alpha$ is not $\mathbb{A}_\alpha$-complete, then $\pi_{\alpha,*}(\theta)$ is not $\mathbb{A}^*$-complete.

For the converse, suppose $\theta(\overline{v}) \in \mathbb{A}_\alpha$ is not $\mathbb{A}_\alpha$-complete. Let $\chi(\overline{v}) \in \mathbb{A}^*$. Choose $\beta \geq \alpha$ and $\psi \in \mathbb{A}_\beta$ such that $\pi_{\beta,*}(\psi) = \chi$. Since $\pi_{\alpha,\beta}(\theta)$ is $\mathbb{A}_\beta$-complete,

$$\mathcal{M}_\beta \models \forall \overline{v}\ (\pi_{\alpha,\beta}(\theta)(\overline{v}) \rightarrow \psi(\overline{v})) \text{ or } \mathcal{M}_\beta \models \forall \overline{v}\ (\pi_{\alpha,\beta}(\theta)(\overline{v}) \rightarrow \neg\psi(\overline{v})).$$

Without loss of generality assume the former, then

$$\mathcal{M}^* \models \forall \overline{v}\ (\pi_{\alpha,*}(\theta)(\overline{v}) \rightarrow \chi(\overline{v}))$$

as desired.

If $\theta(\overline{v})$ isolates the type of $\overline{a}$ in $\mathcal{M}_\alpha$, then $\pi_{\alpha,*}(\theta)$ will isolate the type of $\pi_{\alpha,*}(\overline{a})$ in $\mathcal{M}^*$. □

**Exercise 7.26 [Laskowski]** Suppose $T$ is a complete atomic first order theory in a vocabulary $\tau$ of cardinality $\aleph_1$. We will prove that $T$ has an atomic model, a result due independently to Knight [25], Kueker [28] and Shelah [50].[9]

a) Show that we can find a filtration $(\tau_\alpha : \alpha < \omega_1)$ such that $\tau_\alpha \subseteq \tau_\beta$ for $\alpha < \beta$, $\tau_\alpha = \bigcup_{\beta<\alpha} \tau_\beta$ if $\alpha$ is a limit ordinal and if $\phi(\overline{v})$ is a $\tau_\alpha$-formula such that $T \models \exists \overline{v}\ \phi(\overline{v})$, then there is $\theta(\overline{v})$ a complete $\tau_\alpha$-formula such that $T \models \theta(\overline{v}) \rightarrow \phi(\overline{v})$.

[9] By contrast, Laskowski and Shelah [31] prove that for any $\kappa > \aleph_1$ there is a complete satisfiable atomic theory in a vocabulary of cardinality $\kappa$ with no atomic model.

b) Let $T_\alpha$ be the set of $\tau_\alpha$-sentences in $T$. Show that each $T_\alpha$ is a countable atomic theory. Let $\mathcal{M}_\alpha$ be a prime model of $T_\alpha$.

c) Do a direct limit construction to prove that $T$ has an atomic model.

**The generic Morley tree**

While we can build paths of height $\omega_1$ through the Morley tree and these paths determine satisfiable theories, we might have uncountably many types in $\bigcup_{\alpha<\omega_1} \mathbb{A}_\alpha$ so we can not continue further. We will give a variant of the construction, that allows us to continue our construction up to $\omega_2$ but at a significant cost. It will no longer be clear that we are building satisfiable theories. Indeed, we could do a similar construction beyond $\omega_2$, but we know from Hjorth's Theorem §5.36, that $\phi$ may have no models beyond $\aleph_1$.

Let $P$ be the set of finite functions with domain contained in $\omega$ and image contained in $\omega_1$. Suppose $G$ is a generic filter on $P$, then $\omega_1^{\mathbb{V}}$ is countable in $\mathbb{V}[G]$ and $\omega_1^{\mathbb{V}[G]} = \omega_2^{\mathbb{V}}$. For a, perhaps uncountable, fragment $\mathbb{A} \subset \mathcal{L}_{\omega_2,\omega}$ we will look in $\mathbb{V}[G]$ for satisfiable $\mathbb{A}$-complete theories $T$ containing $\phi$. The next lemma shows that these theories are already in $\mathbb{V}$, but, a priori, we can't conclude they have models in $\mathbb{V}$.

**Lemma 7.27** *Let $\mathbb{A} \in \mathbb{V}$ be a fragment of $\mathcal{L}_{\omega_2,\omega}$ of cardinality at most $\aleph_1$. Suppose $p \in P$, $\dot{T}$ is a $P$-name and $p \Vdash \dot{T}$ is a satisfiable $\mathbb{A}$-complete theory containing $\phi$. Then there is $q \leq p$ such that $q \Vdash \psi \in \dot{T}$ or $q \Vdash \psi \notin \dot{T}$ for every $\psi \in \mathbb{A}$. If $G \subset P$ is a generic filter and $p \in G$, then $\dot{T}_G \in \mathbb{V}$, where $\dot{T}_G$ is the element of $\mathbb{V}[G]$ named by $\dot{T}$.*

*Similarly, if $\dot{t}$ is a $P$-name and $p \Vdash$ "$\dot{t}$ is an $\mathbb{A}$-type", then there is $q \leq p$ that forces $\psi(\overline{v}) \in \dot{t}$ or $\psi(\overline{v})) \in \dot{t}$ for all $\psi(\overline{v}) \in \mathbb{A}$ and $\dot{t}_G \in \mathbb{V}$ for any generic $G \subset P$.*

**Proof** Let $\mathcal{A}$ be a transitive model of ZFC$^-$ of cardinality $\aleph_1$ such that $\phi, \mathbb{A}, P, p$ and $\dot{T}$ are in $\mathcal{A}$. Let $\mathcal{A}_0$ be a countable elementary submodel of $\mathcal{A}$ containing $\phi, \mathbb{A}, P, p$ and $\dot{T}$ and let $\mathcal{B}$ be the Mostowski collapse of $\mathcal{A}_0$. Thus we may assume we have a countable transitive model $\mathcal{B}$ of ZFC$^-$ + $\phi$ is scattered + there is a fragment $\widehat{\mathbb{A}}$ of $\mathcal{L}_{\omega_2,\omega}$ of cardinality at most $\aleph_1$+ $\exists p \in P\ p \Vdash \phi \in \dot{T} \subset \widehat{\mathbb{A}}$ is complete and satisfiable.

We claim that there is $q \in \mathcal{B}$ with $q \leq p$ such that $q \Vdash \phi \in \dot{T}$ or $q \Vdash \neg\phi \in \dot{T}$ for all $\phi \in \widehat{\mathbb{A}}$.

Suppose not. Let $\dot{M} \in \mathcal{B}$ be a $P$-name such that $p \Vdash \dot{M} \models \dot{T}$. Let $D_0, D_1, \dots,$ list all dense subsets of $P$ in $\mathcal{B}$. Working in $\mathbb{V}$ we build a tree $(p_\sigma : \sigma \in 2^{<\omega})$ where all $p_\sigma \in \mathcal{B}$ as follows:

- $p_\emptyset = p$;
- given $p_\sigma$ with $|\sigma| = n$, choose $p'_\sigma \leq p_\sigma$ with $p'_\sigma \in D_n$;
- by assumption there is $\psi \in \widehat{\mathbb{A}}$ such that $p'_\sigma$ does not force either $\psi \in \dot{T}$ or $\neg\psi \in \dot{T}$. Choose $p_{\sigma,0} \leq p'_\sigma$ such that $p_{\sigma,0} \Vdash \psi \in \dot{T}$ and choose $p_{\sigma,1} \leq p'_\sigma$ such that $p_{\sigma,1} \Vdash \neg\psi \in \dot{T}$.

For each $f \in 2^\omega$ we have a $\mathcal{B}$-generic filter $G_f = \{q : \exists n\ p_{f|n} \leq q\}$. Let $T_f = \dot{T}_{G_f}$ be the theory named by $\dot{T}$ in $\mathcal{B}[G_f]$ and let $\mathcal{M}_f \models T_f$ be the structure named by $\dot{M}$ in $\mathcal{B}[G_f]$.

Since $\mathcal{B}[G_f] \models \mathcal{M}_f \models T_f$ and this is absolute, $\mathcal{M}_f$ really is a model of $T_f$, so $T_f$ is satisfiable. For each $\psi \in \widehat{\mathbb{A}}$,

$$\{q : q \Vdash \psi \in \dot{T} \text{ or } q \Vdash \neg\psi \in \dot{T}\}$$

is dense below $p$. Thus $T_f$ is $\widehat{\mathbb{A}}$-complete. We also have $\phi \in T_f$ for all $f$ and, by construction, $T_f \neq T_g$ for $f \neq g$. But this contradicts the fact that $\phi$ is scattered.

Thus there is $q \leq p$ such that in any generic extension

$$\dot{T}_G = \{\phi \in \mathbb{A} : q \Vdash \phi \in \dot{T}\}.$$

But, by the definability of forcing, $\dot{T}_G \in \mathbb{V}$.

The argument for types is identical. □

We now build the *generic Morley tree*. It is worth noting, that even though we are using forcing, the entire construction takes place in $\mathbb{V}$. Let $\mathbb{A}^*_0 = \mathbb{A}_0$. If $\alpha$ is a limit ordinal, then $\mathbb{A}^*_\alpha = \bigcup_{\beta<\alpha} \mathbb{A}^*_\beta$.

Suppose $\alpha < \omega_2$ and $\mathbb{A}^*_\alpha$ is a fragment of $\mathcal{L}_{\omega_2,\omega}$ of cardinality at most $\aleph_1$. Let $\mathcal{T}^*_\alpha = \{T \subset \mathbb{A}^*_\alpha : \phi \in T$ and $p \Vdash$ "$T$ is a satisfiable $\mathbb{A}^*_\alpha$-complete theory and no $T \cap \mathbb{A}^*_\beta$ is complete for $\beta < \alpha$" for some $p \in P\}$. For each $p \in P$ there are at most $\aleph_1$ such $T$. Thus $|\mathcal{T}^*_\alpha| \leq \aleph_1$.

Let $S = \{\gamma(\overline{v}) \subset \mathbb{A}^*_\alpha : p \Vdash$ "$\gamma$ is an $\mathbb{A}_\alpha$-type realized in some $\mathcal{M} \models \phi$" for some $p \in P\}$. As above $|S| \leq \aleph_1$. Let $\mathbb{A}^*_{\alpha+1}$ be the smallest fragment of $\mathcal{L}_{\omega_2,\omega}$ generated by $\mathbb{A}^*_\alpha$ and all formulas $\bigwedge_{\psi\in\gamma}\psi(\overline{v})$ for $\gamma \in S$. Note that $|\mathbb{A}^*_{\alpha+1}| \le \aleph_1$.

**Exercise 7.28** a) Prove that if $p \Vdash$ "$T$ is a satisfiable $\mathbb{A}^*_\alpha$-complete theory and no $T\cap\mathbb{A}^*_\beta$ is complete for $\beta < \alpha$", then this is also forced by $\emptyset$. [Hint: $P$ is *almost-homogeneous*. See Exercises (E1) and (E2) of §VII of [29].]

b) Prove that $\mathcal{T}^*$ is the Morley tree in $\mathbb{V}[G]$.

c) Prove that $\mathbb{A}_\alpha = \mathbb{A}^*_\alpha$ and $\mathcal{T}_\alpha = \mathcal{T}^*_\alpha$ for all $\alpha < \omega_1$.

Suppose $T \in \mathcal{T}^*_\alpha$. In any generic extension $\mathbb{V}[G]$, $T$ is an $\mathbb{A}^*_\alpha$-atomic theory. While, for the moment, we don't know that $T$ is satisfiable in $\mathbb{V}$, it still in many ways behaves like an atomic theory. We say that $\theta(\overline{v}) \in \mathbb{A}^*_\alpha$ is a *generic atom* in $T$ if

- $\exists\overline{v}\ \theta(\overline{v}) \in T$ and
- $\forall\overline{v}\ (\theta(\overline{v}) \to \psi(\overline{v})) \in T$ or $\forall\overline{v}\ (\theta(\overline{v}) \to \neg\psi(\overline{v})) \in T$ for all $\psi(\overline{v}) \in \mathbb{A}^*_\alpha$.

Note that if $T$ is satisfiable the generic atoms are atoms.[10] In $\mathbb{V}[G]$ the generic atoms are exactly the atoms and the atoms are dense. Since the statement that the generic atoms are dense is $\Delta_0$, the generic atoms are still dense in $\mathbb{V}$, i.e., for all $\psi(\overline{v}) \in \mathcal{T}$ if $\exists\overline{v}\ \psi(\overline{v}) \in T$, then there is a generic atom $\theta(\overline{v}) \in \mathbb{A}^*_\alpha$ such that

$$\forall\overline{v}\ (\theta(\overline{v}) \to \psi(\overline{v})) \in T.$$

This will also be true in any transitive substructure containing $T$ and $\mathbb{A}^*_\alpha$.

**Lemma 7.29** *If $T \in \mathcal{T}^*_\alpha$, then $T$ is satisfiable.*

**Proof** To simplify notation we let $\mathbb{A}$ denote $\mathbb{A}^*_\alpha$. The ordinal $\alpha$ plays no further role in the proof.

Let $\mathcal{B}$ be a transitive model of ZFC$^-$ of cardinality $\aleph_1$ such that $\mathbb{A}, T \in \mathcal{B}$. Let $(\mathcal{B}_\beta : \beta < \omega_1)$ be a continuous elementary chain of countable models such that $\mathbb{A}, T \in \mathcal{B}_0$, $\mathbb{A}_0 \subset \mathcal{B}_0$ and $\mathcal{B} = \bigcup_{\beta<\omega_1}\mathcal{B}_\beta$. For each $\beta < \omega_1$ consider $B_\beta$ the transitive collapse of $\mathcal{B}_\beta$. Let $p_\beta : \mathcal{B}_\beta \to B_\beta$ be the Mostowski collapse isomorphism. If $\gamma < \beta$ we have an elementary embedding $\pi_{\gamma,\beta} : B_\gamma \to B_\beta$ given by $\pi_{\gamma,\beta} = p_\beta \circ p_\gamma^{-1}$.

For each $\beta < \omega_1$ let $T_\beta = p_\beta(T)$ and $A_\beta = p_\beta(\mathbb{A})$. Note that $\pi_{\gamma,\beta}$ maps $A_\gamma$ into $A_\beta$ for $\gamma < \beta$.

[10] Here we use *atom* interchangeably with "$\mathbb{A}_\alpha$-complete formula".

**Exercise 7.30** i) Let $\gamma < \beta$. Show that the map $\pi_{\gamma,\beta}$ is a fragment embedding of $A_\gamma$ into $A_\beta$. [Hint: Use the fact that $\pi_{\gamma,\beta} : B_\gamma \to B_\beta$ is elementary.]

ii) Show that if $\beta < \omega_1$ is a limit ordinal, then $A_\beta$ is the direct limit of $(A_\gamma, \pi_{\gamma,\delta} : \gamma < \delta < \beta)$ (where, here, $A_{\omega_1}$ denotes $\mathbb{A}$).

**claim 1** i) $T_\beta$ is a satisfiable $A_\beta$-atomic theory for all $\beta < \omega_1$.

ii) The atoms of $T_\beta$ are exactly the formulas that $B_\beta$ believes are generic atoms.

iii) If $\psi$ is an atom of $T_\gamma$ and $\gamma < \beta$, then $\pi_{\gamma,\beta}(\psi)$ is an atom of $T_\beta$.

$B_\beta \models$ "$\exists p \in P \; p \Vdash T_\beta$ is satisfiable". Since $\mathcal{B}_\beta$ is countable, in $\mathbb{V}$ we can build an $\mathcal{B}_\beta$-generic $G \subset P$ such that $\mathcal{B}_\beta[G] \models \exists \mathcal{N} \models T_\beta$. But then $\mathcal{N}$ really is a model of $T_\beta$ in $\mathbb{V}$! Since $\phi$ is scattered and $A_\beta$ is countable, $T_\beta$ is $A_\beta$-atomic. As $T_\beta$ is satisfiable the generic atoms are exactly the atoms. Moreover being a generic atom is a first order property, thus if $\psi \in A_\gamma$ is an atom, so is $\pi_{\gamma,\beta}(\psi)$.

Let $\mathcal{M}_\beta$ be the $A_\beta$-prime model of $T_\beta$ for $\beta < \omega_1$.

**claim 2** There is an atomic system of twisted elementary embedding $\sigma_{\gamma,\beta} : \mathcal{M}_\gamma \to \mathcal{M}_\beta$ for $\gamma < \beta < \omega_1$.

We build the maps inductively. Suppose $\beta = \gamma + 1$. We will build $\sigma_{\gamma,\beta}$ and let $\sigma_{\delta,\beta} = \sigma_{\gamma,\beta} \circ \sigma_{\delta,\gamma}$ for $\delta < \gamma$.

We build a sequence of finite maps $f_0 \subset f_1 \subset \ldots$ from $\mathcal{M}_\gamma$ to $\mathcal{M}_\beta$ such that if $\overline{a} \in \mathrm{dom} f_n$, then

$$\mathcal{M}_\gamma \models \psi(\overline{a}) \Leftrightarrow \mathcal{M}_\beta \models \pi_{\gamma,\beta}(\psi)(f_n(\overline{a})).$$

If we can do this, we can let $\sigma_{\gamma,\beta} = \bigcup_{n\in\omega} f_n$.

Let $f_0 = \emptyset$. If $\psi$ is a sentence in $T_\gamma$, then $\pi_{\gamma,\beta}(\psi) \in T_\beta$. Thus

$$\mathcal{M}_\gamma \models \psi \Leftrightarrow \mathcal{M}_\beta \models \pi_{\gamma,\beta}(\psi)$$

as desired.

Suppose $\overline{a}$ is the domain of $f_n$ and $b \in \mathcal{M}$. There is $\theta(\overline{v}, w) \in A_\gamma$ isolating the $A_\gamma$-type of $(\overline{a}, b)$. By induction, for any formula $\psi(\overline{v}, w) \in A_\beta$

$$\mathcal{M}_\beta \models \forall \overline{v}, w \; (\pi_{\gamma,\beta}(\theta)(\overline{v}, w) \to \psi(\overline{v}, w))$$

$$\text{or } \mathcal{M}_\beta \models \forall \overline{v}, w \; (\pi_{\gamma,\beta}(\theta)(\overline{v}, w) \to \neg\psi(\overline{v}, w))$$

and

$$\mathcal{M}_\beta \models \exists w \; \pi_{\gamma,\beta}(\theta)(f_n(\overline{a}), w).$$

Then we can extend $f_n$ by sending $b$ to $c$.

Suppose $\beta < \omega_1$ is a limit ordinal and we have constructed a commuting system of twisted elementary $\sigma_{\delta,\gamma} : \mathcal{M}_\delta \to \mathcal{M}_\gamma$ for all $\delta < \gamma < \beta$. By Lemma 7.24 we can build $\mathcal{M}^*$ an an atomic twisted direct limit. But for $\gamma < \beta$ and $\psi \in A_\gamma$,

$$\mathcal{M}^* \models \pi_{\gamma,\beta}(\psi) \Leftrightarrow \mathcal{M}_\gamma \models \psi \Leftrightarrow \psi \in T_\gamma \Leftrightarrow \pi_{\gamma,\beta}(\psi) \in T_\beta.$$

Thus $\mathcal{M}^*$ is an $A_\beta$-atomic model of $T_\beta$, so we can take $\mathcal{M}_\beta$ to be $\mathcal{M}^*$. Let $\sigma_{\gamma,\beta} = \sigma_{\gamma,*}$ be the maps obtained in the twisted direct limit construction.

This completes the proof of construction of an atomic twisted elementary system $(\mathcal{M}_\gamma, \pi_{\gamma,\beta} : \gamma < \beta < \omega_1)$.

Let $\mathcal{M}^*$ be the twisted direct limit of the system. Letting $\pi_{\beta,*} : A_\beta \to \mathbb{A}$ be $p_b^{-1}$ we see that $\mathbb{A}$ is the direct limit of the system $(A_\gamma, \pi_{\gamma,\beta} : \gamma < \beta < \omega_1)$. For $\psi \in \mathbb{A}$. Choose $\beta$ such that $\psi \in \mathcal{B}_\beta$, then

$$\psi \in T \Leftrightarrow p_\beta(\psi) \in T_\beta \Leftrightarrow \mathcal{M}_\beta \models p_\beta(\psi) \Leftrightarrow \mathcal{M}^* \models \pi_{\beta,*}(p_\beta(\psi)) \Leftrightarrow \mathcal{M}^* \models \psi.$$

Thus $\mathcal{M}^*$ is the desired model of $T$. □

**Exercise 7.31** Adapt the argument from Lemma 7.14 to show that if $\alpha < \omega_2$ is a limit ordinal and $\mathcal{M} \models T \in \mathcal{T}^*_\alpha$, then $\mathcal{M}$ has Scott rank at least $\alpha$.

**Corollary 7.32 (Harrington)** *If $\phi$ is a Vaught counterexample, then there are models of $\phi$ of arbitrarily large Scott rank below $\omega_2$.*

**Corollary 7.33** *If $\phi$ is a Vaught counterexample, then $I(\phi, \aleph_1) \geq \aleph_2$.*

This leads to the following question which we state as a conjecture, though the only real evidence is that it is true in the first order case and true if $2^{\aleph_1} = \aleph_2$.

**Conjecture** If $\phi$ is a Vaught counterexample, then $I(\phi, \aleph_1) = 2^{\aleph_1}$.

Interestingly, the number of $\equiv_{\infty,\omega}$-classes of models of size $\aleph_1$ is bounded.

**Proposition 7.34** *If $\phi$ is a Vaught counterexample, then the number of $\equiv_{\infty,\omega}$-classes of models of $\phi$ of cardinality $\aleph_1$ is exactly $\aleph_2$.*

**Proof** By Harrington's Theorem there are at least $\aleph_2$ equivalence classes. Suppose there are more. Let $\kappa > \aleph_2$ and suppose that for $\alpha < \kappa$ we have $\mathcal{M}_\alpha \models \phi$ of cardinality $\aleph_1$ with $\mathcal{M}_\alpha \not\equiv_{\infty,\omega} \mathcal{M}_\beta$ for $\alpha \neq \beta$.

Let $P$ be finite partial functions from $\omega$ to $\omega_1$ and let $G \subset P$ be a generic filter. Each $\mathcal{M}_\alpha$ is countable in $\mathbb{V}[G]$ and for $\mathbb{V}[G] \models \mathcal{M}_\alpha \not\equiv_{\infty,\omega} \mathcal{M}_\beta$ for $\alpha \neq \beta$. Since $\kappa > \aleph_1^{\mathbb{V}[G]}$, this contradicts the fact that $\mathbb{V}[G] \models$"$\phi$ is scattered". $\square$

# 8 Quasiminimal Excellence

## 8.1 Quasiminial Excellence and Categoricity

## 8.2 Excellence Shmexcellece

## 8.3 Covers of $\mathbb{C}^{\times}$

# Part III
# Effective Considerations

## 9 Effective Descriptive Set Theory

A number of insights about the model theory of $\mathcal{L}_{\omega_1,\omega}$ come from the perspective of effective descriptive set theory or generalized recursion theory. We will look at some of these in section 11. In this section and the next, we give a brief introduction to effective descriptive set theory and related results about hyperarithmetic sets. We assume that the reader has some familiarity with the basics of descriptive set theory of Borel, analytic and coanalytic sets as presented in, say, Kechris [21]. Sacks [46] Moschovakis [41], Kechris' portion of [38] and Mansfield and Weitkamp [34] are excellent references for effective descriptive set theory.

### 9.1 Recursion Theory Review

We recall some of the basic ideas we will need from recursion theory. We will take an informal approach where we don't specify a machine model, but all of this could easily be made precise. We assume that the reader has some intuitive idea what a "computer program" is. This could be a simple model such as Turing machine programs or an informal notion like a $\mathrm{C}^{++}$ program.

**Definition 9.1** A partial function $f : \omega \to \omega$ is *partial recursive* if there is a computer program $P$ such that $P$ halts on input $n$ if and only if $n \in \mathrm{dom}(f)$ and if $P$ halts on input $n$, then the output is $f(n)$. We say that a set $A \subseteq \omega$ is *recursive* if and only if its characteristic function is recursive.

We can code computer programs by integers so that each integer codes a program. Let $P_e$ be the machine coded by $e$. Let $\phi_e$ be the partial recursive function computed by $P_e$. We write $\phi_e(n)\downarrow_s$ if $P_e$ halts on input $n$ by stage $s$ and $\phi_e(n)\downarrow$ if $P_e$ halts on input $n$ at some stage. Our enumeration has the following features:

**Fact 9.2** *i)* [**Universal Function**] *The function* $(e, n) \mapsto \phi_e(n)$ *is partial recursive.*

*ii) The set* $\{(e, n, s) : \phi_e(n)\downarrow_s\}$ *is recursive.*

*iii)* [**Halting Problem**] *The set* $\{(e, n) : \phi_e(n)\downarrow\}$ *is not recursive.*

*iv)* [**Parameterization Lemma**] *If $F : \omega^2 \to \omega$ is partial recursive, then there is a total recursive $d : \omega \to \omega$ such that*

$$\phi_{d(x)}(y) = F(x, y)$$

*for all $x, y$.*

**Definition 9.3** We say that $A \subseteq \omega$ is *recursively enumerable* if there is a partial recursive function $f : \omega \to \omega$ such that $A$ is the image of $f$.

**Fact 9.4** *The following are equivalent*
*i) $A$ is recursively enumerable*
*ii) $A$ is the domain of a partial recursive function.*
*iii) $A = \emptyset$ or $A$ is the image of a total recursive function.*
*iv) there is a recursive $B$ such that $A = \{n : \exists m\ (n, m) \in B\}$.*

**Fact 9.5** *a) If $A$ and $B$ are recursively enumerable, then so are $A \cup B$ and $A \cap B$.*
*b) If $A \subseteq \omega \times \omega$ is recursively enumerable so is $\{n : \exists m\ (n, m) \in A\}$.*
*c) If $A$ is recursively enumerable and $f : \omega \to \omega$ is total recursive, then $f^{-1}(A)$ is recursively enumerable.*

**Exercise 9.6** If you have not seen them before prove the statements in the Facts above.

A program with oracle $x \in \omega^\omega$ is a computer program which, in addition to the usual steps, is allowed at any stage to ask the value of $x(n)$.

We say that $f$ is *partial recursive in $x$* if there is a program with oracle $x$ computing $f$ and say that $A \subseteq \omega$ is *recursive in $x$* if the characteristic function of $A$ is recursive in $x$. The facts above relativize to oracle computations. We write $\phi_e^x(n)$ for the value of the partial recursive function in $x$ with oracle program $P_e$ on input $n$. One additional fact is useful.

**Fact 9.7 (Use Principle)** *If $\phi_e^x(n)\downarrow$, then there is $m$ such that if $x|m = y|m$, then $\phi_e^y(n) = \phi_e^x(n)$.*

**Proof** The computation of $P_e$ with oracle $x$ on input $n$ makes only finitely many queries about $x$. Choose $m$ greater than all of the queries made. $\square$

We may also consider programs with finitely many oracles.

For $x, y \in \omega^\omega$ we say $x$ is *Turing-reducible* to $y$ and write $x \leq_T y$ if $x$ is recursive in $y$. For $x, y \in \omega^\omega$ define $x \oplus y$ to be $(x(0), y(0), x(1), y(1), \ldots$. The $x \oplus y$ is a least upper bound for $x$ and $y$ under $\leq_T$. We call $x \oplus y$ the *join* of $x$ and $y$.

There is another useful notion of reducibility which is the analog of Wadge reducibility in descriptive set theory.

**Definition 9.8** We say $A$ is *many-one reducible* to $B$ if there is a total recursive $f$ such that $n \in A$ if and only if $f(n) \in B$ for all $n \in \omega$. We write $A \leq_m B$ if $A$ is many-one reducible to $B$.

Clearly if $A \leq_m B$, then $A \leq_T B$.

There is one subtle fact that will eventually play a key role.

**Theorem 9.9 (Recursion Theorem)** *If $f : \omega \to \omega$ is total recursive, there is an $e$ such that $\phi_e = \phi_{f(e)}$ .*

**Proof** Let

$$F(x, y) = \begin{cases} \phi_{\phi_x(x)}(y) & \text{if } \phi_x(x)\downarrow \\ \uparrow & \text{if } \phi_x(x)\uparrow \end{cases}.$$

By the Parameterization Lemma, there is a total recursive $d$ such that

$$\phi_{d(x)}(y) = F(x, y).$$

Let $\psi = f \circ d$. There is $m$ such that $\psi = \phi_m$. Since $\psi$ is total $\phi_{d(m)} = \phi_{\phi_m(m)}$. Let $e = d(m)$. Then

$$\phi_e = \phi_{d(m)} = \phi_{\phi_m(m)} = \phi_{\psi(m)} = \phi_{f(d(m))} = \phi_{f(e)}.$$

□

For those of you who have not seen this before here are two samples of the many applications of the Recursion Theorem.

Suppose $F : \omega^2 \to \omega$. By the Parameterization Lemma there is a total recursive $d : \omega \to \omega$ such that $\phi_{d(m)}(n) = F(m, n)$ for all $m$ and $n$. By the Recursion Theorem there is an $e$ such that

$$\phi_e(n) = \phi_{d(e)}(n) = F(e, n)$$

for all $n$.

For a second example, consider

$$g(e,n) = \begin{cases} 1 & \text{if } e = n \\ 0 & \text{if } e \neq n \end{cases}.$$

By the Parameterization Lemma, there is a total recursive $f$ such that $\phi_{f(e)}(n) = g(e,n)$. By the Recursion Theorem there is an $e$ such that $\phi_e(n) = 1$ if $n = e$ and $\phi_e(n) = 0$ if $n \neq e$. So this function "recognizes" its own code. The Recursion Theorem will be very useful in §7.

## 9.2 Computable Functions on $\omega^\omega$

There is also a notion of computable function $f : \omega^\omega \to \omega^\omega$.

**Definition 9.10** We say that $f : \omega^\omega \to \omega^\omega$ is *computable* if there is an oracle program $P$ such that if $x \in \omega^\omega$ and $P$ is run with oracle $x$ on input $n$, then $P$ halts and outputs $f(x)(n)$.

We say that $f : \omega^\omega \to \omega^\omega$ is computable from $z$ if there is a two oracle program $P$ such that if $x \in \omega^\omega$ and $P$ is run with oracles $z$ and $x$ on input $n$, then $M$ halts and outputs $f(x)(n)$.

Recall that for $\sigma \in \omega^{<\omega}$ we let $N_\sigma = \{f \in \omega^\omega : \sigma \subset f\}$. The sets $\mathcal{N}_\sigma$ are the basic open sets in the topology on $\omega^\omega$.

**Lemma 9.11** *$f : \omega^\omega \to \omega^\omega$ is continuous if and only if there is $z \in \omega^\omega$ such that $f$ is computable from $z$.*

**Proof**

($\Leftarrow$) Let $P$ be the oracle program computing $f$ from $z$. Suppose $f(x) = y$. By the Use Principle, for any $m$ there is an $n$ such that if $x|n = w|n$, then $f(w)|m = y|m$. Thus $N_{x|n} \subseteq f^{-1}N_{y|m}$ and $f$ is continuous.

($\Rightarrow$) Let $X = \{(\tau,\sigma) : f^{-1}(N_\sigma) \supseteq N_\tau\}$. Since $f$ is continuous, for all $x \in \omega^\omega$ if $f(x) = y$, then for all $n$ there is an $m$ such that $(x|m, y|n) \in X$.

We claim that $f$ is computable from $X$. Suppose we are given oracles $X$ and $x$ and input $n$. We start searching $X$ until we find $(\tau,\sigma) \in X$ such that $\tau \subset x$ and $|\sigma| > n$. Then $f(x)(n) = \sigma(n)$. □

## 9.3 The Arithmetic Hierarchy

For the next few sections we will restrict our attention to Polish spaces $X = \omega^k \times (\omega^\omega)^l \times (2^\omega)^m$ where $k, l.m \geq 0$. Any such space is homeomorphic to one of $\omega, 2^\omega$ or $\omega^\omega$. (In [41] this theory is worked out for "recursively presented Polish spaces".)

Let $X = \omega^k \times (\omega^\omega)^l$. Let $S_X = \{(m_1, \ldots, m_k, \sigma_1, \ldots, \sigma_l) : m_i, \ldots, m_k \in \omega, \sigma_1, \ldots, \sigma_l \in \omega^{<\omega}\}$. For $\sigma = (m_1, \ldots, m_k, \sigma_1, \ldots, \sigma_l) \in S_X$, let

$$N_\sigma = \{(n_1, \ldots, n_k, f_1, \ldots, f_l) \in X : n_i = m_i \text{ if } i \leq k \text{ and } f_i \supset \sigma_i \text{ if } \leq l\}.$$

Then $\{N_\sigma : \sigma \in S_X\}$ is a clopen basis for the topology on $X$. Of course $S_X$ is a countable set and there is a recursive bijection $i \mapsto \sigma_i$ between $\omega$ and $S_X$, such that all natural operations on sequences are given by recursive functions. Thus we can identify $S_X$ with $\omega$ and talk about things like recursive subsets of $S_X$ and partial recursive functions $f : \omega \to S_X$.

**Definition 9.12** We say that $A \subseteq X$ is $\Sigma_1^0$ if there is a partial recursive $f : \omega \to S_X$ such that $A = \bigcup_n N_{f(n)}$.

Note that here we are using a "lightface" $\Sigma_1^0$ rather than the "boldface" $\mathbf{\Sigma}_1^0$ that denotes the open subsets of $X$. Of course every $\Sigma_1^0$ set is open, but there are only countably many partial recursive $f : \omega \to S_X$ thus there are only countably many $\Sigma_1^0$ sets. Thus $\Sigma_1^0 \subset \mathbf{\Sigma}_1^0$. Relativizing these notions we get all open sets.

**Definition 9.13** If $x \in \omega^\omega$ we say that $A \subseteq X$ is $\Sigma_1^0(x)$ if there is $f : \omega \to S_X$ partial recursive in $x$ such that $A = \bigcup_n N_{f(n)}$.

**Lemma 9.14** $\mathbf{\Sigma}_1^0 = \displaystyle\bigcup_{x \in \omega^\omega} \Sigma_1^0(x)$.

We will tend to prove things only for $\Sigma_1^0$ sets. The relativization to $\Sigma_1^0(x)$ sets is usually straightforward.

For the two interesting examples $X = \omega$ and $X = \omega^\omega$ we get slightly more informative characterizations.

**Lemma 9.15** *i) $A \subseteq X$ is $\Sigma_1^0$ if and only if there is a recursively enumerable $W \subseteq S_X$ such that $A = \bigcup_{\eta \in W} N_\eta$. In particular $A \subseteq \omega$ is $\Sigma_1^0$ if and only if $A$ is recursively enumerable.*

*ii) $A \subseteq \omega^\omega$ is $\Sigma_1^0$ if and only if there is a recursive $S \subseteq \omega^{<\omega}$ such that $A = \bigcup_{\sigma \in S} N_\sigma$.*

**Proof**

i) This is clear since the recursively enumerable sets are exactly the images of partial recursive functions.

ii) In this case $S_X = \omega^{<\omega}$. Clearly if $S \subseteq \omega^{<\omega}$ is recursive, there is $f : \omega \to \omega^{<\omega}$ partial recursive with image $S$ and $\bigcup_{\sigma\in\omega^{<\omega}} N_\sigma$ is $\Sigma^0_1$.

Suppose $A = \bigcup_n N_{f(n)}$ where $f$ is partial recursive let $S = \{\sigma :$ there is $n \leq |\sigma|$, the computation of $f(n)$ halts by stage $|\sigma|$ and $f(n) \subseteq \sigma$. It is easy to see that $S$ is recursive. If $\sigma \in S$, then there is an $n$ such that $f(n) \subseteq \sigma$, then $N_\sigma \subseteq N_{f(n)} \subseteq A$. On the other hand if $f$ halts on input $n$, there is $m \geq n, |f(n)|$ such that $f$ halts by stage $m$. If $\tau \supset \sigma$ and $|\tau| \geq m$, then $\tau \in S$. Thus

$$\bigcup_{\sigma\in S} N_\sigma \supset \bigcup_{\tau\supset f(n),|\tau|=m} N_\tau = N_{f(n)}.$$

It follows that $A = \bigcup_{\sigma\in S} N_\sigma$. □

We have natural analogs of the finite levels of the Borel hierarchy.

**Definition 9.16** Let $X = \omega^k \times (\omega^\omega)^l$. We say that $A \subseteq X$ is $\Pi^0_n$ if and only if $X \setminus A$ is $\Sigma^0_n$.

We say that $A \subseteq X$ is $\Sigma^0_{n+1}$ if and only if there is $B \subseteq \omega \times X$ in $\Pi^0_n$ such that

$$x \in A \text{ if and only if } \exists n\ (n, x) \in B.$$

We say that $A$ is $\Delta^0_n$ if it is both $\Sigma^0_n$ and $\Pi^0_n$.

We say that $A \subseteq X$ is *arithmetic* if $A \in \Delta^0_n$ for some $n$.

**Lemma 9.17** *$A \subseteq \omega^\omega$ is $\Pi^0_1$ if and only if there is a recursive tree $T \subseteq \omega^{<\omega}$ such that $A = [T]$.*

**Proof** Clearly,

$$[T] = \{f : \forall n\ f|n \in T\}$$

is $\Pi^0_1$. On the other hand, if $S$ is a recursive set such that $X \setminus A = \bigcup_{\sigma\in S} N_\sigma$, let

$$T = \{\sigma \in \omega^{<\omega} : \forall m \leq |\sigma|\ \sigma|m \notin S\}.$$

Then $T$ is a recursive tree and $A = [T]$. □

Recall that a tree $T$ is *pruned* if for all $\sigma \in T$ there is $f \in [T]$ with $\sigma \subset f$. If $A$ is closed, then we can find a pruned tree such that $A = [T]$, but the next exercises shows that it is not always possible to find a recursive pruned tree.

**Exercise 9.18** a) Show that if $T$ is a recursive pruned tree, then the leftmost path through $T$ is recursive.

b) Let $T = \{\sigma \in \omega^{<\omega} : \text{if } e < |\sigma| \text{ and } \phi_e(e) \text{ halts by stage } |\sigma|, \text{ then } \phi_e(e) \text{ halts by stage } \sigma(e)\}$. Show that $T$ is a recursive tree. Suppose $f \in [T]$. Show that $\phi_e(e)$ halts if and only if it halts by stage $f(e)$. Conclude that there are no recursive paths through $T$ and, using a), that there is no nonempty recursive pruned subtree of $T$.

We show that $\Sigma^0_n$ and $\Pi^0_n$ have closure properties analogous to those for the corresponding levels of the Borel hierarchy. The definition of computable function made in 9.10 makes sense for maps $f : X \to Y$ where both $X$ and $Y$ are of the form $\omega^k \times (\omega^\omega)^l$

**Lemma 9.19** *i) $\Sigma^0_n$ is closed under finite unions, finite intersections, and computable inverse images.*

*ii) If $A \subseteq \omega \times X \in \Sigma^0_n$, then $\{x \in X : \exists n\ (n, x) \in A\} \in \Sigma^0_n$.*

*iii) If $f : X \to \omega$ is computable and $A \subseteq \omega \times X$ is $\Sigma^0_n$ then $\{x \in X : \forall m < f(x)\ (m, x) \in A\} \in \Sigma^0_n$.*

*iv) Similarly $\Pi^0_n$ is closed under union, intersection, computable inverse images, $\forall n$ and $\exists n < f(x)$.*

*v) $\Sigma^0_n \subseteq \Delta^0_{n+1}$.*

**Proof** We prove i)–iii) this for $\Sigma^0_1$ and leave iv), v) and the the induction steps for i)–iii) as an exercise. as an exercise.

i) Suppose $W_0$ and $W_1$ are recursively enumerable subsets of $S_X$ and $A_i = \bigcup_{\eta \in W_i} N_\eta$. Replacing $W_i$ by the recursively enumerable set $\{\nu : \exists \eta \in W_i\ \eta \subseteq \nu\}$ if necessary we may assume that if $\nu \in W_i$ and $\eta \supset \nu$, then $\eta \in W_i$. Then

$$A_0 \cup A_1 = \bigcup_{\eta \in W_0 \cup W_1} N_\eta$$

and

$$A_0 \cap A_1 = \bigcup_{\eta \in W_0 \cap W_1} N_\eta$$

and $W_0 \cup W_1$ and $W_0 \cap W_1$ are recursively enumerable. Thus $A_0 \cup A_1$ and $A_0 \cap A_1$ are $\Sigma^0_1$.

If $f : X \to Y$ is computable with program $P_e$, let

$$G = \{(\eta, \nu) \in S_X \times S_Y : |eta| = |\nu|, x \in N_\eta \Rightarrow f(x) \in N_\nu\}.$$

Then $(\eta, \nu) \in G$ if and only if for all $m < |\nu|$ the program $P_e$ using oracle $\eta$ halts on input $m$ and outputs $\nu(m)$.[11] Thus $G$ is recursively enumerable. Suppose $A = \bigcup_{\nu \in W} N_\nu$ where $W$ is recursively enumerable, let $V = \{\eta : \exists \nu\ (\nu \in W \wedge (\eta, \nu) \in G\}$. Then $V$ is recursively enumerable and $f^{-1}(A) = \bigcup_{\eta \in V} N_\eta$.

ii) Suppose $A \subseteq \omega \times X$ is $\Sigma^0_1$. There is a recursively enumerable $W \subseteq S_{\omega \times X}$ such that $A = \bigcup_{\eta \in W} N_\eta$. Let $V = \{\nu \in S_X : \exists n\ (n, \nu) \in W\}$. Then $V$ is recursively enumerable and

$$\{x : \exists n\ (n, x) \in A\} = \bigcup_{\nu \in V} N_\nu.$$

iii) Suppose $A$ and $W$ are as in ii) and $f : X \to \omega$ is computable by program $P_e$. Let $V = \{\nu \in S_X : \exists k\ P_e$ with oracle $\nu$ halts outputting $k$ and $(m, \nu) \in W$ for all $m \leq k\}$. Then $V$ is recursively enumerable and

$$\{x : \forall m < f(x)\ (m, x) \in A\} = \bigcup_{\nu \in V} N_\nu\}.$$

□

**Exercise 9.20** Complete the proof of 9.19

**Definition 9.21** Let $\Gamma$ be a collection of subsets of $X$. We say that $U \subseteq Y \times X$ is $\Gamma$-*universal* if:

i) $\{x \in X : (y, x) \in U\} \in \Gamma$ for all $y \in Y$;

ii) if $A \in \Gamma$, then there is $y \in Y$ such that $A = \{x \in X : (y, x) \in U\}$.

**Proposition 9.22** *i) There is $U_n \subseteq \omega^\omega \times X$ a $\Sigma^0_n$-set that is $\boldsymbol{\Sigma}^0_n$-universal.*

*ii) There is $V_n \subseteq \omega \times X$ a $\Sigma^0_n$-set that is $\Sigma^0_n$-universal.*

**Proof**

i) Fix $f : \omega \to S_X$ a recursive bijection. The set $U_1 = \{(x, y) : \exists n\ (x(n) = 1 \wedge y \in N_{f(n)})\}$ is $\Sigma^0_1$ and $\boldsymbol{\Sigma}^0_1$-universal.

If $U^*_n \subseteq \omega^\omega \times \omega \times X$ is $\Sigma^0_n$ and $\boldsymbol{\Sigma}^0_n$-universal for $\omega \times X$, then

$$U_{n+1} = \{(x, y) : \exists n\ (x, n, y) \notin U^*_n\}$$

[11] We assume that if the computation makes any queries about numbers $i \geq |\eta|$, then the computation does not halt.

is $\Sigma^0_{n+1}$ and $\mathbf{\Sigma}^0_{n+1}$-universal.

ii) Let $V_1 = \{(n,x) : \exists m\ (\phi_n(m)\downarrow \wedge x \in N_{\phi_n(m)})\}$. Let $g : \omega\times\omega \to S_{\omega\times X}$ be partial recursive such that $g(n,m) = (n, \phi_n(m))$, then

$$V_1 = \bigcup_{n,m} N_{g(n,m)}$$

is $\Sigma^0_1$ and $\mathbf{\Sigma}^0_1$-universal.

An induction as in i) extends this to all levels of the arithmetic hierarchy.
□

Analogous results hold for $\Pi^0_n$. For $X = \omega^\omega$ or $2^\omega$ we know that sets $U_n$ are not $\mathbf{\Pi}^0_n$.

**Corollary 9.23** *There is $A \subseteq \omega$ such that $A$ is $\Sigma^0_n$ but not $\Delta^0_n$.*

**Proof** Suppose $U \subseteq \omega\times\omega$ is $\Sigma^0_n$ and universal $\Sigma^0_n$. Let $A = \{m : (m,m) \notin U\}$. If $U \in \Delta^0_n$, then $A \in \Sigma^0_n$ and $A = \{m : (i,m) \in U\}$ for some $i$. Then

$$i \in A \Leftrightarrow (i,i) \notin U \Leftrightarrow i \notin A,$$

a contradiction. Thus $U \in \Sigma^0_n \setminus \Delta^0_n$. □

Let $\Gamma$ be $\Pi^0_n$, $\Sigma^0_n$ or $\Delta^0_n$.If $A, B \subseteq \omega$, $B \in \Gamma$ and $A \leq_m B$, then $A \in \Gamma$.

We say that $A \subseteq \omega$ is $\Gamma$-*complete* if $A \in \Gamma$ and $B \leq_m A$ for all $B \subseteq \omega$ in $\Gamma$. Here are some well known examples from recursion theory.

**Fact 9.24** *i) $\{e : \mathrm{dom}(\phi_e) \neq \emptyset\}$ is $\Sigma^0_1$-complete.*

*ii) $\{e : \phi_e$ is total$\}$ is $\Pi^0_2$-complete.*

*iii) $\{e : \mathrm{dom}(\phi_e)$ is infinite$\}$ is $\Pi^0_2$-complete.*

*iv) If $U \subseteq \omega\times\omega$ is $\Gamma$-universal, then $U$ is $\Gamma$-complete.*

**Exercise 9.25** Prove the statements in the last fact.

## 9.4 The Effective Projective Hierarchy

We also have effective analogs of the projective point classes. Let $X = \omega^k \times (\omega^\omega)^l$ for some $k, l \in \omega$.

**Definition 9.26** We say that $A \subseteq X$ is $\Sigma^1_1$ if there is a $B \subseteq \omega^\omega \times X$ such that $B \in \Pi^0_1$ and $A = \{x : \exists y\ (y,x) \in B\}$.

We say $A \subseteq X$ is $\Pi^1_n$ if $X \setminus A$ is $\Sigma^1_n$ and we say that $A \subseteq X$ is $\Sigma^1_{n+1}$ if there is a $B \subseteq \omega^\omega \times X$ with $B \in \Pi^1_n$ such that $A = \{x : \exists y\ (y, x) \in B\}$.

We say $A$ is $\Delta^1_n$ if it is both $\Sigma^1_n$ and $\Pi^1_n$.

All of these notions relativize and a set is $\mathbf{\Sigma}^1_n$ if and only $\Sigma^1_1(x)$ for some $x \in \omega^\omega$.

The next theorem summarizes a number of important properties of the classes $\Sigma^1_n$ and $\Pi^1_n$. The proofs are exactly as in classical Descriptive Set Theory, and we leave the proofs as exercises.

**Theorem 9.27** *i) The classes $\Sigma^1_n$ and $\Pi^1_n$ are closed under union, intersection, $\exists n \in \omega$, $\forall n \in \omega$ and computable inverse images.*

*ii) If $A \subseteq X \times \omega^\omega$ is arithmetic, then $\{x : \exists y (x, y) \in A\}$ is $\Sigma^1_1$.*

*iii) There is $U \subseteq \omega^\omega \times X$ a $\Sigma^1_n$-set that is $\mathbf{\Sigma}^1_n$-universal.*

*iv) There is $V \subseteq \omega \times X$ a $\Sigma^1_n$-set that is $\Sigma^1_n$-universal.*

*v) $\Sigma^1_n \subset \Delta^1_{n+1}$, but $\Sigma^1_n \neq \Delta^1_n$.*

*vi) The set* WO *of well orders and the set* WF *of well founded trees are* $\Pi^1_1$.

**Exercise 9.28** Prove 9.27.

Suppose $A \subseteq \omega^\omega$ is $\Sigma^1_1$. Then there is a recursive tree $T \subset \omega^{<\omega} \times \omega^{<\omega}$ such that $x \in A$ if and only if there is $y \in \omega^\omega$ such that $(x, y)$ is a path through $T$. Let

$$T(x) = \{\eta : (x|n, \eta) \in T \text{ where } n = |\eta|\}.$$

Then $x \mapsto T(x)$ is continuous and $x \in A$ if and only if $T(x)$ is not well founded. Note that the map $x \mapsto T(x)$ is computable. Many of the proofs of classical facts about $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$-sets work equally well of $\Sigma^1_1$ and $\Pi^1_1$-sets. Here are statements of the effective versions.

**Theorem 9.29** *i) If $A \subseteq X$ is $\Pi^1_1$, there is a computable $f : X \to Tr$ such that $x \in A$ if and only if $f(x) \in$* WF *for all $x \in X$.*

*ii) $\Pi^1_1$ has the reduction property, i.e., if $A, B$ are $\Pi^1_1$, there are $\Pi^1_1$ $A_0$ and $B_0$ such that $A \cup B = A_0 \cup B_0$ and $A_0 \cap B_0 = \emptyset$.*

*iii) Any two disjoint $\Sigma^1_1$ sets can be separated by a $\Delta^1_1$-set.*

*iv)* [**Uniformization**] *Any $\Pi^1_1$-subset of $\omega^\omega \times \omega^\omega$ can be uniformized by a $\Pi^1_1$-set, i.e., if $A \subseteq \omega^\omega \times \omega^\omega$ is $\Pi^1_1$, there is a $\Pi^1_1$ set $B \subseteq A$ such that $\pi(B) = \pi(A)$ and $B$ is the graph of a function.*

*v)* [**Kreisel Uniformization**] *If $A \subseteq \omega^\omega \times \omega$ is $\Pi^1_1$ and $\pi(A) = \omega^\omega$, then $A$ has a $\Delta^1_1$-uniformization.*

**Proof** We give the proof of v). Let $B$ be a $\Pi^1_1$ uniformization of $A$. Then

$$(x, n) \notin B \Leftrightarrow \exists m\ m \neq n \wedge (x, m) \in B.$$

Thus $B$ is $\Delta^1_1$. □

Further analysis of $\Pi^1_1$-sets will require looking at an effective version of "ordinals".

## 9.5 Recursive Ordinals

The set WF is $\Pi^1_1$. If $A \subseteq \omega^\omega$ is $\Pi^1_1$, we know that $A \leq_w$ WF. We will show that the reduction $f$ can be chosen computable. There is a recursive tree $T$, such that

$$x \in A \Leftrightarrow \forall y\ (x, y) \notin [T] \Leftrightarrow T(x) \in \mathrm{WF}.$$

The function $x \mapsto T(x)$ is computable.

A similar construction gives rise to a $\Pi^1_1$-complete (for $\leq_m$) subset of $\omega$.

Let $O = \{e \in \omega : \phi_e$ is the characteristic function of a well-founded tree $T_e \subseteq \omega^{<\omega}\}$. Then $e \in O$ if and only if

i) $\forall \sigma\ \phi_e(\sigma) \downarrow$ and $\phi_e(\sigma) = 0$ or 1;

ii) $\forall \sigma \in \omega^{<\omega}\ \forall \tau \in \omega^{<\omega}\ ((\sigma \subseteq \tau \wedge \phi_e(\tau) = 1) \rightarrow \phi_e(\sigma) = 1)$;

iii) $\forall f : \omega \rightarrow \omega^{<\omega} \exists n\ (\phi_e(f(n)) = 0 \vee \phi_e(f(n+1)) = 0 \vee f(n) \not\subset f(n+1))$.

Conditions i) and ii) are $\Pi^0_2$ while iii) is $\Pi^1_1$. Thus $O$ is $\Pi^1_1$. The set $O$ is (a variant of) *Kleene's $O$*.

**Proposition 9.30** *$O$ is $\Pi^1_1$-complete.*

**Proof** We will argue that $\omega \setminus O$ is $\Sigma^1_1$-complete. Suppose $A \in \Sigma^1_1$. There is $B \subseteq \omega \times \omega^\omega$ in $\Pi^0_1$ such that $n \in A$ if and only if $\exists x\ (n, x) \in B$. There is a recursive tree $T \subseteq \omega \times \omega^{<\omega}$ such that $(n, x) \in A$ if and only if $(n, x|m) \in T$ for all $m \in \omega$. There is a recursive $f : \omega \rightarrow \omega$ such that $\phi_{f(n)}$ is the characteristic function of $\{\sigma : (n, \sigma) \in T\}$. Then $\phi_{f(n)}$ is the characteristic function of a tree $T_n$ and

$$n \in A \Leftrightarrow T_n \notin \mathrm{WF} \Leftrightarrow f(n) \notin O.$$

□

$O$ will play a very important role in effective descriptive set theory. As a first example, we will show how once we know the complexity of a set, we can find relatively simple elements of the set.

**Lemma 9.31** *Suppose $T \subseteq \omega^{<\omega}$ is a recursive tree. If $[T] \neq \emptyset$, there is $x \in [T]$ with $x \leq_T O$.*

**Proof** We build the "left-most" path though $T$.

For $\sigma \in T$ let $T_\sigma = \{\eta : \sigma\frown\eta \in T\}$. There is a recursive function $f$ such that $\phi_{f(\sigma)}$ is the characteristic function of $T_\sigma$ for all $\sigma \in \omega^{<\omega}$. We build $\emptyset = \sigma_0 \subset \sigma_1 \ldots$ with $\sigma_i \in T$ such that $[T_{\sigma_i}] \neq \emptyset$. Given $\sigma_i$, let $n \in \omega$ be least such that $\sigma_i\frown n \in T$ and $f(\sigma_i\frown n) \notin O$. $\square$

**Corollary 9.32 (Kleene Basis Theorem)** *If $A \subseteq \omega^\omega$ is $\Sigma^1_1$ and nonempty, there is $x \in A$ with $x \leq_T O$.*

**Proof** There is a $\Pi^0_1$-set $B \subseteq \omega^\omega \times \omega^\omega$ such that $x \in A$ if and only if $\exists y\ (x, y) \in B$. By the previous lemma there is $(x, y) \in B$ with $(x, y) \leq_T O$. Clearly $x \leq_T O$. $\square$

Using the Uniformization Theorem, we can find definable elements of $\Pi^1_1$-sets.

**Proposition 9.33** *If $A \subseteq \omega^\omega$ is $\Pi^1_1$, there is $x \in A$ such that $x \in \Delta^1_2$.*

**Proof** Uniformizing $\{0\} \times A$, we find $x \in \omega^\omega$ such that $B = \{(0, x)\}$ is $\Pi^1_1$. Then

$$\begin{aligned} x(n) = m \quad &\Leftrightarrow \exists y\ ((0, y) \in B \wedge y(n) = m) \\ &\Leftrightarrow \forall y\ ((0, y) \notin B \vee y(n) = m) \end{aligned}$$

The first definition is $\Sigma^1_2$, while the second is $\Pi^1_2$. $\square$

We next need to understand the possible heights of recursive trees.

**Definition 9.34** We say that an ordinal $\alpha$ is *recursive* if there is a recursive set $A \subseteq \omega$ and $\prec$ a recursive linear order of $A$ such that $(A, \prec) \cong (\alpha, <)$.

**Lemma 9.35** *i) If $\alpha$ is a recursive ordinal and $\beta < \alpha$, then $\beta$ is a recursive ordinal.*

*ii) If $\alpha$ is a recursive ordinal, then $\alpha + 1$ is a recursive ordinal.*

*iii) Suppose $f : \omega \to \omega$, $g : \omega \to \omega$ are recursive functions such that $P_{f(n)}$ is a program to compute the characteristic function of $A_n$, $P_{g(n)}$ is a program that computes the characterisitic function of $\prec_n$ a well-order of $A_n$ and $(A_n, \prec_n)$ has order-type $\alpha_n$. Then $\sup \alpha_n$ is a recursive ordinal.*

**Proof** a) and b) are routine. For c) let $A = \{(n,m) : \phi_{f(n)}(m) = 1\}$ and let $(n,m) \prec (n',m')$ if and only if $n < n'$ or $n = n'$ and $m \prec_n m'$. Then $(A, \prec)$ is a recursive well-order. Let $\alpha$ be the order type of $A$. Then $\alpha_n \leq \alpha$ for all $n$. Since $\sup \alpha_n \leq \alpha$, $\sup \alpha_n$ is a recursive ordinal. $\square$

There are only countably many recursive well-orders. Thus there are only countably many recursive ordinals.

**Definition 9.36** Let $\omega_1^{\mathrm{ck}}$ be the least non-recursive ordinal. We call this ordinal the *Church–Kleene ordinal*.

More generally, for any $x$ we let $\omega_1^x$ be the least ordinal not recursive in $x$.

**Definition 9.37** We say that a countable ordinal $\alpha$ is *admissible* if $\alpha = \omega_1^z$ for some $z$. More generally, we say that $\alpha$ is $x$-admissible if and only if $\alpha = \omega_1^{x,z}$ for some $z$.[12]

**Definition 9.38** For $\sigma, \tau \in \omega^{<\omega}$ we say $\sigma \lhd \tau$ if $\tau \subset \sigma$ or there is an $n$ such that $\sigma(n) < \tau(n)$, and $\sigma(m) = \tau(m)$ for all $m < n$. We call $\lhd$ the *Kleene–Brower order*.

**Exercise 9.39** a) Show that $\lhd$ is a linear order of $\omega^{<\omega}$.

b) Show that if $T \subseteq \omega^{<\omega}$ is a tree, then $T$ is well-founded if and only if $(T, \lhd)$ is a well-order. [Hint: If $\sigma_0, \sigma_1, \ldots$ is an infinite descending sequence in $(T, \lhd)$, define $x$ inductively by $x(n) =$ least $m$ such that $(x(0), \ldots, x(n-1), m) \lhd \sigma_i$ for some $i$. Prove that $x \in [T]$.]

Recall that $\rho(T)$, the rank of a well founded tree $T$ is defined inductively as follows:

- $\rho(\emptyset) = 0$;
- $\rho(T) = \sup\{\rho(T_n) + 1 : n \in \omega\}$ where $T_n = \{\eta : n\frown\eta \in T\}$.

We say $\rho(T) = \infty$ if $T$ is not well founded.

**Exercise 9.40** Prove that $\omega_1^{\mathrm{ck}} = \sup\{\rho(T) : T \subseteq \omega^{<\omega}$ a recursive well founded tree$\}$.

[12]This is an abuse. The real definition is that $\alpha$ is *admissible* if $\mathbb{L}_\alpha$ is a model of KP, i.e., Kripke–Platek set theory. Sacks proved that a countable ordinal is admissible if and only if it is of the form $\omega_1^x$ for some $x$.

**Theorem 9.41** *i) The set* $\{(S,T) : \rho(S) \leq \rho(T)\}$ *is* $\Sigma^1_1$.

*ii) There is* $R \in \Sigma^1_1$ *such that if* $T \in \mathrm{WF}$, *then* $\{S : (S,T) \in R\} = \{S : \rho(S) < \rho(T)\}$.

**Proof** i) $\rho(S) \leq \rho(T)$ if and only if there is an order preserving $f : S \to T$.

ii) If $T$ is well founded, then $\rho(S) < \rho(T)$ if and only if there is $n$ such that $\rho(S) \leq \rho(T_n)$. □

**Corollary 9.42 (Effective $\Sigma^1_1$-Bounding)** *i) If* $A \subseteq O$ *is* $\Sigma^1_1$, *then there is* $\alpha < \omega_1^{\mathrm{ck}}$ *such that* $\rho(T) < \alpha$ *for all* $T \in A$.

*ii) If* $A \subseteq \mathrm{WF}$ *and* $A \in \Sigma^1_1$, *then there is* $\alpha < \omega_1^{\mathrm{ck}}$ *such that* $\rho(T) < \alpha$ *for all* $T \in A$.

**Proof** If either i) or ii) fails, then $O = \{e : \phi_e$ is the characteristic function of a recursive tree $\exists T\ \forall \sigma \in \omega^{<\omega}((\sigma \in T \leftrightarrow \phi_e(\sigma) = 1)$ and $\exists S \in A\ \rho(T) \leq \rho(S))\}$ is $\Sigma^1_1$, a contradiction. □

**Exercise 9.43** Prove that if $A \subseteq \omega^\omega$ is $\Delta^1_1$, then $A$ is $\mathbf{\Sigma}^0_\alpha$ for some $\alpha < \omega_1^{\mathrm{ck}}$.

# 10 Hyperatithmetic Sets

Our first goal is to try to characterize the $\Delta^1_1$-sets. In particular we will try to formulate the "light-faced" version of

$$\boldsymbol{\Delta}^1_1 = \text{ Borel.}$$

We begin by studying a method of coding Borel sets.

## 10.1 Borel Codes

Let $X = \omega^k \times (\omega^\omega)^k$. Let $S_X$ be as in §9.

**Definition 10.1** A *Borel code* for a subset of $X$ is a pair $\langle T, l\rangle$ where $T \subseteq \omega^{<\omega}$ is a well-founded tree and $l : T \to (\{0\} \times \{0,1\}) \cup (\{1\} \times S_X)$ such that:
i) if $l(\sigma) = \langle 0, 0\rangle$, then $\sigma\widehat{\ }0 \in T$ and $\sigma\widehat{\ }n \notin T$ for all $n \geq 1$;
ii) if $l(\sigma) = \langle 0, 1\rangle$, then $\exists n \sigma\widehat{\ }n \in T$ for all $n \in \omega$;
ii) if $l(\emptyset) = \langle 1, \eta\rangle$, then $\sigma, n \notin T$ for all $n \in \omega$.

Let $BC$ be the set of all Borel codes. It is easy to see that $BC$ is $\Pi^1_1$.

If $x = \langle T, l\rangle$ is a Borel code, we can define $B(x)$ the Borel set coded by $x$. If $\sigma \in T$, recall that $T_\sigma = \{\tau : \sigma\widehat{\ }\tau \in T\}$. We let $l_\sigma : T_\sigma \to \{0\} \times 2 \cup \{1\} \times S_X$ by $l_\sigma(\tau) = l(\sigma\widehat{\ }\tau)$. It is easy to see that $\langle T_\sigma, l_\sigma\rangle$ is also a Borel code.

**Definition 10.2** We define $B(\langle T, l\rangle)$ inductively on the height of $T$.
i) $B(\langle \emptyset, \emptyset\rangle) = \emptyset$.
ii) If $l(\sigma) = \langle 1, \eta\rangle$, then $B(\langle T, l\rangle) = N_\eta$.
iii) If $l(\sigma) = \langle 0, 0\rangle$, then $B(\langle T, l\rangle) = X \setminus B(\langle T_{\langle 0\rangle}, l_{\langle 0\rangle}\rangle)$.
iv) If $l(\sigma) = \langle 0, 1\rangle$, then

$$B(\langle T, l\rangle) = \bigcup_{\langle n\rangle \in T} B(\langle T_{\langle n\rangle}, l_{\langle n\rangle}\rangle).$$

**Exercise 10.3** a) Show that if $x \in BC$, then $B(x)$ is a Borel set.
b) Show that if $A \subseteq X$ is Borel, then there is $x \in BC$ with $B(x) = A$.

**Lemma 10.4** *There are $R \in \Sigma^1_1$ and $S \in \Pi^1_1$ such that if $x \in BC$ then*

$$y \in B(x) \Leftrightarrow (x, y) \in R \Leftrightarrow (x, y) \in S.$$

*In particular $B(x) \in \Delta^1_1(x)$.*

**Proof** We define a set $A$ such that $(x, y, f) \in A$ if and only if $x$ is a pair $\langle T, l \rangle$ where $T \subseteq \omega^{<\omega}$ is a tree, $l : T \to \{0\} \times 2 \cup \{1\} \times S_X$ and $f : T \to 2$ such that for all $\sigma \in T$:

i) if $l(\emptyset) = \langle 1, \eta \rangle$, then $f(\sigma) = 1$ if and only if $y \in N_\eta$;

ii) if $l(\emptyset) = \langle 0, 0 \rangle$, then $f(\sigma) = 1$ if and only if $f(\sigma \frown 0) = 0$;

iii) if $l(\emptyset) = \langle 0, 1 \rangle$, then $f(\sigma) = 1$ if and only if $f(\sigma \frown n) = 1$ for some $n$.

An easy induction shows that if $x = \langle T, l \rangle$ is a Borel code then $(x, y, f) \in A$ if and only if $f$ is the function

$$f(\sigma) = 1 \Leftrightarrow y \in B(\langle T_\sigma, l_\sigma \rangle).$$

It is easy to see that $A$ is arithmetic and if $x \in BC$, then

$$\begin{aligned} y \in B(x) \quad & \Leftrightarrow \exists f \ ((x, y, f) \in A \wedge f(\emptyset) = 1) \\ & \Leftrightarrow \forall f \ ((x, y, f) \notin A \vee f(\emptyset) = 1) \end{aligned}$$

□

**Corollary 10.5** *If $x \in BC$ is recursive, then $B(x)$ is $\Delta^1_1$.*

**Proof** Let $R$ and $S$ be as in the previous lemma. Let $\phi_e = x$. Then

$$\begin{aligned} y \in B(x) \quad & \Leftrightarrow \exists z \ ((\forall n \ \phi_e(n) \downarrow = z(n)) \wedge R(z, y)) \\ & \Leftrightarrow \forall z \ ((\forall n \ \phi_e(n) \downarrow = z(n)) \to S(z, y)). \end{aligned}$$

The first condition is $\Sigma^1_1$ and the second is $\Pi^1_1$. □

## 10.2 Recursively Coded Borel Sets

Our goal is to show that $\Delta^1_1$ is exactly the collection of Borel sets with recursive codes. That will follow from the following two results and $\Sigma^1_1$-Bounding.

**Theorem 10.6** *If $A \subseteq Y$ is a recursively coded Borel set and $f : X \to Y$ is computable, then $f^{-1}(A)$ is a recursively coded Borel set.*

**Proposition 10.7** *If $\alpha < \omega_1^{\mathrm{ck}}$, then $\mathrm{WF}_\alpha$ is a recursively coded Borel set.*

**Corollary 10.8** *Suppose $A \subseteq X$. The following are equivalent:*

*i) $A$ is $\Delta^1_1$;*

*ii) $A$ is a recursively coded Borel set.*

**Proof** We have already shown that every recursively coded Borel set is $\Delta^1_1$. Suppose $A$ is $\Delta^1_1$. Since $A$ is $\Pi^1_1$, there is a computable $f : X \to Tr$ such that $x \in A$ if and only if $f(x) \in \mathrm{WF}$. The set

$$f(A) = \{y : \exists x\ x \in A \wedge f(x) = y\}$$

is a $\Sigma^1_1$-subset of WF. By $\Sigma^1_1$-Bounding, there is $\alpha < \omega_1^{\mathrm{ck}}$ such that $f(A) \subseteq \mathrm{WF}_\alpha$. By 10.7 $\mathrm{WF}_\alpha$ is recusively coded, and by 10.6 $A = f^{-1}(\mathrm{WF}_\alpha)$ is recursively coded. □

For notational simplicity we will assume $X = \omega^\omega$, but all our arguments generalize easily.

Let $BC_{\mathrm{rec}} = \{(e, x) : \phi^x_e$ is a total function and $\phi^x_e \in BC\}$. Then

$$(e, x) \in BC_{\mathrm{rec}} \Leftrightarrow \phi^x_e \text{ is total } \wedge \forall z\ (\forall n\ \phi^x_e(n) = z(n)) \to z \in BC).$$

Thus $BC_{\mathrm{rec}}$ is $\Pi^1_1$.

If $(e, x) \in BC_{\mathrm{rec}}$, then $B_{\mathrm{rec}}(e, x)$ is the Borel set coded by $\phi^x_e$. A similar argument shows that there are $R_{\mathrm{rec}} \in \Sigma^1_1$ and $S_{\mathrm{rec}} \in \Pi^1_1$ such that if $(e, x) \in BC_{\mathrm{rec}}$ then

$$y \in B_{\mathrm{rec}}(e, x) \Leftrightarrow R_{\mathrm{rec}}(e, x, y) \Leftrightarrow S_{\mathrm{rec}}(e, x, y)$$

We say $e \in BC_{\mathrm{rec}}$ and $x \in B_{\mathrm{rec}}(e)$ if $(e, \emptyset) \in BC_{\mathrm{rec}}$ and $x \in B_{\mathrm{rec}}(e, \emptyset)$.

The proofs of both 10.6 and 10.7 will use the Recursion Theorem to do a transfinite induction.

We begin with the base case of the induction.

**Lemma 10.9** *There is a recursive function $F : \omega \times S_Y \to \omega$ such that if $f : X \to Y$ is computable and $e$ is a code for the program computing $f$ , then $B_{\mathrm{rec}}(F(e, i)) = f^{-1}(N_\eta)$.*

**Proof** For notational simplicity we assume $X = Y = \omega^\omega$, this is no loss of generality. Let

$$W = \{\nu \in \omega^{<\omega} : \forall m < |\eta|\ \exists s \leq |\nu|\ \phi^\nu_e(m) \downarrow_s = \eta(m)\}.$$

Then $W$ is recursive and $f^{-1}(N_\eta) = \bigcup_{\nu \in W} N_\nu$. Let $\nu_0, \nu_1, \ldots$ be a recursive enumeration of $\omega^{<\omega}$. Let $T = \{\emptyset\} \cup \{\langle n \rangle : \sigma_n \in W\}$ and let $l(\emptyset) = \langle 0, 1 \rangle$, $l(\langle n \rangle) = \langle 1, \nu \rangle$. Then $x = \langle T, l \rangle$ is a recusive code. Given $e$ and $\eta$ we can easily compute $F(e, \eta) = i$ such that $\phi_i = x$. □

**Lemma 10.10** *i) There is a total recursive function $H_c : \omega \to \omega$ such that if $e \in BC$, then $B_{\rm rec}(H_c(e)) = \omega^\omega \setminus B_{\rm rec}(e)$.*

*ii) There is a total recursive function $H_u : \omega \to \omega$ such that if $\phi_e(n) \in BC_{\rm rec}$ for all $n$, then $B_{\rm rec}(H_u(e)) = \bigcup_n B_{\rm rec}(\phi_e(n))$.*

**Proof** i) $\phi_e$ is a code for a pair $\langle T, l\rangle$. Let

$$T' = \{\emptyset\} \cup \{0\widehat{\ }\eta : \eta \in T\}$$

and $l'(\emptyset) = \langle 0, 0\rangle$, $l'(0\widehat{\ }\eta) = l(\eta)$. It is easy to find $H_c$ such that $H_c(e)$ codes $\langle T', l'\rangle$ and that if $e \in BC_{\rm rec}$, then $H_c(e)$ is a code for the complement.

ii) Suppose $\phi_e(n)$ code a pair $\langle T_n, l_n\rangle$. Let

$$T = \{\emptyset\} \cup \{n\widehat{\ }\sigma : \sigma \in T_n\}$$

and let $l(\emptyset) = \langle 0, 1\rangle$ and $l(n\widehat{\ }\sigma) = l_n(\sigma)$. It is easy to find $H_u$ such that $H_u(e)$ codes $\langle T, l\rangle$. If each $\langle T_n, l_n\rangle$ is a Borel code, then $\langle T, l\rangle$ codes their union. $\square$

Theorem 10.6 follows from the next lemma.

**Lemma 10.11** *If $x = \langle T, l\rangle$ is a recursive Borel code, there is a recursive function $G : \omega \times T \to \omega$ such that if $f : \omega^\omega \to \omega^\omega$ is a computed by program $P_e$, then $G(e, \sigma) \in BC_{\rm rec}$ is a Borel code for $f^{-1}(B(\langle T_\sigma, l_\sigma\rangle)$ for all $\sigma \in T$.*

**Proof**

We define a recursive function $g : \omega \times \omega \times T \to \omega$ as follows:

i) If $l(\sigma) = \langle 1, \eta\rangle$, then $g(i, e, \sigma) = F(e, \eta)$;

ii) If $l(\sigma) = \langle 0, 0\rangle$, then $g(i, e, \sigma) = H_c(\phi_i(e, \sigma, 0))$;

iii) Suppose $l(\sigma) = \langle 0, 1\rangle$. Choose $j$ such that $\phi_j(n) = \phi_i(e, \sigma, n)$. Then $g(i, e, \sigma) = H_u(j)$.

By the Recursion Theorem, there is $\widehat{i}$ such that $\phi_{\widehat{i}}(e, \sigma) = g(\widehat{i}, e, \sigma)$ for all $e, \sigma$. Let $G(e, \sigma) = \phi_{\widehat{i}}(e, \sigma)$.

We prove by induction on $T$, that $G(e, \sigma)$ is a code for $f^{-1}(B(\langle T_\sigma, l_\sigma\rangle))$. By i) this is clear if $l(\sigma) = \langle 1, \eta\rangle$. We assume the claim is true for all $\tau \supset \sigma$.

If $l(\sigma) = \langle 0, 0\rangle$, then

$$G(e, \sigma) = g(\widehat{i}, e, \sigma) = H_c(\phi_{\widehat{i}}(e, \sigma)) = H_c(G(e, \sigma, 0)).$$

By induction, $H_c(G(e, \sigma\widehat{\ }n)$ is a code for

$$f^{-1}(B(\langle T_\sigma, l_\sigma\rangle) = X \setminus f^{-1}(B(\langle T_{\sigma\widehat{\ }0}, l_{\sigma\widehat{\ }0}\rangle)).$$

If $l(\sigma) = \langle 0, 1 \rangle$, then $G(e, \sigma, n)$ is a Borel code for $A_n = f^{-1}(B(\langle T_{\sigma \frown n}, l_{\sigma \frown n} \rangle))$. We choose $j$ such that $\phi_j(n)$ is a code for $A_n$ and $G(e, \sigma) = H_u(j)$ is a code for $\bigcup A_n$. $\square$

Theorem 10.7 follows from the next lemma.

**Lemma 10.12** *If $T$ is a recursive well founded tree, then there is a recursive function $G : T \to BC_{\text{rec}}$, such that $B_{\text{rec}}(G(\sigma)) = \{S \in Tr : \rho(S) \leq \rho(T_\sigma)\}$.*

**Proof** For $\sigma \in \omega^{<\omega}$ let $f_\sigma : Tr \to Tr$ be the computable function $S \mapsto S_\sigma$.

Note that $\rho(S) \leq \rho(T)$ if and only if for all $n \in \omega$ there is $m \in \omega$ such that $\rho(S_{\langle n \rangle}) \leq \rho(T_{\langle m \rangle})$. Thus

$$\{S \in Tr : \rho(S) \leq \rho(T_\sigma)\} = \bigcap_{n \in \omega} \bigcup_{m \in N} f^{-1}_{\langle n \rangle}(\{S \in Tr : \rho(S) \leq \rho(T_{\sigma \frown m})\}).$$

Fix $c$ such that $B_{\text{rec}}(c) = \emptyset$. We define a recursive function $g : \omega \times T \to \omega$ as follows.

i) If $\sigma \notin T$, then $g(i, \sigma) = c$.

ii) Otherwise $g(i, \sigma)$ is a Borel code for

$$\bigcap_n \bigcup_m f^{-1}_{\langle n \rangle}(B_{\text{rec}}(\phi_i(\sigma, m))).$$

We can do this using the functions $F, H_u$ and $H_c$ above. Of course for some $i$, this may well be undefined.

By the Recursion Theorem there is $\widehat{i}$ such that $\phi_{\widehat{i}}(\sigma) = g(\widehat{i}, \sigma)$ for all $\sigma$.

An easy induction shows that $G = \phi_{\widehat{i}}$ is the desired function. $\square$

## 10.3 Hyperarithmetic Sets

**Definition 10.13** We say $x \in \omega^\omega$ is *hyperarithmetic* if $x \in \Delta^1_1$. We say that $x$ is *hyperarithmetic in* $y$, and write $x \leq_{\text{hyp}} y$ if $x \in \Delta^1_1(y)$.

We sometimes let HYP denote the hyperarithmetic elements of $\omega^\omega$.

**Exercise 10.14** i) Show that if $x \leq_{\text{hyp}} y \leq_{\text{hyp}} z$, then $x \leq_{\text{hyp}} z$.

ii) Show that if $x \leq_T y$, then $x \leq_{\text{hyp}} y$.

**Lemma 10.15** *i) $\{(x, y) : x \leq_{\text{hyp}} y\}$ is $\Pi^1_1$. In particular,* HYP *is $\Pi^1_1$.*

**Proof** $x \leq_{\text{hyp}} y$ if and only if $\exists e\ (BC_{\text{rec}}(e,y) \wedge \forall n \forall m\ (x(n)=m \leftrightarrow (n,m) \in B_{\text{rec}}(e,y))$.

This definition is $\Pi^1_1$. □

**Theorem 10.16** *Suppose $A \subseteq \omega^\omega \times \omega^\omega$ is $\Pi^1_1$. Then $B = \{x : \exists y \leq_{\text{hyp}} x\ (x,y) \in A\}$ is $\Pi^1_1$.*

**Proof** $x \in B$ if and only if

$\exists e \in \omega\, \forall z \in \omega^\omega (\phi_e = z \rightarrow (z \in BC \wedge (\forall n \forall m\ ((y(n) = m \rightarrow S((n,m),z)) \wedge (y(n) \neq m \rightarrow \neg R((n,m),z)))) \wedge (x,y) \in A)$

This definition is $\Pi^1_1$. □

We next give a refinement of Kleene's Basis Theorem. First note that we can a recursive ill-founded tree need not have a hyperarithmetic branch.

**Lemma 10.17** *There is a recursive ill founded tree with no infinite hyperarithmetic branch.*

**Proof** Let $A = \{e : e$ codes a recursive tree and $\forall f \in \text{HYP}$ *existsn* $f(n) \not\subset f(n+1)\}$. Then $A$ is $\Sigma^1_1$ and $O \subseteq A$. Thus there is $e \in A \setminus O$. □

**Lemma 10.18** *If $\omega_1^{\text{ck}} < \omega_1^x$, then $O \leq_{\text{hyp}} x$.*

**Proof** Clearly $O$ is $\Pi^1_1(x)$. There is $T$ recursive in $x$ such that $T \in \text{WF}$ and $\rho(T) > \omega_1^{\text{ck}}$. Then

$$O = \{e : e \text{ codes a recursive tree } S \text{ and } \rho(S) < \rho(T)\}$$

is $\Sigma^1_1(x)$. Thus $O \leq_{\text{hyp}} x$. □

**Theorem 10.19 (Gandy's Basis Theorem)** *If $A \subseteq \omega^\omega$ is $\Sigma^1_1$ and nonempty, there is $x \in A$ such that $x \leq_T O$, $x <_{\text{hyp}} O$ and $\omega_1^{\text{ck}} = \omega_1^x$.*

**Proof** Let $B = \{(x,y) : x \in A \wedge y \not\leq_{\text{hyp}} x\}$. By 10.15 $B$ is $\Sigma^1_1$. By Kleene's Basis Theorem 9.32 there is $(x,y) \in B$ with $(x,y) \leq_T O$. If $O \leq_{\text{hyp}} x$, then $y \leq_T O \leq_{\text{hyp}} x$, so $y \leq_{\text{hyp}} x$, a contradiction. By the previous lemma $\omega_1^{\text{ck}} = \omega_1^x$. □

**Exercise 10.20** Suppose $A \subseteq \omega^\omega$ is $\Sigma^1_1$ and uncountable. Show that there is a perfect set $P \subset A$ such that $\omega_1^x = \omega_1^{\text{ck}}$ for all $x \in P$.

*add remarks on the equivalence of hyperarithmetic and being recursive in $0^{(\alpha)}$ for some $\alpha < \omega_1^{\text{ck}}$ and $x \in \mathbb{L}_{\omega_1^{\text{ck}}}$.*

## 10.4 The Effective Perfect Set Theorem

The following theorem is very important.

**Theorem 10.21 (Harrison)** *Let $A \subseteq \omega^\omega$ be $\Sigma^1_1$. If $A$ is countable, then every element of $A$ is hyperarithmetic. In particular, if $A$ contains a nonhyperarithmetic element, then $A$ contains a perfect set.*

There are many ways to prove this. The approach we will take is by first looking at the problem of finding strategies in closed games. We review the basics on determinacy in closed games.

Suppose $A \subseteq \omega^\omega$. We consider the game $G(A)$ where Player's I and II alternate playing elements of $\omega$.

| Player I | Player II |
|---|---|
| $x_0$ | |
| | $x_1$ |
| $x_2$ | |
| | $x_3$ |
| $\vdots$ | $\vdots$ |

Together they play $x = (x_0, x_1, \ldots)$ and Player I wins if $x \in A$.

**Definition 10.22** A *strategy* for Player I is a function $\tau : \omega^{<\omega} \to \omega$.

Player I uses the strategy by opening with $\tau(\emptyset)$. If Player II responds with $x_0$, then Player I replies $\tau(x_0)$. If Player II next plays, $x_1$, then Player II replies $\tau(x_0, x_1)$. ...

The full play looks like:

| Player I | Player II |
|---|---|
| $\tau(\emptyset)$ | |
| | $x_0$ |
| $\tau(x_0)$ | |
| | $x_1$ |
| $\tau(x_0, x_1)$ | |
| | $x_2$ |
| $\tau(x_0, x_1, x_2)$ | |
| $\vdots$ | $\vdots$ |

**Definition 10.23** We say that $\tau$ is a *winning strategy* for Player I if Player I wins any game played using the strategy $\tau$, i.e., for any $x_0, x_1, x_2, \ldots \in \omega$, the sequence

$$\tau(\emptyset), x_0, \tau(x_0), x_1, \tau(x_0, x_1), x_2, \tau(x_0, x_1, x_2), \ldots$$

is in $A$.

There are analogous definitions of strategies and winning strategies for Player II.

**Definition 10.24** We say that the game $G(A)$ is *determined* if either Player I or Player II has a winning strategy.

**Theorem 10.25 (Gale-Stewart Theorem)** *If $A \subseteq \omega^\omega$ is closed, then $G(A)$ is determined.*

**Proof** Let $T$ be a tree such that $A = [T]$. Suppose Player II has no winning strategy. We will show that Player I has a winning strategy. Suppose $\sigma \in \omega^{<\omega}$ and $|\sigma|$ is even. We consider the game $G_\sigma(A)$ where Players I and II alternate playing elements of $\omega$ to build $x \in \omega^\omega$ and Player I wins if $\sigma\widehat{\ }x \in A$.

Let $P = \{\sigma : |\sigma| \text{ is even and Player II has a winning strategy in } G_\sigma(A)\}$. If $\sigma \notin T$, then Player II has already won $G_\sigma(A)$. In particular, always playing 0 is a winning strategy for Player II. Thus $\omega^{<\omega} \setminus T \subseteq P$.

**claim** Suppose that for all $n \in \omega$ there is $m \in \omega$ such that $\sigma\widehat{\ }n\widehat{\ }m \in P$. Then $\sigma \in P$.

Player II has a winning strategy in $G_\sigma(A)$; namely if Player I plays $n$ and Player II plays the least $m$ such that Player II has a winning strategy in $G_{\sigma\widehat{\ }n\widehat{\ }m}$, and then uses the strategy in this game.

We describe a winning strategy for Player I. This strategy can be sumarized as "avoid losing postions".

Since Player II does not have a winning strategy $\emptyset \notin P$. Player I's strategy is to avoid $P$. If we are in position $\sigma$ where $\sigma \notin P$ and $|\sigma|$ is even, then by the claim there is a least $n$ such that $\sigma\widehat{\ }n\widehat{\ }m \notin P$ for all $m$. Player I plays $n$. No matter what $m$ Player II now plays the new position is not in $P$. If Player I continues Playing playing this way they will play $x \in \omega^\omega$ such that $x|2n \notin P$ for all $n$. In particular $x|2n \in T$ for all $n$. Thus $x \in [T]$ and this is a winning strategy for Player I. $\square$

For our purposes it will be useful to understand the complexity of the winning strategies.

**Theorem 10.26 (Strategic Basis Theorem)** *Let $A$ be $\Pi^0_1$. If Player II has a winning strategy in $G(A)$, then Player II has a hyperarithmetic winning strategy.*

**Proof** Let $T$ be a recursive tree such that $A = [T]$. Suppose Player II does not have a hyperarithmetic winning strategy. We will show that Player I has a winning strategy. Suppose $\sigma \in \omega^{<\omega}$ and $|\sigma|$ is even. We consider the game $G_\sigma(A)$ where Players I and II alternate playing elements of $\omega$ to build $x \in \omega^\omega$ and Player I wins if $\sigma \widehat{\ } x \in A$.

Let $P = \{\sigma : |\sigma|$ is even and Player II has a hyperarithmetic winning strategy in $G_\sigma(A)\}$. If $\sigma \notin T$, then Player II has already won. In particular, always playing 0 is a hyperarithmetic winning strategy for Player II. Thus $\omega^{<\omega} \setminus T \subseteq P$.

**claim** Suppose that for all $n \in \omega$ there is $m \in \omega$ such that $\sigma, n, m \in P$. Then $\sigma \in P$.

Let $B = \{(n, m, e) : e$ is a hyperarithmetic code for $\tau$ and $\forall y$ if we play $G_{\sigma,n,m}(A)$ where Player I plays $y$ and Player II plays using $\tau$, then the result is in $A\}$. The set $B$ is $\Pi^1_1$. and $\forall n \exists m \exists e (n, m, e) \in B$. By selection there is a $\Delta^1_1$-function $f : \omega \to \omega^2$ such that $(n, f(n)) \in B$ for all $n \in \omega$. Player II has a hyperarithmetic winning strategy in $G_\sigma(A)$; namely if Player I plays $n$ and $f(n) = (m, e)$, then Player II plays $m$, and then uses the strategy coded by $e$.

We describe a winning strategy for Player I.

Since Player II does not have a hyperarithmetic winning strategy $\emptyset \notin P$. Player I's strategy is to avoid $P$. If we are in position $\sigma$ where $\sigma \notin P$ and $|\sigma|$ is even, then by the claim there is a least $n$ such that $\sigma, n, m \notin P$ forall $m$. Player I plays $n$. No matter what $m$ Player II now plays the new postion is not in $P$. If Player I continues playing this way they will play $x \in \omega^\omega$ such that $x|2n \notin P$ for all $n$. In particular $x|2n \in T$ for all $n$. Thus $x \in [T]$ and this is a winning strategy for Player I. $\square$

**Exercise 10.27** Suppose $A$ is $\Pi^0_1$ and Player I has a winning strategy in $G(A)$. Then Player I has a winning strategy hyperaritmetic in $O$.

**Proof of Effective Perfect Set Theorem** Suppose $A$ is $\Sigma^1_1$. Let $B \subseteq \omega^\omega \times \omega^\omega$ be $\Pi^0_1$ such that

$$A = \{x : \exists y \ (x, y) \in B\}.$$

Consider the following game. At stage $s$ Player I plays $(\sigma_{s,0}, \sigma_{s,1}, y_s)$ where $\sigma_{s,0}, \sigma_{s,1} \in \omega^{<\omega}$ are incomparable and $y_s \in \omega$. Player II plays $i_s = 0, 1$. At the end of the game let

$$x = \sigma_{0,i_0} {}^\frown \sigma_{1,i_1} {}^\frown \sigma_{2,i_2} \ldots \text{ and } y = y_0, y_1, y_2, \ldots$$

Player 1 wins if $(x, y) \in B$. This is an open game.

**claim 1** If Player I has a winning strategy, then $A$ has a perfect subset.

Consider the function $f : 2^\omega \to \omega^\omega \times \omega^\omega$ where $f(z) = (x(z), y(z))$ is the play of the game where Player II plays $z(i)$ at stage $i$ and Player I uses her winning strategy. Then $f$ is continuous and if $z_1 \neq z_2$ and $s$ is least such that $z_1(s) \neq z_2(s)$, then we insure at stage $s$ that $x(z_1) \neq x(z_2)$. Thus $f$ is a continuous injection from $2^\omega$ into $A$, so $A$ has a perfect subset.

Suppose $\tau$ is a winning strategy for Player II. Consider a position $p$ in the game where it is Player I's turn to move at stage $s+1$. Let $\mu = \sigma_{0,i_0} {}^\frown \ldots {}^\frown \sigma_{s,i_s}$ and then $\nu = (y_0, \ldots, y_s)$ be the initial segments of the final play determined at this position. Suppose $(x, y) \in B$, $x \supset \mu$ and $y \supset \nu$. We say that $(x, y)$ is *rejected* at position $p$ if for any possible move $(\sigma_{s+1,0}, \sigma_{s+1,1}, y_{s+1})$ by Player I, Player II using $\tau$ will make reply so that $x \not\subset \mu {}^\frown \sigma_{s+1,i_{s+1}}$.

**claim 2** If $(x, y) \in B$, there is a position $p$ such that $(x, y)$ is rejected at $p$.

Suppose not. Consider a play of the game where Player II uses $\tau$. Whenever it is Player I's turn to move we are at a position where $(x, y)$ has not been rejected. Since $(x, y)$ is not rejected at the empty position, there is some move by Player I where he plays $y(0)$ such that when II responds we will still have an initial segment of $x$. Since $(x, y)$ is not rejected at this new position, there is some move that Player I can make, so that II's response will still be an initial segment of $(x, y)$. Continuing this way the eventual play of the game will be $(x, y)$ and Player I will win, a contradiction.

**claim 3**: For each position $p$ there are at most countably many $x$ such that some $(x, y)$ is rejected at position $p$ and each is arithmetic in $\tau$.

For each possible $k$, we will show that there is at most one $(x, y) \in B$ with $y \supset \nu {}^\frown k$ such that $y$ is rejected at $p$, where $\nu$ is as above.

Suppose $(x, y)$ is rejected at position $p$ at stage $s+1$. Let $\mu$ be as above and suppose that we have determined that $x \subset \mu {}^\frown \eta$. There is a unique $n$ such that for all $m$, if Player I plays $(\eta {}^\frown n, \eta {}^\frown m, k)$ at stage $s+1$, then Player II, using $\tau$ will play 1. Then we know that $x \subset \mu {}^\frown \eta {}^\frown 1$. This argument shows $x \leq_T \tau'$.

Since every element of $A$ is rejected at one of the countably many possible positions, $A$ must be countable and every element is arithmetic in $\tau$

We can easily modify the proof of Theorem 10.26 to show that either Player I has a winning strategy or Player II has a hyperarithmetic winning strategy.

Thus if $A$ is countable, $A \subseteq \mathrm{HYP}$.

**Exercise 10.28** Show that if $A$ is $\Sigma^1_1$ and uncountable, then there is a continuous injection $f : 2^\omega \to A$ with $f$ computable in $x$ for some $x \leq_{\mathrm{hyp}} O$.

**Exercise 10.29** Modify the proof of the Effective Perfect Set Theorem, using the Banach–Mazur game, to prove that if $A \subseteq \omega^\omega$ is a nonmeager $\Delta^1_1$-set, then there is a hyperarithmetic $x \in A$.

**Corollary 10.30** *Suppose $A \subseteq \omega^\omega \times \omega^\omega$ is $\Delta^1_1$ and $\{y : (x, y) \in A\}$ is countable for all $x \in \omega^\omega$. Then*

*i) the projection $\pi(A) = \{x : \exists y\ (x, y) \in A\}$ is $\Delta^1_1$ and*

*ii) $A$ has a $\Delta^1_1$-uniformization*

**Proof**

i) Clearly $\pi(A)$ is $\Sigma^1_1$, but by Harrison's Theorem

$$\exists y\ (x, y) \in A \leftrightarrow \exists y \leq_{\mathrm{hyp}} x\ (x, y) \in A.$$

The later condition is $\Pi^1_1$.

ii) Let

$$A^* = \{(x, e) : e \in BC_{\mathrm{rec}}(x) \wedge \forall y(y = B_{\mathrm{rec}}(e, x) \to (x, y) \in A\}.$$

Then $A^*$ is $\Pi^1_1$ and has a $\Pi^1_1$ uniformization $B$. But

$$(x, e) \notin B \Leftrightarrow x \notin \pi(A) \vee \exists i \neq e\ (x, i) \in B.$$

Thus $B$ is $\Delta^1_1$. Let

$$C = \{(x, y) : \exists e\ (x, e) \in B \wedge y = B_{\mathrm{rec}}(e, x)\}.$$

Then $C$ is a $\Delta^1_1$-uniformization of $A$. □

Relativizing these corollaries lead to interesting results about Borel sets.

**Corollary 10.31** *Suppose $A \subseteq \omega^\omega \times \omega^\omega$ is a Borel set such that every section is countable. Then $X$ the projection of $X$ is Borel and $X$ can be uniformized by a Borel set.*

**Corollary 10.32** *Suppose $f : \omega^\omega \to \omega^\omega$ is continuous, $A$ is Borel and $f|A$ is one-to-one. Then $f(A)$ is Borel.*

**Proof** $f(A)$ is the projection of $\{(x,y) : x \in A \land f(x) = y\}$ and sections are singletons. □

If $x \in \omega^{\omega^2}$ we identify $x$ with $(x_0, x_1, \ldots)$ in $(\omega^\omega)^\omega$ where $x_n(m) = x(n,m)$.

**Lemma 10.33** *Suppose $A$ is a $\Delta^1_1$-subset of* HYP*. There is a hyperarithmetic $x \in \omega^{\omega^2}$ such that $A \subseteq \{x_0, x_1, \ldots\}$.*

**Proof** Let

$$B = \{(x,i) \in \omega^\omega \times \omega : x \in A \land i \in BC_{\rm rec} \land \forall n \forall m\, (x(n) = m \leftrightarrow (n,m) \in B_{\rm rec}(i))\}.$$

Then $B$ is $\Pi^1_1$ and $\pi(B) = A$. By selection, there is a $\Delta^1_1$ function $s : A \to \omega$, uniformizing $B$.

Let $C = \{i : \exists x \in A\ s(x) = i\}$. Clearly $C$ is $\Sigma^1_1$. Since

$$i \in C \leftrightarrow \exists x \in \mathrm{HYP}\ (x \in A \land s(x) = i),$$

$C$ is $\Delta^1_1$. Let

$$x(i,n) = \begin{cases} 0 & \text{if } i \notin C \\ m & \text{if } i \in D \land (n,m) \in B_{\rm rec}(i) \end{cases}.$$

Then $A \subseteq \{x_0, x_1, \ldots\}$. □

**Exercise 10.34** Show that the same is true if $A$ is $\Sigma^1_1$. [Hint: First show that any $\Sigma^1_1$ subset of HYP is contained in an $\Delta^1_1$ subset of HYP.]

# 11 Effective Aspects of $\mathcal{L}_{\omega_1,\omega}$

## 11.1 Coding $\mathcal{L}_{\omega_1,\omega}$-formula

Before discussing effective aspects of the model theory of $\mathcal{L}_{\omega_1,\omega}$ we should take the time to discuss one method of coding $\mathcal{L}_{\omega_1,\omega}$-formulas. Throughout this section we will only consider formulas with finitely many free variables– though this requirement could easily be relaxed.

Fix $\tau$ a countable vocabulary. We can define a coding of $\mathcal{L}_{\omega_1,\omega}(\tau)$-formulas that is analogous to the construction of Borel codes.

We assume that we have fixed a Gödel coding $\ulcorner\phi\urcorner$ of atomic $\tau$-formulas.

A *labeled tree* is a non-empty tree $T \subseteq \omega^{<\omega}$ with functions $l$ and $v$ with domain $T$ such that for any $\sigma \in T$ one of the following holds:

- $\sigma$ is a terminal node of $T$ and $l(\sigma) = \ulcorner\psi\urcorner$ where $\psi$ is an atomic $\tau$-formula and $v(\sigma)$ is the set of free variables in $\psi$;
- $l(\sigma) = \neg$, $\sigma^\frown 0$ is the unique successor of $\sigma$ in $T$ and $v(\sigma) = v(\sigma^\frown 0)$;
- $l(\sigma) = \exists v_i$, $\sigma^\frown 0$ is the unique successor of $\sigma$ in $T$ and $v(\sigma) = v(\sigma^\frown 0) \setminus \{i\}$;
- $l(\sigma) = \bigwedge$ and $v(\sigma) = \bigcup_{\sigma^\frown i \in T} v(\sigma^\frown i)$ is finite.

A *formula code* $\phi$ is a well founded labeled tree $(T, l, v)$. A *sentence code* is a formula where $v(\emptyset) = \emptyset$.

**Exercise 11.1** The set of labeled trees is arithmetic. The set of formula codes is $\Pi^1_1$ as is the set of sentence codes.

**Proposition 11.2** *There is $R(x,y) \in \Pi^1_1$ and $S(x,y) \in \Sigma^1_1$. Such that if $\phi$ is a sentence and $\mathcal{M} \in \mathbb{X}_\tau$, then*

$$\mathcal{M} \models \phi \Leftrightarrow R(\mathcal{M}, \phi) \Leftrightarrow S(\mathcal{M}, \phi).$$

*In particular, $\{(\mathcal{M}, \phi) : \phi$ is a sentence and $\mathcal{M} \models \phi\}$ is $\Pi^1_1$, but for any fixed $\phi$, $\mathrm{Mod}(\phi) = \{\mathcal{M} \in \mathbb{X}_\tau : \mathcal{M} \models \phi\}$ is $\Delta^1_1(\phi)$.*

**Proof** We define a predicate "$f$ is a *truth definition* for the labeled tree $(T, l, v)$ in $\mathcal{M}$" as follows.

- The domain of $f$ is pairs $(\sigma, \eta)$ where $\sigma \in T$ and $\eta : v(\sigma) \to \mathcal{M}$ is an assignment of the free variables at node $\sigma$ and $f(\sigma, \eta) \in \{0, 1\}$.

- If $l(\sigma) = \ulcorner\psi\urcorner$ an atomic $\tau$-formula, then $f(\sigma, \eta) = 1$ if and only if $\psi$ is true in $\mathcal{M}$ when we use $\eta$ to assign the free variables.

- If $l(\sigma) = \neg$, then $f(\sigma, \eta) = 1$ if and only if $f(\sigma\widehat{\ }0, \eta) = 0$.

- If $l(\sigma) = \exists v_i$ there are two cases. If $v_i \in v(\sigma\widehat{\ }0)$, then $f(\sigma, \eta) = 1$ if and only if there is $a \in \mathcal{M}$ such that $f(\sigma\widehat{\ }0, \eta^*) = 1$, where $\eta^* \supset \eta$ is the assignment where $\eta^*(v_i) = a$. Otherwise, $f(\sigma, \eta) = f(\sigma\widehat{\ }0, \eta)$.

- If $l(\sigma) = \bigwedge$, then $f(\sigma, \eta) = 1$ if and only if $f(\sigma\widehat{\ }i, \eta|v(\sigma\widehat{\ }i)) = 1$ for all $i$ such that $\sigma\widehat{\ }i \in T$.

This predicate is arithmetic. If $\phi$ is a sentence, there is a unique truth definition $f$ for $\phi$ in $\mathcal{M}$. Let

$R(x, y) \Leftrightarrow x \in \mathbb{X}_\tau$ and $y$ is a labeled tree, $v(\emptyset) = \emptyset$ and $f(\emptyset, \emptyset) = 1$ for all truth definition $f$ for $y$ in $x$

and

$S(x, y) \Leftrightarrow y$ is a labeled tree, $v(\emptyset) = \emptyset$ and there is a truth definition $f$ for $y$ in $x$ such that $f(\emptyset, \emptyset) = 1$. $\square$

We will say that a sentence is recursive (hyperarithmetic) if it has a recursive (hyperarithmeic) code. We can sharpen some of our earlier theorems.

**Theorem 11.3** *Let $\tau = (<, \ldots)$ be a recursive vocabulary and let $\phi$ be a hyperarithmetic $\mathcal{L}_{\omega_1,\omega}$-sentence such that every model of $\phi$ is well ordered by $<$. There is $\alpha < \omega_1^{\mathrm{ck}}$ such that every model of $\phi$ has order type at less that $\alpha$.*

**Proof** Let $A = \{x : x$ codes a linear order and $\exists \mathcal{M} \in \mathrm{Mod}(\phi) \exists f\ f$ is an order embedding of $x$ into $\mathcal{M}\}$. Then $A$ is a $\Sigma^1_1$ set of well orders. Thus, by $\Sigma^1_1$-Bounding, there is a recursive bound on the order types in $\mathrm{Mod}(\phi)$. $\square$

**Exercise 11.4** For each $\alpha < \omega_1^{\mathrm{ck}}$, there is a recursive $\phi$ such that any model of $\phi$ is isomorphic to $\alpha$. Thus the above bound is optimal.

**Effective construction of models**

If $T$ is a first order theory in a recursive language then we can find an arithmetic model of $T$. In fact, we can find a completion $T^*$ of $T$ with $T^* \leq_T T'$, the recursion theoretic jump of $T$, and $\mathcal{M} \models T^*$ with $\mathcal{M} \leq_T T^*$. For $\mathcal{L}_{\omega_1,\omega}$, this is more complicated.

Suppose $\phi \in \mathcal{L}_{\omega_1,\omega}$ is hyperarithmetic. Using Kleene's Basis Theorem 9.32 there is $\mathcal{M} \models \phi$ with $\mathcal{M} \leq_T O$.

**Definition 11.5** For any $\tau$-structure $\mathcal{M}$ we let $\omega_1^{\mathcal{M}}$ be the infimum of $\omega_1^z$ where $z$ codes an isomorphic copy of $\mathcal{M}$.

**Proposition 11.6** *There is a satisfiable recursive sentence with no hyperarithmetic models.*

**Proof** By Corollary 10.17 there is an ill-founded recursive tree $T \subseteq \omega^{<\omega}$ with no hyperarithmetic paths. Let $\tau$ be the vocabulary where we have a unary predicate $\mathcal{P}_\eta$ for all $\eta \in T$. Let $\phi$ be the conjunction of

$$\forall x \bigwedge_{\eta \in T} (P_\eta(x) \rightarrow \bigwedge_{\nu \subset \eta} P_\nu(x))$$

and

$$\forall x \bigwedge_{n=1}^{\infty} \bigvee_{\eta \in T \cap \omega^n} P_\eta(x).$$

If $\mathcal{M} \models T$, then there is a path through $T$ recursive in $\mathcal{M}$. Thus $\phi$ has no hyperarithmetic models. $\square$

Nadel showed that it easier to find models of complete sentences.

**Theorem 11.7** *If $\phi$ is a complete sentence, then there is $\mathcal{M} \models \phi$ with $\mathcal{M} \leq_{\text{hyp}} \phi$.*

**Proof** Add countably many new constant symbols $c_0, c_1, \ldots$ and let $\mathbb{A}$ be the smallest fragment containing $\phi$ such that if $\theta(v_1, \ldots, v_n) \in \mathbb{A}$, then $\theta(c_{i_1}, \ldots, c_{i_n}) \in \mathbb{A}$ for all $i_0, \ldots, i_n$. The fragment $\mathbb{A}$ is arithmetic in $\phi$. Let $\Sigma$ be all finite sets $\sigma$ of $\mathbb{A}$-sentences such that if $\overline{c}$ are the new constants occurring in $\sigma$ then $\phi \models \exists \overline{v} \bigwedge_{\psi(\overline{c}) \in \sigma} \psi(\overline{v})$.

Since $\phi$ is complete,

$$\phi \models \theta \iff \forall \mathcal{M} \ (\mathcal{M} \models \phi \rightarrow \mathcal{M} \models \theta) \iff \exists \mathcal{M}(\mathcal{M} \models \phi \wedge \mathcal{M} \models \theta).$$

Thus $\Sigma$ is $\Delta^1_1$ in $\phi$. It is also easy to check that $\Sigma$ is a consistency property.

The proof of the Model Existence Theorem shows we can build $\mathcal{M} \models \phi$ with $\mathcal{M}$ recursive in $\Sigma$ and $\mathbb{A}$. Thus we have $\mathcal{M} \models \phi$ with $\mathcal{M} \leq_{\mathrm{hyp}} \phi$ □

## 11.2 Kreisel–Barwise Compactness

Kreisel found an interesting version of the compactness theorem holds when we look at theories that are $\Pi^1_1$-sets of recursive sentences and think of the $\Delta^1_1$-subsets as "finite".

Throughout this section we assume that $\tau$ is a recursive vocabulary.

**Lemma 11.8** *Suppose $I$ is a $\Pi^1_1$ set of hyperarithmetic Borel codes and*

$$\bigcap_{x \in I_0} B(x) \neq \emptyset$$

*for any $I_0 \subseteq I$ with $I_0 \in \Delta^1_1$. Then*

$$\bigcap_{n \in I} B(x) \neq \emptyset.$$

**Proof** Suppose not. Then $\forall x \exists y \in \mathrm{HYP}\ y \in I \wedge x \notin B(y)$. The set $A = \{(x, e) : \exists y \in \mathrm{HYP}\ e \text{ codes } y \wedge y \in I \wedge x \notin B(y)\}$ is $\Pi^1_1$ By Kreisel's Uniformization Theorem 9.29 v), it has a $\Delta^1_1$ uniformization $P$. Thus $X = \{y : \exists x \exists n\ P(x, n) \wedge n \text{ codes y}\}$ is a $\Sigma^1_1$ subset of $I$. By the $\Sigma^1_1$-Separation Theorem, there is a $\Delta^1_1$ set $Y$ such that $X \subseteq Y \subseteq I$. But then there is $x \in \bigcap_{y \in Y} B(y)$, but there is $y \in X$ such that $x \notin B(y)$, a contradiction. □

**Corollary 11.9 (Kreisel–Barwise Compactness)** *Suppose $T$ is a $\Pi^1_1$ set of hyperarithmetic sentences in $\mathcal{L}_{\omega_1,\omega}$ and every $\Delta^1_1$ subset of $T$ has a model. Then $T$ has a model.*

**Proof** We can easily pass from a hyperarithmetic sentence $\phi$ to a hyperarithmetic code for $\mathrm{Mod}(\phi)$. □

**Corollary 11.10** *Let $T$ be as above and suppose every $\Delta^1_1$ subset of $T$ has a recursive model. Then $T$ has a recursive model.*

**Proof** For notational simplicity assume that $\tau = \{R\}$ a single binary relation and let $\tau^* = \tau \cup \{c \cup s\}$ where $c$ is a constant and $s$ is a unary function. If $x \in 2^{\omega \times \omega}$ is recursive let $\Psi_x$ be the conjunction of

- $s$ is injective with no cycles and every element except $c$ is in the image of $s$;
- $\bigwedge_{x(i,j)=1} R(s^i(c), s^j(c)) \wedge \bigwedge_{x(i,j)=0} \neg R(s^i(c), s^j(c))$.

If $(\omega, R, c, s) \models \Psi_x$, then $(\omega, R)$ has an isomorphic recursive copy. Let $\Theta$ be the hypeartithmetic sentence

$$\bigvee_{x \text{ recursive}} \Psi_x$$

and let $T^* = T \cup \{\Theta\}$. Every $\Delta^1_1$ subset of $T^*$ has a model. Thus $T^*$ has a model $\mathcal{M}$. Since $\mathcal{M} \models \Psi_x$ for some recursive $x$, $T$ has a recursive model. □

Suppose $T$ is a $\Pi^1_1$ set of hyperarithmetic codes for $\mathcal{L}_{\omega_1,\omega}$-sentences and $T$ has a model. Let $X = \{\mathcal{M} : \forall \phi \ (\phi \in T \to \mathcal{M} \models \phi)\}$. Then $X$ is a $\Sigma^1_1$ set and by the Gandy Basis Theorem there is $\mathcal{M} \models T$ with $\omega_1^{\mathcal{M}} = \omega_1^{\text{ck}}$.

We will use Kreisel–Barwise Compactness to construct a recursive ordering of order type $\omega_1^{\text{ck}} + \mathbb{Q} \times \omega_1^{\text{ck}}$.

**Lemma 11.11** *Suppose $(\omega, \prec)$ is an ill-founded recursive linear ordering with no infinite hyperarithmetic descending chains. Then $(\omega, \prec)$ is isomorphic to $\omega_1^{\text{ck}} + \mathbb{Q} \times \omega_1^{\text{ck}} + \beta$ for some recursive ordinal $\beta$.*

**Proof** Let $I = \{n :$ there is no infinite descending chain in $\{m : m \prec n\}$. Clearly $I$ is $\Pi^1_1$.

**claim** $I$ is not $\Sigma^1_1$.

Suppose $I$ is $\Sigma^1_1$. Since $\prec$ is ill-founded there is $m \in \omega$ with $I \prec m$. Consider

$$A = \{(x, y) : y \prec x \prec m \wedge y \notin I\}$$

Since $I$ is $\Sigma^1_1$, $A$ is $\Pi^1_1$ and we can find a $\Pi^1_1$-uniformization $B$. Pick $x_0$ such that $I \prec x_0 \prec m$. Let $x_{n+1}$ be the unique element such that $(x_n, x_{n+1}) \in B$.

We claim that $x_0, x_1, x_2, \ldots$ is a hyperarithmetic descending sequence.

Let $C = \{\sigma \in \omega^{<\omega} : \sigma(0) = x_0 \wedge \forall n < |\sigma| - 1 \ (\sigma(n), \sigma(n+1) \in B\}$. $C$ is clearly $\Pi^1_1$. But

$$\sigma \notin C \Leftrightarrow \sigma(0) \neq x_0 \vee \exists n < |\sigma| - 1 \exists m \neq \sigma(n+1) \ (\sigma(n), m) \in B.$$

Thus $C$ is $\Delta^1_1$ and

$$x_n = m \Leftrightarrow \exists \sigma \in C \ \sigma(n) = m.$$

Since there are no hyperarithmetic descending sequences this is a contradiction.

**claim** $I$ is isomorphic to $\omega_1^{\mathrm{ck}}$.

Clearly the order type of $I$ is at most $\omega_1^{\mathrm{ck}}$, for otherwise there would be a recursive initial segment of order type $\omega_1^{\mathrm{ck}}$. Suppose $\alpha$ is a recursive ordinal greater than the order type of $I$. Then $x \in I$ if and only if there is an order preserving map from $\{y : y \prec x\}$ into $(\omega, R)$ and $I$ is $\Sigma^1_1$, a contradiction.

Thus $I$ is isomorphic to $\omega_1^{\mathrm{ck}}$.

Define an equivalence relation $x \sim y$ if and only if the interval

$$[\min_{\prec}(x, y), \max_{\prec}(x, y)]$$

is well ordered. This is a $\Pi^1_1$-equivalence relation.

**claim** Each $\sim$ class has a least element.

Suppose the sim class of $x$ has no least element. Let

$$A = \{(n, m) : n \sim x \wedge y \sim x \wedge y \prec x\}.$$

There is $B$ a $\Pi^1_1$-uniformization of $A$ and arguing as above we can build an infinite hyperarithmetic descending chain.

If $n$ is the least element of it's $\sim$-class, then we can consider the restriction of $\prec$ to $\{m : n \preceq m\}$. By our first claim either this set is well ordered or it has an initial segment of order type $\omega_1^{\mathrm{ck}}$.

If there is a maximal $\sim$-class, that class must be well ordered, and hence has order type $\alpha$ for some $\alpha < \omega_1^{\mathrm{ck}}$. Every other $\sim$-class must have order type $\omega_1^{\mathrm{ck}}$.

If the $\sim$-class of $x$ has order type $\omega_1^{\mathrm{ck}}$, then there must be $x \prec y$ with $x \sim y$. Otherwise we would get a recursive well order of order type $\omega_1^{\mathrm{ck}}$.

Suppose $x \prec y$ and $y$ is the least element of it's $\sim$-class. Then there is an infinite descending chain $x \prec \ldots x_n \prec x_1 \prec x_0 \prec y$. The $\sim$-class of any $x_i$ is strictly between the $\sim$-classes of $x$ and $y$.

We have shown that $(\omega, \prec)$ is order isomorphic to $\omega_1^{\mathrm{ck}} + (\mathbb{Q} \times \omega_1^{\mathrm{ck}}) + \alpha$ for some recursive $\alpha$. $\square$

**Corollary 11.12 (Harrison Order)** *There is a recursive ordering of $\omega$ of order type $\omega_1^{\mathrm{ck}}+(\mathbb{Q}\times\omega_1^{\mathrm{ck}})$ with no hyperarithmetic infinite descending chains.*

**Proof**
**claim** If $\alpha$ is a recursive ordinal, then there is a recursive formula $\varphi_\alpha(x)$ such that in any linear order $\mathcal{M}$

$$\mathcal{M}\models\varphi_\alpha(a)\Leftrightarrow\{m:m<a\}\text{ has order type }\alpha.$$

We define $\varphi_\alpha$ be effective transfinite recursion.

$$\varphi_0(\overline{x})=\forall y\ x\le y$$

$$\varphi_\alpha(\overline{x})=\forall y<x\ \bigvee_{\beta<\alpha}\varphi_\beta(y)\wedge\bigvee_{\beta<\alpha}\exists y<x\ \varphi_\beta(y).$$

We can do this in such a way that there is a function $f$ with $\Pi^1_1$-graph such that if $e$ is the code for a recursive well ordering of order type $\alpha$, then $f(e)$ is a code for $\varphi_\alpha$.

Let $\tau$ be the vocabulary with a binary relation symbol $<$, constants $c_0,c_1,\ldots$. We will define a $\tau$-theory $T$ asserting:

- $<$ is a linear order;
- $\forall x\ \bigvee_{i\in\omega}x=c_i$;
- $\bigvee_{i\in\omega}c_{f(i+1)}\not<c_{f(i)}$ for $f\in\omega^\omega$ hyperarithmetic;
- $\exists x\ \varphi_\alpha(x)$ for $\alpha<\omega_1^{\mathrm{ck}}$.

**Exercise 11.13** a) There is a function $f$ with $\Pi^1_1$-graph such that if $e$ is the code for a recursive well ordering of order type $\alpha$, then $f(e)$ is a code for $\varphi_\alpha$.

b) Show that the theory $T$ is $\Pi^1_1$.

c) Show that if $T'\subset T$ is $\Delta^1_1$, then $T'$ only contains $\exists x\ \varphi_\alpha(\overline{x})$ for $\alpha<\beta$ for some recursive $\beta$. Conclude that $T'$ has a recursive model.

By Kreisel–Barwise Compactness, there is a computable model $\mathcal{H}$ of $T$. Clearly $\mathcal{H}$ has an initial segment of order type $\omega_1^{\mathrm{ck}}$. Since $\omega_1^{\mathrm{ck}}$ is not a recursive ordinal, $\mathcal{H}$ must be ill-founded. The map $n\mapsto c_n^{\mathcal{H}}$ is recursive, so a hyperarithmetic descending chain in $\mathcal{H}$ will give rise to a hyperarithmetic descending chain $c_{f(0)},c_{f(1)},\ldots$, a contradiction. By the previous Lemma,

$\mathcal{H} \cong \omega_1^{\text{ck}} + \mathbb{Q} \times \omega_1^{\text{ck}} + \alpha$ for some recursive $\alpha$. By taking an initial segment, we may assume $\alpha = 0$. $\square$

This ordering is called the *Harrison order*. Let $\prec$ be the Harrison order on $\omega$. Let $T$ be the tree set of finite sequences $(n_0, \ldots, n_m)$ such that $n_i \succ n_{i+1}$ for $i = 0, \ldots, m-1$. Then $T$ is a recursive ill founded tree with no hyperarithmetic paths.

**Exercise 11.14** Show that the existence of the Harrison ordering also follows directly from Lemmas 10.17 and 11.11.

## 11.3 Effective Analysis of Scott Rank

Let $\tau$ be a recursive vocabulary. Let $\mathbb{X}_\tau$ be the Polish space of countable $\tau$-structures with universe $\omega$.

**Lemma 11.15** *For any $\mathcal{M} \in \mathbb{X}_\tau$, $\overline{a} \in M$ and $\beta < \omega_1^{\mathcal{M}}$, the sentence $\Phi^{\mathcal{M}}_{\overline{a},\beta}$ is recursive in $\mathcal{M}$.*

**Proof** We will actually give a sentence $\widehat{\Phi}^{\mathcal{M}}_{\overline{a},\beta}$ that is equivalent to $\Phi^{\mathcal{M}}_{\overline{a},\beta}$ The inductive definition is a bit more cumbersome because we don't distinguish succesor and limit ordinals. [13]

Fix an ordering recursive in $\mathcal{M}$ of order type $\beta + 1$. We give a definition by effective transfinite recursion. We will build an $\mathcal{M}$-recursive map $(\overline{a}, \alpha) \mapsto$ a code for $\widehat{\Phi}^{\mathcal{M}}_{\overline{a},\alpha}$ for $\overline{a} \in \mathcal{M}$ and $\alpha \leq \beta$.

$$\widehat{\Phi}^{\mathcal{M}}_{\overline{a},0}(\overline{v}) = \Phi^{\mathcal{M}}_{\overline{a},0}(\overline{v}).$$

For $\alpha > 0$

$$\widehat{\Phi}^{\mathcal{M}}_{\overline{a},\alpha}(\overline{v}) = \bigwedge_{\gamma<\alpha} \left[ \widehat{\Phi}_{\overline{a},\gamma}(\overline{v}) \wedge \bigwedge_{b\in\mathcal{M}} \exists w \; \widehat{\Phi}^{\mathcal{M}}_{\overline{a},b,\gamma}(\overline{v}, w) \wedge \forall \overline{w} \bigvee_{b\in\mathcal{M}} \widehat{\Phi}^{\mathcal{M}}_{\overline{a},b,\gamma}(\overline{v}, w) \right].$$

It is easy to see that $\widehat{\Phi}^{\mathcal{M}}_{\overline{a},\alpha}(\overline{v})$ is equivalent to $\Phi^{\mathcal{M}}_{\overline{a},\alpha}(\overline{v})$.

Here is a sketch of the details. Let $R$ be a recursive well order of $\omega$ of order type $\beta + 1$. Suppose $f : \omega \times \omega^{<}\omega \to \omega$ is partial recursive we define a new partial recursive $\widehat{f} : \omega \times \omega^{<\omega} \to \omega$. We think of $\widehat{f}(n, \overline{a})$ and $f(n, \overline{a})$ as coding formulas let $\psi_e$ denote the formula coded by $e$. Let

[13] We could avoid this by beeing more careful about ordinal notations.

• $\widehat{f}(0,\overline{a}) = g(\overline{a})$ a code for the conjunction of all atomic and negated atomic sentence true about $\overline{a}$ in $\mathcal{M}$;
• $\widehat{f}(n,\overline{a})$ codes

$$\bigwedge_{nRm}\left[\psi_{f(n,\overline{a})}(\overline{v}) \wedge \bigwedge_{b\in\mathcal{M}} \exists w\ \psi_{f(m,\overline{a},b)}(\overline{v},w) \wedge \forall w \left(\bigvee_{b\in\mathcal{M}} \psi_{f(m,\overline{a},b)}(\overline{v},w)\right)\right].$$

Note, this is just a formal manipulation of potential codes and makes sense even $f$ is not outputting actual codes. These manipulations are clearly computable. Thus there is a total recursive $F:\omega\to\omega$ such that if $f=\phi_e$, then $\widehat{f}=\phi_{F(e)}$. By the Recursion Theorem 9.9, there is $\widehat{e}$ such that $\phi_{\widehat{e}} = \phi_{F(\widehat{e})}$.

We can then show by induction that $\phi_e(n,\overline{a})$ codes $\widehat{\Phi}^{\mathcal{M}}_{\overline{a},\alpha}$ where $\{m : mRn\}$ has order type $\alpha$. □

**Lemma 11.16** *Suppose $\mathcal{M}$ is computable in $z$ and $\mathbb{A}$ is the fragement of $\mathcal{L}_{\omega_1,\omega}$-formulas recursive in $z$. If $\mathcal{M} \equiv_{\mathbb{A}} \mathcal{N}$, then $\mathcal{M} \equiv_{\omega_1^{\mathcal{M}}} \mathcal{N}$.*

**Proof** For all $\beta<\alpha$, $\Phi^{\mathcal{M}}_{\beta}$ is recursive in $z$. Since $\mathcal{N}\models\Phi^{\mathcal{M}}_{\beta}$, $\mathcal{N}\equiv_{\beta}\mathcal{M}$. Since $\mathcal{N}\equiv_{\beta}\mathcal{M}$ for all $\beta<\omega_1^{\mathcal{M}}$, $\mathcal{N}\equiv_{\omega_1^{\mathcal{M}}}\mathcal{M}$ □

**Theorem 11.17 (Nadel [43])** *If $\mathcal{M}\in\mathbb{X}_\tau$, then* $\mathrm{SR}(\mathcal{M})\le\omega_1^{\mathcal{M}}+1$.

Before proving Nadel's Theorem, we argue that it is best possible. Consider the Harrision ordering $\mathcal{H}=(\omega,\prec)\cong\omega_1^{\mathrm{ck}}+(\mathbb{Q}\times\omega_1^{\mathrm{ck}})$, where $\mathbb{Q}\times\omega_1^{\mathrm{ck}}$ is ordered lexicographically. Suppose $a,b\in\omega$ correspond to elements of the form $(q,0)$ for $q\in\mathbb{Q}$. Then there is an automorphism of $\mathcal{H}$ taking $a$ to $b$. Also note that any automorphism of $\mathcal{H}$ fixes the $\omega_1^{\mathrm{ck}}$ initial segment of $\mathcal{H}$. Suppose $\beta<\omega_1^{\mathrm{ck}}$ and

$$\Phi^{\mathcal{H}}_{a,\beta}(v)\models\Phi^{\mathcal{H}}_{a,\alpha}(v)$$

for all $\alpha$. The sentence $\Phi^{\mathcal{H}}_{a,\beta}$ is recursive. But using this formula we can define the well-founded initial segment of $\mathcal{H}$ by

$$X=\{n:\forall v\ \Phi^{\mathcal{H}}_{a,\beta}(v)\to n<v\}$$

This set is $\Delta^1_1$ and $e\in O$ if and only if $e$ codes a linear order embedable into $X$. But this gives a $\Sigma^1_1$-definition of $O$, a contradiction. Thus $\mathrm{SR}(\mathcal{H})\ge\omega_1^{\mathrm{ck}}+1$.

We now do some preparation of the proof of Theorem 11.17.

Let $\mathrm{WO}^*$ be the set of all linear orders $R$ with domain $\omega$ such that:

i) 0 is the $R$-least element of $\omega$;

ii) if $x$ is not $R$-maximal, then there is $y$ such that $xRy$ and there is no $z$ such that $xRz$ and $zRx$, we say $y$ is the $R$-successor of $x$ and write $y = s_R(x)$. If $n \neq 0$ is not an $R$-successor we say it is an $R$-limit.

Note that $\mathrm{WO}^*$, $s_R(n) = m$ and "$n$ is an $R$-limit" are arithmetic.

We say that $z$ is an *$R$-analysis of $\mathcal{M}$ and $\mathcal{N}$* if

i) $z \subseteq \omega \times \bigcup_{n\in\omega}(\omega^n \times \omega^n)$;

ii) $(0, \overline{a}, \overline{b}) \in z$ if and only if $\mathcal{M} \models \phi(\overline{a}) \leftrightarrow \mathcal{N} \models \phi(\overline{b})$ for all quantifier free $\phi$;

iii) if $(n, \overline{a}, \overline{b})$ and $mRn$, then $(m, \overline{a}, \overline{b})$;

iv) $(s_R(n), \overline{a}, \overline{b}) \in z$ if and only if for all $c \in \omega$ there is $d \in \omega$ such that $(n, \overline{a}^\frown c, \overline{b}^\frown d) \in z$ and for all $d \in \omega$ there is $c \in \omega$ such that $(n, \overline{a}^\frown c, \overline{b}^\frown d) \in z$;

v) if $n$ is an $R$-limit, then $(n, \overline{a}, \overline{b}) \in z$ if and only if $(m, \overline{a}, \overline{b}) \in z$ for all $mRn$.

Note:

- $\{(z, R, M, N) : z$ is an $R$-analysis of $\mathcal{M}$ and $\mathcal{N}\}$ is arithmetic.
- Suppose $R$ is a well-order of order type $\alpha$. Let $\beta(n) < \alpha$ be the order type of $\{m : mRn\}$. If $z$ is an $R$-analysis of $\mathcal{M}$ and $\mathcal{N}$, then

$$(n, \overline{a}, \overline{b}) \in z \text{ if and only if } (\mathcal{M}, \overline{a}) \sim_{\beta(n)} (\mathcal{N}, \overline{b}).$$

- Suppose $R \in \mathrm{WO}^*$ is not a well-ordering and $z$ is an $R$-analysis of $\mathcal{M}$ and $\mathcal{N}$. If $\dots n_2 R n_1 R n_0$ is a descending chain and $(n_0, \overline{a}, \overline{b}) \in z$, then there is an isomorphism $\sigma : \mathcal{M} \to \mathcal{N}$ with $\sigma(\overline{a}) = \overline{b}$. [Do a back and forth building $\overline{a}_0 \subset \overline{a}_1 \subset \overline{a}_2 \subset \dots$ and $\overline{b}_0 \subset \overline{b}_1 \subset \overline{b}_2 \subset \dots$ such that $(n_i, \overline{a}_i, \overline{b}_i) \in z$ and $\bigcup \overline{a}_i = \bigcup \overline{b}_i = \omega$. Then $\overline{a}_i \mapsto \overline{b}_i$ is the desired isomorphism.]
- We say that $z$ is an $R$-analysis of $\mathcal{M}$ if it is an analysis of $\mathcal{M}$ with itself.

**Proof of Theorem 11.17**

We need to show that if $\overline{a}, \overline{b} \in \omega$ and $(\mathcal{M}, \overline{a}) \not\sim (\mathcal{M}, \overline{b})$, then there is $\alpha < \omega_1^{\mathcal{M}}$ such that that $(\mathcal{M}, \overline{a}) \not\sim_\alpha (\mathcal{M}, \overline{b})$. Fix $\overline{a}, \overline{b}$ with $(\mathcal{M}, \overline{a}) \not\sim (\mathcal{M}, \overline{b})$.

$S = \{R \in \mathrm{WO}^* : \exists z$ an $R$-analysis of $\mathcal{M}$ and $n \in \omega$ such that $n$ is $R$-maximal and $(n, \overline{a}, \overline{b}) \in z\}$.

If $R \in S$ and $R$ is not well-founded, then by the remarks above there is an automorphism $\sigma$ of $\mathcal{M}$ with $\sigma(\overline{a}) = \overline{b}$ and $(\mathcal{M}, \overline{a}) \sim (\mathcal{M}_y, \overline{b})$, a contradiction. Thus $S \subseteq \mathrm{WO}$.

But $S$ is $\Sigma^1_1(\mathcal{M})$. Thus, by $\Sigma^1_1$-Bounding, there is $\alpha < \omega_1^{\mathcal{M}}$ such that every element of $R$ has order type $\leq \alpha$. But then $(\mathcal{M}, \overline{a}) \not\sim_\alpha (\mathcal{M}, \overline{b})$.

Hence $\mathrm{sr}(\mathcal{M}) \leq \omega_1^{\mathcal{M}}$ and $\mathrm{SR}(\mathcal{M}) \leq \omega_1^{\mathcal{M}} + 1$. □

**Theorem 11.18 (Nadel [43])** *Suppose $\mathcal{M}, \mathcal{N} \in \mathbb{X}_\tau$ are recursive in $x$ and $\mathcal{M} \equiv_{\mathbb{A}} \mathcal{N}$ where $\mathbb{A}$ is the fragement of $\mathcal{L}_{\omega_1,\omega}$-formulas recursive in $x$. Then $\mathcal{M} \cong \mathcal{N}$.*

**Proof** Let $A = \{R \in \mathrm{WO}^* : \exists z\ z$ is an $R$-analysis of $\mathcal{M}$ and $\mathcal{N}$ and $(n, \emptyset, \emptyset) \in R$ for all $n \in \omega\}$. If some $R \in A$ is not a well ordering then $\mathcal{M} \cong \mathcal{N}$ as desired. So assume that $A \subseteq \mathrm{WO}$. Since $A$ is $\Sigma^1_1 \in x$, there is $\beta < \omega_1^x$ such that every $R \in A$ has order type less that $\beta$. But then $\mathcal{M} \not\sim_\beta \mathcal{N}$. But $\Phi^{\mathcal{M}}_\beta \in \mathbb{A}$. Thus $\mathcal{N} \models \Phi^{\mathcal{M}}_\beta$ and $\mathcal{M} \sim_\beta \mathcal{N}$, a contradiction. □

**Theorem 11.19** *Suppose $\tau$ is a recursive vocabulary and $\phi$ is a hyperarithmetic $\mathcal{L}_{\omega_1,\omega}$-sentence. Then $\cong$ is $\Delta^1_1$ on $\mathrm{Mod}(\phi)$ if and only if there is $\alpha < \omega_1^{\mathrm{ck}}$ such that every model of $\phi$ has Scott rank at most $\alpha$.*

*Relativizing, for any $\phi \in \mathcal{L}_{\omega_1,\omega}$, $\cong$ is a Borel equivalence relation on* $\mathrm{Mod}(\phi)$ *if and only if there is a countable bound on Scott rank.*

**Proof** ($\Leftarrow$) If all models have Scott rank below $\alpha$, then $\cong$ is $\sim_\alpha$ a $\Delta^1_1$ equivalence relation.

($\Rightarrow$) Let $A = \{R \in \mathrm{WO}^* : \exists \mathcal{M}, \mathcal{N} \in \mathrm{Mod}(\phi)\ \mathcal{M} \ncong \mathcal{N} \wedge \exists z\ z$ is an $R$-analysis of $\mathcal{M}$ and $\mathcal{N}$ and $(n, \emptyset, \emptyset) \in \mathbb{Z}$ for all $\in \omega\}$. Then $A$ is $\Sigma^1_1$. As above if $R \in A$ is not a well-ordering and $z$ is an $R$-analysis of $\mathcal{M}$ and $\mathcal{N}$ where $(n, \emptyset, \emptyset) \in z$ for all $n$, then $\mathcal{M} \cong \mathcal{N}$. Thus $A$ is a $\Sigma^1_1$ set of well-orders and, by $\Sigma^1_1$-Bounding, there is $\alpha < \omega_1^{\mathrm{ck}}$ such that every element of $A$ has order type less that $\alpha$. It follows that we can bound Scott rank below $\omega_1^{\mathrm{ck}}$. □

**Exercise 11.20** Show that $\{\phi \in \mathcal{L}_{\omega_1,\omega} : \phi$ is complete$\}$ is $\Pi^1_1$. [Hint: Argue that $\phi$ is complete if and only if

- $\exists \mathcal{M} \leq_{\mathrm{hyp}} \phi\ \mathcal{M} \models \phi$;
- for all $\mathcal{M}, \mathcal{N}\ \exists R \exists z \leq_{\mathrm{hyp}} \phi\ R \in WO^*$ and $z$ is an $R$-analysis of $\mathcal{M}$ and $\mathcal{N}$ and there is $n$ such that if $\overline{a}, c \in \mathcal{M}$ and $\overline{b} \in \mathcal{N}$ with $(n, \overline{a}, \overline{b}) \in z$ then there is $d \in \mathcal{N}$ such that $(n, \overline{a}, c, \overline{b}, d) \in z$.]

**Theorem 11.21 (Sacks [45])** *Suppose* $\mathrm{SR}(\mathcal{M}) \leq \omega_1^{\mathcal{M},\phi}$ *for all* $\mathcal{M} \in \mathrm{Mod}(\phi)$, *then there is $\alpha < \omega_1^\phi$ such that* $\mathrm{SR}(\mathcal{M}) < \alpha$ *for all $\mathcal{M} \models \phi$. In particular, $\cong$ is a Borel equivalence relation on* $\mathrm{Mod}(\phi)$ *and Vaught's Conjecture holds for $\phi$.*

**Proof** For notational simplicity we assum $\phi \in$ HYP. Suppose $\mathrm{SR}(\mathcal{M}) \leq \omega_1^{\mathcal{M}}$ for all $\mathcal{M} \in \mathrm{Mod}(\phi)$. Then for all $\mathcal{M}$ and $\overline{a} \in \mathcal{M}$ there is $\alpha < \omega_1^{\mathcal{M}}$ such that if $\overline{b} \in \mathcal{M}$ and $(\mathcal{M}, \overline{a}) \sim_\alpha (\mathcal{M}, \overline{b})$, then $(\mathcal{M}, \overline{a}) \sim (\mathcal{M}, \overline{b})$. We let $r(\overline{a})$ denote the least such $\alpha$.

Let $S_1 = \{(\mathcal{M}, \overline{a}, e) : e \in \omega$ and if $\mathcal{M} \models \phi$, then
$phi_e^{\mathcal{M}}$ is the characteristic function of a well-order $R$ and for any $R$-analysis $z$ of $\mathcal{M}$ there is $n$ such that for all $\overline{b}$ if $(n, \overline{a}, \overline{b}) \in z$, then $(s_R(n), \overline{a}, \overline{b})\}$.

The set $S_1$ is $\Pi_1^1$. If $\mathcal{M} \not\models \phi$, then $(\mathcal{M}, \overline{a}, e) \in S_1$ for all $e \in \omega$. If $\mathcal{M} \models \phi$ then $(\mathcal{M}, \overline{a}, e) \in S_1$ if and only if $\{e\}^{\mathcal{M}}$ is the characteristic function of an ordinal $\alpha > r(\overline{a})$.

By Kreisel's uniformization Theorem 9.29, there is $f : \mathbb{X}_\tau(\phi) \times \bigcup_n \omega^n \to \omega$ such that $(\mathcal{M}, \overline{a}, f(\mathcal{M}, \overline{a})) \in S_1$ for all $x, \overline{a}$. Since the range is $\omega$, $f(\mathcal{M}, \overline{a}) \neq e$ if and only if $\exists m \neq n \ f(x, \overline{a}) = m$, and $f$ is $\Delta_1^1$.

Let $S_2 = \{R : \exists \mathcal{M} \exists \overline{a} \ \mathcal{M} \models \phi \wedge R$ is isomorphic to the ordering coded by $\phi_{f(\mathcal{M}, \overline{a})}^{\mathcal{M}}\}$. Then $S_2$ is a $\Sigma_1^1$ set of well-orderings and, by $\Sigma_1^1$-Bounding, there is $\alpha < \omega_1^{\mathrm{ck}}$ such every $R$ in $S_1$ has order type at most $\alpha$. Thus for all $\mathcal{M}_x \models \sigma$ we have $r(\overline{a}) < \alpha$ for all $\overline{a}$. Thus $\mathrm{SR}(\mathcal{M}) \leq$
$alpha$ for all $\mathcal{M} \models \sigma$. □

Note: This argument would work just as well for $\mathrm{PC}_{\omega_1, \omega}$-classes.

**Exercise 11.22** In Exercise 2.21 we showed that for and $\alpha < \omega_1$ there is a formula $\sigma_\alpha \in \mathcal{L}_{\omega_1, \omega}(\tau)$ such

$$\mathcal{M} \models \sigma_\alpha \Leftrightarrow \mathrm{SR}(\mathcal{M}) \geq \alpha$$

for any $\tau$-structure $\mathcal{M}$. Show that if $\alpha < \omega_1^x$, then $\sigma_\alpha$ is computable in $x$ and $\{\sigma_\alpha : \alpha < \omega_1^x\}$ is $\Pi_1^1(x)$.

We will need the following result in §13.

**Theorem 11.23** *Suppose $\phi \in$ HYP, $\alpha$ is admissible and for all $\beta < \alpha$, $\phi$ has a model of Scott rank at least $\beta$. Then there is $\mathcal{M} \models \phi$ with* $\mathrm{SR}(\mathcal{M}) \geq \omega_1^{\mathcal{M}} = \alpha$.

**Proof** Choose $x$ such that $\omega_1^x = \alpha$.

Define $\Theta_0^n(\overline{u}, \overline{v})$ $|u| = |v| = n$ to be

$$\bigwedge_{\psi \text{ atomic}} \psi(\overline{u}) \leftrightarrow \psi(\overline{v})$$

and define $\Theta^n_\beta(\overline{u},\overline{v})$ to be

$$\bigwedge_{\gamma<\beta}\left[\Theta^n_\gamma(\overline{u},\overline{v})\wedge\forall x\exists y\ \Theta^{n+1}_\gamma(baru,x,\overline{v},y)\wedge\forall y\exists x\ \Theta^{n+1}_\gamma)\overline{u},x,\overline{v},y)\right].$$

Note that if $\overline{a},\overline{b}\in\mathcal{M}$ and $|\overline{a}|=|\overline{b}|=n$, then $\overline{a}\sim_\beta\overline{b}$ if and only if $\mathcal{M}\models\Theta^n_\beta(\overline{a},\overline{b})$. Let $\sigma_\beta$ be the sentence

$$\bigvee_{n<\omega}\exists\overline{u}\exists\overline{v}\ \Theta^n_\beta(\overline{u},\overline{v})\wedge\neg\Theta^n_{\beta+1}(\overline{u},\overline{v}).$$

Then $\mathcal{M}\models\sigma_\beta$ if and only if $\mathrm{SR}(\mathcal{M})>\beta$.

As in the proof of Theorem **??**, if $\beta<\alpha$, then we can find a codes for $\sigma_\beta$ recursive in $x$ and we can find a $\Pi^1_1(x)$ set of codes for $\{\sigma_\beta:\beta<\alpha\}$. By Kreisel–Barwise Compactness there are models of $\{\phi\}\cup\{\sigma_\beta:\beta<\alpha\}$ and, by Gandy's Basis Theorem 10.19 there is $\mathcal{M}\models\{\phi\}\cup\{\sigma_\beta:\beta<\alpha\}$ with $\omega_1^{\mathcal{M}}\leq\alpha$. Since $\mathrm{SR}(\mathcal{M})\leq\omega_1^{\mathcal{M}}+1$, we must have $\mathrm{SR}(\mathcal{M})\geq\omega_1^{\mathcal{M}}=\alpha$. □

**Exercise 11.24** Show that for any Vaught counterexample $\phi$ and $z\geq_T\phi$ there is $\mathcal{M}\in\mathrm{Mod}(\phi)$ with $\mathrm{SR}(\mathcal{M})\geq\omega_1^{\mathcal{M},z}=\omega_1^z$. [Hint: Relativize the previous proof.]

Finally, we give the proof promised in Proposition 3.20.

**Proposition 11.25** *Suppose there is a perfect set of non-isomorphic countable models of $\phi$. Then for some $\alpha<\omega_1$ there are uncountably many $\sim_\alpha$-classes.*

**Proof** Let $P$ be a perfect set of non-isomorphic models. Let $A=\{R\in\mathrm{WO}^*:\exists\mathcal{M},\mathcal{N}\in P$ distinct $\wedge\exists z$ an $R$-analysis of $\mathcal{M}$ and $\mathcal{N}$ and $(n,\emptyset,\emptyset)\in z$ for all $n\in\omega\}$. As usual, if $A$ contains an ill-founded order $R$ and $z$ is an $R$-analysis of $\mathcal{M}$ and $\mathcal{N}$ as above, then we can do a back-and-forth argument to show $\mathcal{M}\cong\mathcal{N}$, a contradiction. Thus $A\subseteq\mathrm{WO}$. But $A$ is $\mathbf{\Sigma}^1_1$. Thus by $\mathbf{\Sigma}^1_1$-bounding, there is $\alpha<\omega_1$ such that every $R\in A$ has order type less than $\alpha$. It follows that if $\mathcal{M},\mathcal{N}\in P$ and $\mathcal{M}\neq\mathcal{N}$, then $\mathcal{M}\not\sim_\alpha\mathcal{N}$. □

We can also give a stronger version of this.

**Proposition 11.26** *If there is a perfect set of non-isomorphic countable models of $\phi$, then for some $\alpha<\omega_1$ there is a perfect set of models of Scott rank $\alpha$.*

**Proof** Suppose $T$ is a perfect tree such that $[T]$ the paths thru $T$ is a perfect set of non-isomorphic models of $T$. By Exercise 10.20 there is a perfect subset $P$ of $[T]$ such that $\omega_1^x \leq \omega_1^T$ for all $x \in P$. But then, by Nadel's Theorem 11.17, each model coded in $P$ has Scott rank at most $\omega_1^T$. □

**Exercise 11.27** Give a second proof of Proposition 11.26 using Proposition 11.25.

# 12 Admissibility

## 12.1 Admissible Sets

## 12.2 Model Theory and Admissible Fragments

To be added later

# 13 Spectra of Vaught Counterexamples

Suppose $\mathcal{M}$ is a countable $\tau$-structure.

**Definition 13.1** The *degree spectrum* of $\mathcal{M}$ is

$$\mathrm{Sp}(\mathcal{M}) = \{z : \exists \mathcal{N} \leq_T z \ \mathcal{M} \cong \mathcal{N}\}$$

i.e., the set of reals where there is a computable isomorphic copy of $\mathcal{M}$.

Suppose $\alpha$ is a countable ordinal. Then $z \in \mathrm{Sp}(\alpha)$ if and only if the ordinal $\alpha$ is recursive in $z$. Thus $z \in \mathrm{Sp}(\alpha)$ if and only if $\omega_1^z \geq \alpha$.

If we review the counterexamples we have seen to Vaught's Conjecture for $\mathrm{PC}_{\omega_1,\omega}$ classes in Exercises 3.10 and 3.11. We see there too that for any $\mathcal{M}$ there is an ordinal $\alpha$ such that $\mathrm{Sp}(\mathcal{M}) = \{z : \omega_1^z \geq \alpha\}$. This corresponds to our intuition that the models of Vaught counterexamples in someway behave like ordinals.

In this section we will prove a warm-up result to the main theorem Montalbán [39] where, under the additional assumption PD (i.e., that all projective games are determined), he makes this intuition surprisingly precise.

**Theorem 13.2 (Montalbán [39])** (PD) *The following are equivalent:*
*i) $\phi$ is a Vaught counterexample;*
*ii) there is an oracle relative to which*

$$\{\mathrm{Sp}(\mathcal{M}) : \mathcal{M} \models \phi\} = \{\{z : \omega_1^z \geq \alpha\} : \alpha < \omega_1\}.$$

It is unknown if this result can be proved in ZFC alone. We will prove the *i)* $\Rightarrow$ *ii)* direction under the additional assumption that $\phi$ is a minimal counterexample.

## 13.1 Determinacy and Turing Degrees

Montalbán's work relies on several results of Martin [37] using determinacy to understand the structure of the Turing degrees. We begin with a survey of these results. The results of this section are due to Martin, though some appear first in [53].

**Pointed perfect sets**

**Definition 13.3** We say that a tree $T \subset 2^{<\omega}$ is a *perfect* if $T \neq \emptyset$ and for any $\sigma \in T$ there are incomparable $\sigma_0, \sigma_1 \in T$ with $\sigma_0, \sigma_1 \supset \sigma$.

A perfect tree $T$ is *pointed* if $T \leq_T f$ for all $f \in [T]$, where we recall that $[T]$ is the set of infinite paths through $T$. We say that $P \subseteq 2^\omega$ is a *pointed perfect set* if $P = [T]$ for some pointed perfect tree.

**Exercise 13.4** Suppose $f : 2^\omega \to 2^\omega$ is continuous, one-to-one. Let

$$T = \{\eta \in 2^{<\omega} : \exists x \in 2^\omega \ \eta \subset f(x)\}.$$

Show that $T$ is a perfect tree.

**Lemma 13.5** *Suppose $f$ is computable. Then the tree $T$ in the exercise above is computable.*

**Proof** There is a program $Q$ that using oracle $x$ and input $i$ computes $f(x)(i)$. For $\nu \in 2^{<\omega}$ we say $f(\nu)(i) = j$ if the computation of $Q$ using oracle $\nu$ and input $i$ halts in at most $|\nu|$ steps making no oracle queries outside the domain of $\nu$ and outputs $j$. Then

$$\eta \in T \Leftrightarrow \exists \nu \forall i < |\eta| \ f(\nu)(i) = \eta(i)$$

and

$$\eta \notin T \Leftrightarrow \exists n \forall \nu \in 2^n \exists i < |\eta| \ f(\nu)(i) \neq \eta(i)$$

It follows that $T$ is computable. □

**Corollary 13.6** *Suppose $f : 2^\omega \to 2^\omega$ is continuous, one-to-one, computable in $d$ and $d \leq_T f(x)$ for all $x \in 2^\omega$. Then the image of $f$ is a pointed perfect set.*

**Lemma 13.7** *If $T$ is a pointed perfect tree, then there is a $T$-computable continuous one-to-one $f : 2^\omega \to [T]$. Further, $f$ $z \geq_T T$, then $f(z) \equiv_T z$.*

**Proof** Let $x \in 2^\omega$. We build a path $\emptyset = \eta_0 \subset \eta_1 \subset \ldots$ through $T$ where $|\eta_i| = i$. Let $c_0 = 0$; $c$ will count how many choices we have made so far.

If $\eta_s$ has only one extension in $T$, let $\eta_{s+1}$ be this unique extension and let $c_{s+1} = c_s$. If not, let $\eta_{s+1} = \eta_s{}^\frown x(c_s)$ and $c_{s+1} = c_s + 1$. Define $f(x) = \bigcup \eta_s$.

Clearly if $z \geq_T T$, then $f(z) \leq_T z$, since $f$ is $T$-computable. On the other hand, since $T$ is pointed, $f(x) \geq_T T$. Using $f(x)$ and $T$ we can recover $x$. Start building $\eta_0 \subset \eta_1 \ldots$ as above. By noticing what we did when we had choices, we can compute $x$. □

**Turing determincay**

Recall that if $A \subset 2^\omega$, then $G(A)$ is the game where players alternate playing 0 or 1

$$\begin{array}{c|c} \text{I} & \text{II} \\ \hline n_0 & \\ & n_1 \\ n_2 & \\ & n_3 \\ \vdots & \vdots \end{array}$$

and Player II wins if $x \in A$ where $x(i) = n_i$ for all $i$.

If $\sigma$ is a strategy for Player II, define $\sigma^* : 2^\omega \to 2^\omega$ by $\sigma^*(x)$ is the final play of the game when Player I plays $x(0), x(1), \ldots$ and Player II uses $\sigma$.

$$\begin{array}{c|c} \text{I} & \text{II} \\ \hline x(0) & \\ & \sigma(x(0)) \\ x(1) & \\ & \sigma(x(0), x(1)) \\ \vdots & \vdots \end{array}$$

If $\sigma$ is a winning strategy for Player II, then $\sigma^* : 2^\omega \to A$. We can define analogous maps from strategies for Player I.

**Lemma 13.8** *If Player II has a winning strategy in $G(A)$, then $A$ contains a pointed perfect set.*

**Proof** Let $\sigma$ be a winning strategy for Player II. Suppose $\sigma \equiv_T b \in 2^\omega$. Let $g : 2^\omega \to A$ be the function

$$g(x) = \sigma^*(x(0), b(0), x(1), b(1), \ldots).$$

The image of $g$ is the set of plays in the game where Player II uses $\sigma$ and Player I plays

$$x(0), b(0), x(1), b(1), \ldots$$

Then $g$ is continuous, injective and computable in $\sigma$. Moreover, $\sigma \equiv_T b \leq_T g(x)$ for all $x \in 2^\omega$. Thus $A$ contains a pointed perfect set. $\square$

A similar argument shows that if Player I has a winning strategy, then $2^\omega \setminus A$ contains a pointed perfect set.

Note in the argument above that $g(x) \leq_T (x,\sigma)$, $x \leq_T g(x)$ and $\sigma \leq_T g(x)$.

**Definition 13.9** A subset $C$ of $2^\omega$ is a *cone* if there is $d \in 2^\omega$ such that $C = \{x \in 2^\omega : d \leq_T x\}$.

We say that $A \subseteq 2^\omega$ is *Turing invariant* if $y \in A$ whenever there is $x \in A$ with $x \equiv_T y$.

**Exercise 13.10** Let $A \subseteq 2^\omega$ be Turing-invariant. At most one of $A$ and $2^\omega \setminus A$ contains a cone.

**Corollary 13.11 (Turing Determinacy)** (PD) *If $A \subseteq 2^\omega$ is projective and Turing-invariant, then one of $A$ and $2^\omega \setminus A$ contains a cone.*

**Proof** If $\sigma$ is, say, a winning strategy for Player II in $G(A)$ and $g$ is as above, then for any $x \geq_T \sigma$, $x \equiv_T g(x)$, so $x \in A$. Thus $A$ contains the cone above $\sigma$. □

**Exercise 13.12** Assume AD the Axiom of Determinacy, i.e., that $G(A)$ is determined for all $A \subset 2^\omega$. Define $\mathcal{U} \subset \mathcal{P}(\omega_1)$ by

$$A \in \mathcal{U} \Leftrightarrow \{x : \omega_1^x \in A\} \text{ contains a cone.}$$

Show that $\mathcal{U}$ is a non-principle $\sigma$-complete ultrafilter on $\omega_1$. This shows that AD$\Rightarrow \aleph_1$ is a measurable cardinal.

**Lemma 13.13** (PD) *Suppose $A \subseteq 2^\omega$ is projective and cofinal in the Turing degrees (i.e., for all $x$ there is $x \leq_T y$ with $y \in A$). Then there is a pointed perfect set $P \subseteq A$.*

**Proof** Consider the game where Player I plays $e \in \omega$ and then plays $x \in 2^\omega$, and Player II plays $y \in 2^\omega$. Player I wins if $x \in A$ and $y = \phi_e^x$ (i.e., $y$ is computable using Turing machine $e$ and oracle $x$). We claim that Player I has a winning strategy. Suppose, for purposes of contradiction, that Player II has a winning strategy $\sigma$. Choose $x \in A$ such that $\sigma \leq_T x$. Consider the $x$-recursive function $f$ where $f(e,n)$ is the $n^{\text{th}}$ move by Player II using $\sigma$ if Player I first plays $e$ and then plays $x(0), x(1), \ldots$, i.e, if Player II uses $\sigma$ this is a play of the game

| I | II |
|---|---|
| $e$ | |
| | $f(e,0)$ |
| $x(0)$ | |
| | $f(e,1)$ |
| $x(1)$ | |
| | $f(e,2)$ |
| $\vdots$ | $\vdots$ |

By the Recursion Theorem 9.9, there is an $e$ such that $\phi_e^x(n) = f(e,n)$ for all $n$. Let $y = \phi_e^x$. But since $x \in A$, Player I wins by playing $e$ followed by $x$, contradicting the fact $\sigma$ is a winning strategy of Player II.

Thus Player I has a winning strategy $\sigma$. Let $e$ be the first move made by Player I using this strategy. Let $z \in 2^\omega$ be such that $z \equiv_T \sigma$. Let $g : 2^\omega \to 2^\omega$ be such that $e^\frown g(y)$ is the element of $2^\omega$ played by Player I if Player II plays

$$z(0), y(0), z(1), y(1), \ldots$$

and one uses $\sigma$. Since $y = \phi_e^{g(y)}$, $g$ is one-to-one. It is not immediately clear that the image of $g$ contains a perfect pointed set. But consider the map $h : 2^\omega \to 2^\omega$ where $h(x) = g(z(0), x(0), z(1), x(1), \ldots)$. Then $h$ is also continuous, one-to-one and computable from $\sigma$. Note that $(z(0), x(0), z(1), x(1), \ldots) = \phi_e^{h(x)}$, thus $\sigma$ is computable from $h(x)$, for all $x \in 2^\omega$. Thus the image of $h$ is a pointed perfect set. □

**Corollary 13.14** (PD) *Suppose $f : 2^\omega \to \omega$ is projective. Then there is a pointed perfect set $P$ such that $f$ is constant on $P$.*

**Proof** If $f^{-1}(i)$ is cofinal in the Turing degrees, then it contains a pointed perfect set. Otherwise there is $z_i$ such that if $z_i \leq_T x$, then $f(x) \neq i$. If no $f^{-1}(i)$ contains a pointed perfect set, then we can find $x_T \geq z_i$ for all $i$. But then $f(x) \neq i$ for any $i$, a contradiction. □

**Theorem 13.15** (PD) *Suppose $f : 2^\omega \to \omega_1$ is Turing-invariant and there is a projective $g : 2^\omega \to WO$ such that $f(x)$ is the order type of the well ordering $g(x)$. If $f(x) < \omega_1^x$ for all $x$, then $f$ is constant on a cone.*

**Proof** There is $h : 2^\omega \to \omega$ projective such that $\phi_{h(x)}^x \in WO$ has order type $f(x)$. By the last Corollary, there is a pointed perfect tree $T$ and $e \in \omega$ such that $h(x) = e$ for $x \in [T]$. The map $x \mapsto \phi_e^x$ is continuous on $[T]$. Thus by $\mathbf{\Sigma}_1^1$-bounding, there is $\alpha < \omega_1$ such that the order type of $\phi_e^x < \alpha$ for all $x \in [T]$.

For $\beta < \alpha$, let $A_\beta = \{x \geq_T T : f(x) = \beta\}$. This is a Turing-invariant, projective set. If it contains a cone we are done, if not there is a $z_\beta$ such that $x \notin A_\beta$ if $x \geq_T z_\beta$. But then we can find $x \geq_T T$ such that $x \geq z_\beta$ for all $\beta < \alpha$. We can find $w \in [T]$ such that $w \equiv_T x$, but then $|f(w)| \neq \beta$ for all $\beta < \alpha$, a contradiction. $\square$

## 13.2 Montalbán's Theorem

**Definition 13.16** We say that an equivalence relation $E$ on a set $X \subseteq 2^\omega$ is *ranked* if there is an $E$-invariant $r : X \to \omega_1$. We say that this is a *projective ranking* if there is a projective $g : X \to WO$ such that $r(x)$ is the order type of $g(x)$.

We say that $(E, r)$ is *scattered* if $r^{-1}(\alpha)$ has only countably many $E$-classes for all $\alpha < \omega_1$.

Of course we are thinking about the case where $X = \mathrm{Mod}(\phi)$, $E$ is isomorphism and $r(\mathcal{M})$ is the Scott rank of $\mathcal{M}$. Then $\phi$ is scattered if and only if $(E, r)$ is scattered.

**Theorem 13.17** (PD) *Let $(E, r)$ be a scattered projective ranked equivalence relation. Then there is a cone of $z$ such that if $x \in X$, if $r(x) < \omega_1^z$, then there is $yEx$ such that $y \leq_T z$.*

**Proof** Suppose not. Then for every cone $C$ there is $z \in C$ and $x$ such that $r(x) < \omega_1^z$ but there is no $z$-computable $yEx$. By Turing Determinacy, there is a cone $C_0$ of such $z$. For $z \in C_0$ let $\alpha_z$ be least such that there is $x$ with $r(x) = \alpha_z < \omega_1^z$ but no $z$-computable $yEx$. By Theorem 13.15, there is a cone $C_1 \subseteq C$ such that $\alpha_z = \alpha$ for $z \in C_1$. But there are only $\aleph_0$ classes of rank at most $\alpha$. Thus we can find $(y_i : i \in \omega)$ a set of representatives. But then we could find $z$ in the cone greater than all of the $y_i$. But for any $x$, if $r(x) \leq \alpha$, then there is $y_i \leq_T z$ with $y_i E x$, a contradiction. $\square$

We will use the following recursion theoretic lemma which we state without proof. This is Lemma 3.6 of [39] where Montalbán proves it using hyperarithmetic Cohen forcing and the Harrison order. Harington (Theorem 2.10 of [15]) proved a similar result by Steel forcing.

**Lemma 13.18** *If $\omega_1^x = \omega_1^y$, then there is $z$ such that*

$$\omega_1^x = \omega_1^{x,z} = \omega_1^z = \omega_1^{y,z} = \omega_1^y.$$

**Lemma 13.19** *For any $z \in 2^\omega$ the set of $z$-admissible ordinals contains a closed unbounded subset.*

**Proof** The set of $\{\alpha : \mathbb{L}_\alpha[z] \prec \mathbb{L}_{\omega_1}[z]\}$ is a closed unbounded set of $z$-admissible ordinals. $\square$

For the following proof, recall that for any structure $\mathcal{M}$ we let $\omega_1^{\mathcal{M}}$ be the minimum of $\{\omega_1^z : z \text{ codes an isomorphic copy of } \mathcal{M}\}$.

**Theorem 13.20** (PD) *Suppose $\phi$ is a minimal Vaught counterexample. There is $z_0 \in 2^\omega$ such that*

$$z \in \mathrm{Sp}(\mathcal{M}) \Leftrightarrow \omega_1^z \geq \omega_1^{\mathcal{M},z_0}$$

*for every $z_0 \leq_T z$ and every $\mathcal{M} \in \mathrm{Mod}(\phi)$.*

**Proof** By Theorem 13.17, there is a cone $\mathcal{C}_4$ with base $z_4$ such that if $z \in \mathcal{C}_4$, $\mathcal{M} \in \mathrm{Mod}(\phi)$ and $\mathrm{SR}(\mathcal{M}) < \omega_1^z$, then $z \in \mathrm{Sp}(\mathcal{M})$. Without loss of generality, we may assume $\phi \leq_T z_4$.

By Exercise 11.24, for any $y \geq_T \phi$, there is $\mathcal{M} \in \mathrm{Mod}(\phi)$ such that $\mathrm{SR}(\mathcal{M}) \geq \omega_1^{\mathcal{M},y} = \omega_1^y$. Thus, taking $z = \mathcal{M} \oplus y$

$$\{z : \exists \mathcal{M} \in \mathrm{Mod}(\phi)\ \mathcal{M} \leq_T z \wedge \mathrm{SR}(\mathcal{M}) \geq \omega_1^{\mathcal{M},z} = \omega_1^z\}$$

is cofinal in the Turing degrees, projective and Turing invariant, and hence contains a cone $\mathcal{C}_3$ with base $z_3$. Without loss of generality, we may assume $\mathcal{C}_3 \subseteq \mathcal{C}_4$.

Let

$$\mathcal{C}_0 = \{z \geq_T \phi : \forall \mathcal{M} \in \mathrm{Mod}(\phi)\ [\mathrm{SR}(\mathcal{M}) \geq \omega_1^{\mathcal{M},z_3} \Rightarrow \exists \mathcal{N} \leq_T z\ \mathcal{M} \cong \mathcal{N}]\}.$$

**claim** $\mathcal{C}_0$ is cofinal in the Turing degrees.

Let $z_2 \geq_T z_3$. We will find $z \in \mathcal{C}_0$ with $z \geq_T z_2$. Recall from Corollary 7.15 that the set $X$ of $\alpha$ such that if $\mathcal{M}, \mathcal{N} \in \mathrm{Mod}(\phi)$ and $\mathrm{SR}(\mathcal{M}), \mathrm{SR}(\mathcal{N}) \geq \alpha$, then $\mathcal{M} \equiv_\alpha \mathcal{N}$ is closed unbounded. Since the set of $z_2$-admissible ordinals is also closed unbounded, we can find $\alpha \in X$ where $\alpha$ is $z_2$-admissible.

Let $z_1 \geq_T z_2$ such that $\omega_1^{z_1} = \alpha$. We claim that $z_1 \in \mathcal{C}_0$. By Exercise 11.24, there is $\mathcal{M} \in \mathrm{Mod}(\phi)$ with $\mathrm{SR}(\mathcal{M}) \geq \omega_1^{\mathcal{M},z_1} = \omega_1^{z_1}$. Let $y = \mathcal{M} \oplus z_1$.

Then $\mathcal{M} \leq_T y$ and $\omega_1^y = \omega_1^{z_1}$. Since $z_1 \in \mathcal{C}_3$, there is $\mathcal{N} \leq_T z_1$ with $\mathrm{SR}(\mathcal{N}) \geq \omega_1^{\mathcal{N},z_1} = \omega_1^{z_1}$. By Nadel's Theorem 11.17, $\mathrm{SR}(\mathcal{N}) \leq \omega_1^{\mathcal{N}} + 1$, thus we must have $\omega_1^{z_1} = \omega_1^{\mathcal{N}}$.

We will show that $\mathcal{M} \cong \mathcal{N}$. By Lemma 13.18, there is $g \geq z_3$ such that

$$\alpha = \omega_1^{z_1} = \omega_1^{z_1,g} = \omega_1^g = \omega_1^{y,g} = \omega_1^y.$$

Since $g \in \mathcal{C}_3$, there is $\mathcal{M}' \in \mathrm{Mod}(\phi)$ with $\mathrm{SR}(\mathcal{M}') \geq \omega_1^{\mathcal{M}'} = \omega_1^g$. Note that $\mathcal{M} \leq_T y$, $\mathcal{N} \leq_T z_1$ and $\mathcal{M}' \leq_T g$. All of these structures have Scott rank at least $\alpha$, thus they are $\equiv_\alpha$-equivalent.

By Theorem 11.18, since $\mathcal{M}, \mathcal{M}' \leq_T y \oplus g$, $\alpha = \omega_1^{y,g}$ and $\mathcal{M} \equiv_\alpha \mathcal{M}'$, $\mathcal{M} \cong \mathcal{M}'$. By an analogous argument, $\mathcal{N} \cong \mathcal{M}'$. Thus $\mathcal{M} \cong \mathcal{N}$. This proves that $\mathcal{C}_0$ is unbounded in the Turing degees.

By Turing determinacy, $\mathcal{C}_0$ contains a cone with base $z_0$. Let $z \geq_T z_0$ and let $\mathcal{M} \in \mathrm{Mod}(\phi)$ such that $\omega_1^{\mathcal{M},z_0} \leq \omega_1^z$. If $\mathrm{SR}(\mathcal{M}) < \omega_1^z$, then since $z \in \mathcal{C}_4$, $z \in \mathrm{Sp}(\mathcal{M})$. If $\mathrm{SR}(\mathcal{M}) \geq \omega_1^z$, that

$$\omega_1^z + 1 \geq \omega_1^{\mathcal{M},z_0} + 1 \geq \omega_1^{\mathcal{M},z_3} + 1 \geq \mathrm{SR}(\mathcal{M}) \geq \omega_1^z.$$

Thus $\mathrm{SR}(\mathcal{M}) \geq \omega_1^{\mathcal{M},z_3} = \omega_1^z$. Since $z \in \mathcal{C}_\prime$, $z \in \mathrm{Sp}(\mathcal{M})$. $\square$

# A $\aleph_1$-Free Abelian Groups

We will prove that there are $2^{\aleph_1}$ non-isomorphic $\aleph_1$-free abelian groups of cardinality $\aleph_1$. The proof works by coding stationary sets into the groups. This is a standard trick in many model arguments. We need the following set theoretic fact.

**Lemma A.1** *There is a family $(X_\alpha : \alpha < \omega_1)$ of disjoint stationary subsets of $\omega_1$. For $A \subseteq \omega_1$, let $X_A = \bigcup_{\alpha \in A} X_\alpha$. Then $(X_A : A \subset \omega_1)$ is a family of $2^{\aleph_1}$ stationary subsets of $\omega_1$ such that $X_A \triangle X_B$ is stationary for $A \neq B$.*

See, for example, [35] 5.3.10. The following exercise gives a simple example of how we can use stationary sets to build many non-isomorphic models.

**Exercise 1.2** Recall that a linear order $(X, <)$ is $\aleph_1$-*like* if and only if $|X| = \aleph_1$ and $|\{y : y < x\}| \leq \aleph_0$ for all $x \in X$. We will show that there are $2^{\aleph_1}$ non-isomorphic $\aleph_1$-like dense linear orders.

Let $(S_A : A \subset \omega_1)$ be a family of stationary subsets of $\omega_1$ such that $S_A \triangle S_B$ is stationary for $A \neq B$. Fix $(L, <)$ a countable dense linear order with least element but no greatest element. For $A \subseteq \omega_1$ define a dense linear order $(X^A, <)$ as follows. For $\alpha < \omega_1$, let

$$\mathbb{X}^A_\alpha = \begin{cases} (\mathbb{Q}, <) & \text{if } \alpha \in S_A \\ (L, <) & \text{if } \alpha \notin S_A \end{cases}$$

and let $X^A = \mathbb{Q} + \sum_{\alpha < \omega_1} X^A_\alpha$. Let $X^A_{<\alpha} = \mathbb{Q} + \sum_{\beta < \alpha} X^A_\beta$. We may assume each $X^A$ has underlying set $\omega_1$.

a) Show that each $X^A$ is an $\aleph_1$-like dense linear order.

b) Show that $\{\alpha < \omega_1 : \text{the underlying set of } X^A_{<\alpha} = \alpha\}$ is closed unbounded.

c) Suppose that $f : \omega_1 \to \omega_1$ is a bijection. Then $\{\alpha : f|\alpha \text{ is a bijection onto } \alpha\}$ is closed unbounded.

d) Show that $X^A \not\cong X^B$ for $A$ and $B$ distinct subsets of $\omega_1$. [Hint: Suppose $f$ is an isomoprhim. Find $\alpha \in S_A \triangle S_B$ such that $f|X^A_{<\alpha}$ is an isomorphism onto $f|X^B_{<\alpha}$. Find a contradiction.

We say that an abelian group is is $\aleph_1$-free if every countable subgroup is free.

We describe the construction of abelian groups of size $\aleph_1$ that are $\aleph_1$-free but not free. Throughout this appendix all groups are abelian.

Throughout we use the basic fact that a subgroup of a free abelian group is free.

We will need the following algebraic lemma.

**Lemma A.3** *There are free abelian groups $K_1 \subset K_2 \subset \ldots \subset K_n \subset \ldots$, $K = \bigcup K_n$, and $F \supset K$ such that:*

*i) $F/K_m$ is free for all $m$, but $F/K$ is not;*

*ii) each $K_m$, $F/K$, $K_{m+1}/K_m$ has rank $\aleph_0$.*

**Proof** Let $G$ be the subgroup of $\mathbb{Q}$ generated by $\{\frac{1}{2^n} : n \in \omega\}$. Let $\widehat{F}$ be the free abelian group on generators $x_0, x_1, \ldots$ and let $f : \widehat{F} \to G$ be the surjective homomorphism $x_i \mapsto \frac{1}{2^i}$. Let $\widehat{K}$ be the kernel of $F$. Then $\widehat{K}$ is the free abelian group on generators $\{x_0 - 2x_1, x_1 - 2x_2, \ldots\}$. Then $F/K \cong G$ is not free.

Let $\widehat{K}_m$ be the free abelian group on $\{x_0 - 2x_1, \ldots, x_{m-1} - 2x_m\}$. Then $\widehat{F}/\widehat{K}_m$ is isomorphic to the free abelian group on $x_m, x_{m+1}, \ldots$.

The groups we have constructed satisfy i), but not ii). It is easy to make ii) true by adding in large free factors. Here are the details. Let $H$ be a free abelian group on $\aleph_0$ generators.

Let

$$K_m = \widehat{K}_m \oplus \oplus_{i=0}^m H,$$

let

$$K = \bigcup_{m=0}^{\infty} K_m = \widehat{K} \oplus \oplus_{i=0}^{\infty} H$$

and let $F = \widehat{F} \oplus \oplus_{i=0}^{\infty} H \oplus H$. These groups have the desired properties. $\square$

**Theorem A.4** *There are $2^{\aleph_1}$ nonisomorphic $\aleph_1$-free groups of cardinality $\aleph_1$.*

**Proof** We fix a family $\mathcal{F}$ of $2^{\aleph_1}$ stationary subsets of $\omega_1$ such that if $S_1, S_2 \in \mathcal{F}$ are distinct, then $S_1 \triangle S_2$ is stationary. We may assume each $S \in \mathcal{F}$ is a set of limit ordinals. Let $S \in \mathcal{F}$. We construct a sequence of countable free abelian groups

$$G_0 \subset G_1 \subset \ldots G_\alpha \subset \ldots$$

for $\alpha < \omega_1$. We will do this so that:

$(*)$ if $\beta < \alpha$, then $G_\alpha$ is free over $G_{\beta+1}$.

i) $G_0 = \mathbb{Z}$.

ii) If $\alpha$ is a limit ordinal, let $G_\alpha = \bigcup_{\gamma<\alpha} G_\gamma$. Choose $\gamma_0 < \gamma_1 < \ldots$ with $\sup \gamma_n = \alpha$ such that each $\gamma_n$ is a successor ordinal. By $(*)$ $G_{\gamma_{n+1}}$ is free over $G_{\gamma_n}$. Thus $G_\alpha$ is free. Indeed $G_\alpha$ is free over each $G_{\gamma_n}$. If $\beta < \alpha$, choose $n$ such that $\beta < \gamma_n$. Then $G_{\gamma(n)}$ is free over $G_{\beta+1}$ and $G_\alpha$ is free over $G_{\gamma(n)}$. Thus $G_\alpha$ is free over $G_{\beta+1}$ and $(*)$ holds.

iii) If $\alpha \notin S$, then $G_{\alpha+1} = G_\alpha \oplus H$, where $H$ is free abelian on $\aleph_0$-generators. Clearly $(*)$ holds.

iv) Suppose $\alpha \in S$. Choose $\gamma_0 < \gamma_1 < \ldots$ successor ordinals with $\alpha = \sup \gamma_n$.

Let $F, K, K_0, K_1, \ldots$ be as in the Lemma. Since $G_{\gamma_0}$ and $K_0$ are both free abelian on $\aleph_0$-generators, we can find an isomorphism $\phi_0 : G_{\gamma_0} \to K_0$. Since $G_{\gamma_{n+1}}$ is free over $G_\gamma$ of rank $\aleph_0$ and $K_{n+1}/K_n$ is free of rank $\aleph_0$, we can extend $\phi_n$ to an isomorphism $\phi_{n+1} : G_{\gamma_n} \to K_n$. Then $\phi = \bigcup \phi_n$ is an isomorphism from $G_\alpha$ to $K$. We then define $G_{\alpha+1} \supset G_\alpha$ such that $\phi$ extends to an isomorphism from $G_{\alpha+1} \to F$. Thus we have

$(**)$ if $\alpha \in S$, then $G_{\alpha+1}/G_\alpha$ is not free.

On the other hand if $\beta + 1 < \alpha + 1$, there is an $n$ such that $\beta + 1 < \gamma_n$. Then $G_{\gamma_n}$ is free over $G_{\beta+1}$ and, by construction, $G_{\alpha+1}$ is free over $G_{\gamma_n}$. Thus $G_{\alpha+1}$ is free over $G_{\beta+1}$. Thus $(*)$ holds.

This concludes the construction. If $A \subseteq G$ is countable, there is an $\alpha$ such that $A \subseteq G_\alpha$. Since $G_\alpha$ is free abelian so is $A$. Thus $G$ is $\aleph_1$-free.

Suppose $S_1$ and $S_2$ are distinct elements of $\mathcal{F}$. Let $G_1$ and $G_2$ be groups we constructed. We claim that $G_1 \ncong G_2$. Suppose not. Let $f : G_1 \cong G_2$ be an isomorphism. Then $C = \{\alpha : f$ is an isomorphism between $G_{1,\alpha}$ and $G_{2,\alpha}\}$ is closed unbounded.

Since $S_1 \triangle S_2$ is stationary, we can, without loss of generality, find $\alpha \in (C \cap S_1) \setminus S_2$.

Then $f$ is an isomorphism between $G_1$ and $G_2$ such that $f$ maps $G_{1,\alpha}$ onto $G_{2,\alpha}$. But $G_1/G_{1,\alpha}$ is not free while $G_2/G_{2,\alpha}$ is free, a contradiction. $\square$

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