3 Extensions of Rings and Valuations

When studying the model theory of certain theories of valued fields our first step will usually be to prove quantifier elimination in an appropriate language. Proofs of quantifier elimination in algebraic theories usually require some algebraic extension results. That is particular true in valued fields. In this section we will prove some basic results and then will use them in §4 to begin the study of the model theory of algebraically closed valued fields. In §5 we will focus on extension results for henselian valued fields.

For more details on some of the background results from commutative algebra see, for example [12] or [19]. All of the results we will be proving on extensions of valuations can be found in [13]. To be careful we will tend to state most results for domains even though many are true in more generality.

3.1 Integral extensions

We begin by reviewing some facts about the integral extensions.

Recall that a domain A is local if and only if A has a unique maximum ideal $\mathfrak{m}$ which is exactly the nonunits of A.

Definition 3.1 If $A \subset B$ are domains, we say that $b \in B$ is *integral* over A, if there are $a_0, \ldots, a_{n-1} \in A$ such that

$$b^n + a_{n-1}b^{n-1} + \cdots + a_1 b + a_0 = 0$$

for some n. We say that B *is integral over* A if every element of B is integral over A.

Lemma 3.2 *Let $A \subset B$ be domains and $b \in B$. The following are equivalent.*
i) b is integral over A.
ii) $A[b]$ is a subring of B that is a finitely generated A-module.
iii) $A[b]$ is contained in a finitely generated A-module.

Proof i) $\Rightarrow$ ii) If $b^m = \sum_{n=0}^{m-1} a_n b^n$ where $a_0, \ldots, a_{m-1} \in A$. Then $A[b]$ is generated over A by $1, b, \ldots b^{m-1}$.

ii) $\Rightarrow$ iii) is clear.

iii) $\Rightarrow$ i) Let $x_1, \ldots, x_m$ generate a submodule containing $A[b]$ over A. For $i = 1, \ldots m$ we can find $a_{i,1}, \ldots, a_{i,m} \in A$ such that

$$bx_i = \sum_{j=1}^{m} a_{i,j} x_j.$$

Let M be the matrix

$$\begin{pmatrix} a_{1,1} - b & a_{1,2} & \ldots & a_{1,m} \\ a_{2,1} & a_{2,2} - b & \ldots & a_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \ldots & a_{m,m} - b \end{pmatrix}$$

i.e, the matrix with $a_{i,i} - b$ along the diagonal and $a_{i,j}$ everywhere else. Then $M(x_1, \ldots, x_m)^T = 0$. Let $\mathrm{Adj}(M)$ be the adjoint of M. Then

$$\mathrm{Adj}(M)M(x_1, \ldots, x_m)^T = (\det M x_1, \ldots, \det M x_m)^T = (0, \ldots, 0)^T.$$[2]

Thus we must have $\det M = 0$. But $\det M$ is a monic polynomial in $A[b]$. □

Corollary 3.3 *If $A \subset B \subset C$ are domains, B is an integral extension of A and C is an integral extension of B, then C is an integral extension of C.*

Proof Let $c \in C$. There are $b_0, \ldots, b_{n-1} \in B$ such that $c^n + \sum b_i c^i = 0$. Then $A[b_0, \ldots, b_{n-1}, c]$ is a finitely generated A-module and c is integral over A. □

The next lemma is a simple but useful tool.

Lemma 3.4 *If A is a local subring of a field K, $x \in K^\times$ and $1 = a_0 + \frac{a_1}{x} + \ldots \frac{a_n}{x^n}$ where $a_0 \in \mathfrak{m}$ and $a_1, \ldots, a_n \in A$, then x is integral over A.*

Proof Then $(1 - a_0)x^n - a_1 x^{n-1} - \cdots - a_n = 0$. Since $a_0 \in \mathfrak{m}$, $1 - a_0 \notin \mathfrak{m}$. Since A is local, $1 - a_0$ is a unit and x is integral over A. □

Lemma 3.5 *If $A \subset B$ are domains and B is integral over A, then A is a field if and only if B is a field.*

Proof ($\Leftarrow$) Suppose B is a field and $a \in A$ is nonzero. Then there are $c_0, \ldots, c_{m-1} \in A$ such that

$$(a^{-1})^m + \sum_{n=0}^{m-1} c_n (a^{-1})^n = 0.$$

Multiplying by a^{m-1} we see that

$$a^{-1} = -\sum_{n=0}^{m-1} c_n a^{m-n-1} \in A.$$

Thus A is a field.

($\Rightarrow$) Suppose A is a field and $b \in B$ is nonzero. Then, by Lemma 3.2 $A[b]$ is a finitely generated vector space over A. The map $z \mapsto bz$ is an injective linear transformation of $A[b]$ and, since $A[b]$ is a finite dimensional vector space must be surjective. Thus there is $z \in A[b]$ with $zb = 1$. □

Definition 3.6 Let $A \subset B$ be domains and let $P \subset A$, $Q \subset B$ be prime ideals. We say that Q *lies over* P if $A \cap Q = P$.

Corollary 3.7 *Let $A \subset B$ be domains with B integral over A and let $P \subset A$ and $Q \subset B$ be prime ideals such that Q lies over P. Then P is maximal if and only if Q is maximal.*

[2]Remember Cramer's Rule!

Proof Since $P = A \cap Q$ we can view B/Q as an integral extension of A/P. By the last lemma, A/P is a field if and only if B/Q is a field. $\square$

Lemma 3.8 *Suppose $A \subset B$ are domains, B is integral over A, Q is a prime ideal in B and $P = Q \cap A$. Then then BA_P is integral over A_P.*

Proof Consider b/t where $b \in B$ and $t \in A \setminus Q$. There are $a_0, \ldots, a_{m-1} \in A$ with $b^m + \sum a_i b^i = 0$. But then

$$(b/t)^m + \sum (a_i/t^{m-i})(b/t^i) = 0.$$

$\square$

Lemma 3.9 *Suppose $A \subset B$ are domains, B is integral over A, $P \subset A$ is a prime ideal and $Q_1 \subseteq Q_2$ are prime ideals in B lying over P. Then $Q_1 = Q_2$.*

Proof Consider the localization A_P and the integral extension BA_P. Then Q_1A_P and Q_2A_P are prime ideals of BA_P lying over PA_P. But PA_P is maximal. Thus each Q_iA_P is maximal and we must have $Q_1A_P = Q_2A_P$. But if $x \in Q_2 \setminus Q_1$, then $x \notin Q_1A_P$. If we did have $x = q/t$ for some $q \in Q_1$ and $t \in A \setminus P$. Then $xt \in Q_1$ and since $x \notin Q_1$ and Q_1 is prime, we would have $t \in Q_1 \cap A = P$, a contradiction. $\square$

Theorem 3.10 (Lying Over Theorem) *Suppose $A \subset B$ are domains, B is integral over A and P is a prime ideal of A. There is a prime ideal Q of B such that $A \cap Q = P$.*

Proof First, suppose A was a local ring then P is the unique maximal of A. If $Q \subset B$ is any maximal idea extending P, then, by Corollary 3.7, $Q \cap A$ is maximal. But then $Q = P$.

In general, we pass to the localization A_P. As above, if Q_0 is any maximal ideal in BA_P, then $Q_0 \cap A_P = PA_P$. So $Q_0 \cap A = P$. Let $Q = Q_0 \cap B$. Then $Q \cap A = P$ and, since Q_0 is prime, Q is prime. $\square$

3.2 Extensions of Valuations

Theorem 3.11 (Chevalley's Theorem) *Suppose A is a subring of a field K and $P \subset A$ is a prime ideal. Then there is a valuation ring $\mathcal{O}$ of K with $A \cap \mathcal{M}_{\mathcal{O}} = P$*

Proof Replacing A by A_P we may assume that A is a local ring with maximal ideal P. Let $\mathcal{P}$ be the set of all local subrings B of K with $\mathfrak{m}_B \cap A = P$. Clearly $\mathcal{P}$ is partially ordered by $\subset$ and if $(B_i : i \in I)$ increasing chain in $\mathcal{P}$ then $\bigcup_{i \in I} B_i$ is an upper bound. Thus by Zorn's Lemma, $\mathcal{P}$ has maximal elements. Let $\mathcal{O} \in \mathcal{P}$ be maximal. Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}$. We will argue that $\mathcal{O}$ is a valuation ring.

Suppose $x, 1/x \in K \setminus \mathcal{O}$. If x is integral over $\mathcal{O}$, then we can find a maximal ideal of $\mathcal{O}[x]$ lying over $\mathfrak{m}$ contradicting the maximality of $\mathcal{O} \in \mathcal{P}$. Thus x is not integral over $\mathcal{O}$.

By Lemma 3.4, $1 \notin \mathfrak{m}\mathcal{O}[1/x]$. Thus there is a maximal ideal Q of $\mathcal{O}[1/x]$ that lies over $\mathfrak{m}$, contradicting the maximality of $\mathcal{O}$. Thus for all $x \in K$ at least one of x and $1/x$ is in $\mathcal{O}$. $\square$

Exercise 3.12 Show that if $v : K^\times \to \Gamma$ is a valuation and $L \supset K$ is an extension field, there is $\Gamma' \supseteq \Gamma$ and $w : L\times \to \Gamma'$ extending v.

integral closures and valuations

Definition 3.13 We say that A is *integrally closed in B* if no element of $B \setminus A$ is integral over A. We say that A is *integrally closed* if it is integrally closed in its fraction field.

The integral closure of A is the smallest integrally closed ring containing A.

Lemma 3.14 *If (K, v) is a valued field, then the valuation ring $\mathcal{O}$ is integrally closed.*

Proof Suppose $b \in K$ and $b^n + a_{n-1}b^{n-1} + \cdots + a_1 b + a_0 = 0$ where $a_0, \ldots, a_{n-1} \in \mathcal{O}$. If $b \notin \mathcal{O}$, then $v(b) < 0$ and

$$v(a_i b^i) = v(a) + iv(b) < nv(b)$$

since $v(a_i) \geq 0$ for all i. Thus $v(b^n + a_{n-1}b^{n-1} + \cdots + a_1 b + a_0) = nv(b) < 0$, a contradiction. $\square$

We can use valuation rings to find the integral closure of a local subring.

Lemma 3.15 *Let A be a local subring of a field K with maximal ideal $\mathfrak{m}$. The integral closure of $A \in K$ is the intersection of all valuation rings $\mathcal{O} \subset K$ with $\mathfrak{m}_{\mathcal{O}}$ lying over $\mathfrak{m}$.*

Proof Suppose $x \in K$ be nonintegral over A. Then by Lemma 3.4, $1 \notin \mathfrak{m}A[1/x] + \frac{1}{x}A[1/x]$. Thus we can find a maximal ideal Q of $A[1/x]$ lying over $\mathfrak{m}$ with $1/x \in Q$. Let $\mathcal{O} \supseteq A[1/x]$ be a maximal local subring of K. Then, as in the proof of Theorem 3.11, $\mathcal{O}$ is a valuation ring, $\mathfrak{m}_{\mathcal{O}}$ lies over $\mathfrak{m}$ and $1/x \in \mathfrak{m}_{\mathcal{O}}$. Thus $x \notin \mathcal{O}$. $\square$

Algebraic Extensions

Suppose $K \subset L$ are fields and v is a valuation on K. Then v restricts to a valuation on K. Let $\mathcal{O}_L, \Gamma_L, \boldsymbol{k}_L$ and $\mathcal{O}_K, \Gamma_K, \boldsymbol{k}_K$ denote the respective valuation rings, value groups and residue fields.

Lemma 3.16 *Then Γ_L is contained in the divisible hull of Γ_K and $\boldsymbol{k}_L$ is an algebraic extension of $\boldsymbol{k}_K$.*

Proof Let $x \in L \setminus K$. There are $a_0, \ldots, a_n \in K$ such that $\sum a_i x^i = 0$. There must be $i \neq j$ such that $v(a_i x^i) = v(a_j x^j)$. But then $v(x) = \frac{v(a_i) - v(a_j)}{j-i}$.

Suppose $x \in L$ and the residue $\overline{x} \in \boldsymbol{k}_L \setminus \boldsymbol{k}_K$. There is a polynomial $f[X] \in \mathcal{O}_K(X)$ such that $\overline{f(x)} = 0$. Let $f(X) = \sum a_i X^i$. Suppose a_j has minimal value and let $g(X) = \sum \frac{a_i}{a_j} X^i$. Then $\overline{g(x)} = 0$ and $\overline{g}(X)$ is not identically zero as some coefficient is 1. Thus $\overline{x}$ is algebraic over K. $\square$

Corollary 3.17 *i) If k is an algebraically closed field, Γ is a divisible ordered abelian group and $K = k(((\Gamma)))$, then K is algebraically closed.*

ii) If k is a real closed, Γ is a divisible ordered abelian group and $K = k(((\Gamma)))$, then K is real closed.

Proof If K is not algebraically closed field let L/K be an algebraic extension, then we can extend the valuation to L and since $\boldsymbol{k}_L/k$ is algebraic and $\Gamma(L)$ is contained in the divisible hull of $\Gamma(K)$ by Exercise 1.8 (see also Lemma 3.16). But k is algebraically closed and Γ is divisible, thus L/K is immediate. But we saw in Lemma 2.46 that Hahn fields have no proper immediate extensions. Thus K is algebraically closed.

ii) If k is real closed, then $k^{\mathrm{alg}}(((\Gamma)))$ is a degree 2 algebraic extension of $k(((\Gamma)))$. Thus by the work of Artin and Schreier (see for example [19] XI §2 Proposition 3), $k(((\Gamma)))$ is real closed. $\square$

We will prove much more general of these results later.

If L/K is a finite algebraic extension and $[L : K] = d$, then the argument above shows that $[\Gamma_L : \Gamma_K] \leq d$ and $[\boldsymbol{k}_L : \boldsymbol{k}_K] \leq d$. We will prove a much sharper bound. We let $e = [\Gamma_L : \Gamma_K]$ be the *ramification index* and $f = [\boldsymbol{k}_L : \boldsymbol{k}_K]$ be the *residue degree* . Note that if $e = f = 1$, then L is an immediate extension of K.

Theorem 3.18 (Fundamental Inequality) *If L/K is a finite algebraic extension of degree d then $ef \leq d$.*

Proof Choose $x_1, \ldots, x_e \in L$ such that $v(x_1), \ldots, v(x_n)$ represent distinct cosets of Γ_L/Γ_K. Choose $y_1, \ldots, y_f \in L$ such that $\overline{y}_1, \ldots, \overline{y}_f$ is a basis for $\boldsymbol{k}_L/\boldsymbol{k}_K$. It suffices to show that $(x_i y_j : i \leq e, j \leq f)$ are linearly independent over K.

Suppose
$$\sum_{i \leq e, j \leq f} a_{i,j} x_i y_j = 0$$
where not all $a_{i,j} = 0$. Pick $\widehat{i}$ and $\widehat{j}$ such that
$$v(a_{\widehat{i},\widehat{j}} x_{\widehat{i}}) = \min\{v(a_{i,j} x_i) : i \leq e, j \leq f\}.$$
Suppose $i \neq \widehat{i}$ and $j \leq f$. We claim that $v(a_{\widehat{i},\widehat{j}} x_{\widehat{i}}) < v(a_{i,j} x_i)$. If they were equal then
$$v(x_{\widehat{i}}) - v(x_i) = v(a_{i,j}) - v(a_{\widehat{i},\widehat{j}}) \in \Gamma_K,$$

contradicting that $v(x_{\widehat{i}})$ and $v(x_i)$ represent different cosets. Thus $v(a_{\widehat{i},\widehat{j}}x_{\widehat{i}}) < v(a_{i,j}x_i)$ for $i \neq \widehat{i}$.

Let $b_{i,j} = \frac{a_{i,j}}{a_{\widehat{i},\widehat{j}}}x_{\widehat{i}}$. Then

$$0 = \sum_{j=1}^{f}\sum_{i=1}^{e} b_{i,j}\frac{x_i}{x_{\widehat{i}}}y_j$$

and $b_{i,j}\frac{x_i}{x_{\widehat{i}}} \in \mathfrak{m}_L$ for $i \neq \widehat{j}$. Thus

$$\sum_{j=1}^{f}\frac{a_{\widehat{i},j}}{a_{\widehat{i},\widehat{j}}}y_j = -\sum_{j=1}^{f}\sum_{i\neq\widehat{i}} b_{i,j}x_iy_j \in \mathfrak{m}_L.$$

Let $c_{\widehat{i},j} = \mathrm{res}(a_{\widehat{i},j}/a_{\widehat{i},\widehat{j}})$. Then $c_{\widehat{i},\widehat{j}} = 1$ and

$$\sum_{j=1}^{f} c_{i,j}\overline{y}_j = 0,$$

contradicting that $\overline{y}_1, \ldots, \overline{y}_f$ are linearly independent over $\boldsymbol{k}_K$. $\square$

Exercise 3.19 Show that even if L/K is an infinite algebraic extension the argument above shows that if $(x_i : i \in I)$ represent distinct cosets of Γ_L/Γ_K and $(y_j : j \in J)$ are such that $(\overline{y}_j : j \in J)$ are linearly independent over $\boldsymbol{k}_K$, then $(x_iy_j : i \in I, j \in J)$ are linearly independent and $v(\sum a_{i,j}x_iy_j) = \min v(a_{i,j}x_iy_j)$.

Definition 3.20 If $K \subset L$ are fields and L/K is algebraic, we say that L/K is *normal* if L is a splitting field for every irreducible $f \in K[X]$ with a zero in L.

A separable normal extension is a *Galois extension.* Thus in characteristic 0 normal and Galois are the same. But in characteristic p we can build nonseparable normal extensions by taking p^{th}-roots.

Our goal for the rest of this section is to show that if L/K is a normal extension and $\mathcal{O}$ is a valuation ring of K, then the valuation rings of L extending $\mathcal{O}$ are all conjugate under the action of the Galois group.

We need a form of the Chinese Remainder Theorem.

Lemma 3.21 *Let A be a domain and let $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ be distinct maximal ideals of A. Then for any $a_1, \ldots, a_n$ we can find $a \in A$ such that $a = a_i(\mathrm{mod}\ \mathfrak{m}_i)$ for all i.*

Proof

claim For each i we can find b_i such $b_i = 1(\mathrm{mod}\ \mathfrak{m}_i)$ but $b_i \in \mathfrak{m}_j$ for $j \neq i$.

For notational simplicity assume $i = 1$. If $j \neq 1$ then $\mathfrak{m}_1 + \mathfrak{m}_j = A$, as otherwise $\mathfrak{m}_1 + \mathfrak{m}_j$ is an ideal, contradicting maximality. Thus there is $c_j \in \mathfrak{m}_1$ and $d_j \in \mathfrak{m}_j$ such that $c_j + d_j = 1$. Then

$$1 = \prod_{j\neq 1}(c_j + d_j) = \prod_{j\neq 1} d_j(\mathrm{mod}\ \mathfrak{m}_1).$$

Let $b_1 = \prod_{j \neq 1} d_j$. Then $b_1 = 1 (\mathrm{mod}\ \mathfrak{m}_1)$ but $b_1 \in \mathfrak{m}_i$ for $i \neq \jmath$.

Let $a = \sum a_i b_i$. Then $a = a_i (\mathrm{mod}\ \mathfrak{m}_i)$ for all i. □

Lemma 3.22 *Let A be a local domain integrally closed in its fraction field K and let L/K be normal. Let B be the integral closure of A in L. Then any two maximal ideals of B are conjugate under* $\mathrm{Gal}(L/K)$.[3]

Proof It suffices to prove this when L/K is finite. Let $\mathfrak{m}_0$ and $\mathfrak{m}_1$ be maximal ideal of B and suppose there is no $\sigma \in \mathrm{Gal}(L/K)$ with $\sigma(\mathfrak{m}_1) = \mathfrak{m}_2$. Let $X_i = \{\sigma(\mathfrak{m}_i) : \sigma \in \mathrm{Gal}(L/K)\}$ then $X_0 \cap X_1 = \emptyset$. By the Chinese Remainder Theorem, we can find $b \in B$ such that $b \in \mathfrak{m}$ for $\mathfrak{m} \in X_0$ and $b = 1 (\mathrm{mod}\ \mathfrak{m})$ for $\mathfrak{m} \in X_1$. Thus $\sigma(b) \in \mathfrak{m}_0 \setminus \mathfrak{m}_1$ for all $\sigma \in \mathrm{Gal}(L/K)$.

For the remainder of the proof we will assume that our fields have characteristic zero. One needs to be slightly more careful in characteristic p when we have an inseparable extension. Suppose $f(X) = X^d + \sum_{n=0}^{d-1} a_i X^i$, $a_0, \dots, a_{d-1} \in A$ be the minimal polynomial of b over K.. Since L/K is normal, $f(X) = \prod_{i=1}^d (X - \beta_i)$ where $\beta_1, \dots, \beta_d \in L$ are the distinct roots or f, i.e., the set of conjugates of b under $\mathrm{Gal}(L/K)$. Without loss of generality, we assume $L = K(\beta_1, \dots, \beta_d)$. Then

$$\prod_{\sigma \in \mathrm{Gal}(L/K)} \sigma(b) = \prod_{i=1}^{d} \beta_i = a_0 \in A.$$

Each $\sigma(b) \in \mathfrak{m}_0$. Thus $a_0 \in \mathfrak{m}_0 \cap A = \mathfrak{m}_A \subseteq \mathfrak{m}_1$. But no $\sigma(b) \in \mathfrak{m}_1$, thus, since $\mathfrak{m}_1$ is prime $a_0 \notin \mathfrak{m}_1$, a contradiction. □

Lemma 3.23 *Let A be a valuation ring with fraction field K, let $L \supseteq K$ be an algebraic extension and let B be the integral closure of A in L. For every valuation ring $\mathcal{O} \subset K$ with $\mathfrak{m}_A \subseteq \mathfrak{m}_\mathcal{O}$ there is $\mathfrak{n}$ a maximal ideal of B with $\mathcal{O} = B_\mathfrak{n}$.*

Moreover, for every maximal ideal $\mathfrak{n} \subset B$, $B_\mathfrak{n}$ is a valuation ring.

Proof Let $\mathcal{O}$ be a valuation ring of L with $\mathfrak{m}_A \subseteq \mathfrak{m}_\mathcal{O}$. Since $\mathcal{O}$ is integrally closed in L, $B \subseteq \mathcal{O}$. Let $\mathfrak{n} = \mathfrak{m}_\mathcal{O} \cap B$.

If $x \in B \setminus \mathfrak{n}$, then $1/x \in \mathcal{O}$. Thus $B_\mathfrak{n} \subseteq \mathcal{O}$. Let $x \in \mathcal{O}$. Since L/K is algebraic, there are $a_0, \dots, a_d \in A$ not all zero such that $\sum a_i x^i = 0$. Let $m \leq d$ be maximal such that $v(a_m) = \min(v(a_i) : i = 0, \dots, d)$ and divide $\sum a_i x^i$ by $a_m x^m$. Thus, letting $b_i = a_i / a_m$ we have

$$\sum_{i=m+1}^{d} b_i x^{i-m} + 1 + \sum_{i=0}^{m-1} b_i x^{i-m} = 0.$$

Note that $b_0, \dots, b_{m-1} \in A$ and $b_{m+1}, \dots, b_d \in \mathfrak{m}_A$. Let $y = \sum_{i=m+1}^{d} b_i x^{i-m} + 1$ and $z = \sum_{i=0}^{m-1} b_i x^{i-m+1}$. Then $xy = -z$ and y is a unit in $\mathcal{O}$.

[3]We use $\mathrm{Gal}(L/K)$ to denote the group of automorphism of L/K even when L/K is not necessarily a Galois extension.

We claim that $y, z \in B$. Since B is the integral closure of A in L, by Lemma 3.15, it suffices to show that $x, y \in V$ for any valuation ring $V \subset \mathcal{L}$ with $\mathfrak{m}_V \cap A = \mathfrak{m}_A$. If $x \in V$, then $y \in V$ and $z = -xy \in V$. If $x \notin V$, then $1/x \in V$, $z = \sum_{i=0}^{m-1} b_i x^{i-m+1} \in V$ and $y = -z/x \in V$.

Since y is a unit in $\mathcal{O}$, $y \notin \mathfrak{n}$. Thus $x = -z/y \in B_\mathfrak{n}$. Thus $B_\mathfrak{n} = \mathcal{O}$.

To prove the last claim of the lemma we need to show that if $\mathfrak{n}$ is a maximal ideal of B, then $B_\mathfrak{n}$ is a valuation ring extending A. Clearly $\mathfrak{n} \cap A = \mathfrak{m}_A$. By Chevalley's Theorem, there is a valuation ring $\mathcal{O}$ such that $B \cap \mathfrak{m}_\mathcal{O} = \mathfrak{n}$. Then by the first part of the lemma $\mathcal{O} = B_\mathfrak{n}$. □

We summarize the last few lemmas.

Theorem 3.24 *Let A be a valuation ring with fraction field K, let $L \supseteq K$ be an algebraic extension and let B be the integral closure of A in L. There is a bijective correspondence $\mathfrak{m} \mapsto B_\mathfrak{m}$ between maximal ideals of B and valuation rings $\mathcal{O} \subset L$ with $\mathfrak{m}_O \cap A = \mathfrak{m}_A$. Moreover, if L/K is normal, then any two such valuation rings are conjugate under* $\mathrm{Gal}(L/K)$.

Corollary 3.25 *Let $(K, \mathcal{O})$ be a valued field and let L/K be a purely inseparable algebraic extension of K. Then there is a unique valuation ring $\mathcal{O}^*$ on L with $(K, \mathcal{O}) \subseteq (L, \mathcal{O}^*)$.*

Proof L is obtained from K by adjoining p^{th}-roots where K has characteristic p. Then L/K is normal but there are no nontrivial automorphisms of L fixing K. □