Descriptive Set Theory

David Marker

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Contents

These are informal notes for a course in Descriptive Set Theory given at the University of Illinois at Chicago in Fall 2002. While I hope to give a fairly broad survey of the subject we will be concentrating on problems about group actions, particularly those motivated by Vaught's conjecture. Kechris' *Classical Descriptive Set Theory* is the main reference for these notes.

Notation: If A is a set, $A^{<\omega}$ is the set of all finite sequences from A. Suppose $\sigma = (a_0, \ldots, a_m) \in A^{<\omega}$ and $b \in A$. Then $\sigma\widehat{\ }b$ is the sequence $(a_0, \ldots, a_m, b)$. We let $\emptyset$ denote the empty sequence. If $\sigma \in A^{<\omega}$, then $|\sigma|$ is the length of σ. If $f : \mathbb{N} \to A$, then $f|n$ is the sequence $(f(0), \ldots, f(n-1))$.

If X is any set, $\mathcal{P}(X)$, the power set of X is the set of all subsets X.

If X is a metric space, $x \in X$ and $\epsilon > 0$, then $B_\epsilon(x) = \{y \in X : d(x, y) < \epsilon\}$ is the open ball of radius ϵ around x.

Part I
Classical Descriptive Set Theory

1 Polish Spaces

Definition 1.1 Let X be a topological space. We say that X is *metrizable* if there is a metric d such that the topology is induced by the metric. We say that X is *separable* if there is a countable dense subset.

A *Polish space* is a separable topological space that is metrizable by a complete metric.

There are many classical examples of Polish spaces. Simple examples include $\mathbb{R}^n$, $\mathbb{C}^n$, $\mathbb{I} = [0, 1]$, the unit circle $\mathbb{T}$, and $\mathbb{Q}_p^n$, where $\mathbb{Q}_p$ is the p-adic field.

Example 1.2 *Countable discrete sets are Polish Spaces.*

Let X be a countable set with the discrete topology. The metric

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

is a complete metric inducing the topology.

If d is a metric on X, then

$$\widehat{d}(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

is also a metric, $\widehat{d}$ and d induce the same topology and $\widehat{d}(x, y) < 1$ for all x.

Example 1.3 *If $X_0, X_1, \ldots$ are Polish spaces, then $\prod X_n$ is a Polish space.*

Suppose d_n is a complete metric on X_n, with $d_n < 1$, for $n = 0, 1, \ldots$. Define $\widehat{d}$ on $\prod X_n$ by

$$\widehat{d}(f,g) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} d_n(f(n), g(n)).$$

If $f_0, f_1, \ldots$ is a Cauchy-sequence, then $f_1(i), f_2(i), \ldots$ is a Cauchy-sequence in X_i for each i. Let $g(n) = \lim_{i \to \infty} f_i(n)$. Then g is the limit of $f_0, f_1, \ldots$.

Suppose $x_0^i, x_1^i, \ldots$ is a dense subset of X_i. For $\sigma \in \mathbb{N}^{<\omega}$ let

$$f_\sigma(n) = \begin{cases} x^n_{\sigma(n)} & \text{if } i < |\sigma| \\ x^n_0 & \text{otherwise} \end{cases}.$$

The $\{f_\sigma : \sigma \in \mathbb{N}^{<\omega}\}$ is dense in $\prod X$.

In particular, the *Hilbert cube* $\mathbb{H} = \mathbb{I}^{\mathbb{N}}$ is Polish. Indeed, it is a universal Polish space.

Theorem 1.4 *Every Polish space is homeomorphic to a subspace of* $\mathbb{H}$.

Proof Let X be a Polish space. Let d be a compatible metric on X with $d < 1$ and let $x_0, x_1, \ldots$ a dense set. Let $f : X \to \mathbb{H}$ by $f(x) = (d(x, x_1), d(x, x_2), \ldots)$. If $d(x, y) < \epsilon/2$, then $|d(x, x_i) - d(y, x_i)| < \epsilon$ and $d(f(x), f(y)) < \sum \frac{1}{2^{n+1}} \epsilon < \epsilon$. Thus f is continuous. If $d(x, y) = \epsilon$ choose x_i such that $d(x, x_i) < \epsilon/2$. Then $d(y, x_i) > \epsilon/2$, so $f(x) \neq f(y)$.

We need to show that f^{-1} is continuous. Let $\epsilon > 0$. Choose n such that $d(x, x_n) < \epsilon/3$. If $|y - x_n| > 2\epsilon/3$, then $d(f(x), f(y)) \geq \frac{1}{3(2^{n+1})}$. Thus if $d(f(x), f(y)) < \frac{1}{3(2^{n+1})}$, then $d(x, y) < \epsilon$. Hence f^{-1} is continuous.

Function spaces provide other classical examples of Polish spaces. Let $C(\mathbb{I})$ be the continuous real-valued functions on $\mathbb{I}$, with $d(f, g) = \sup\{|f(x) - g(x)| : x \in \mathbb{I}\}$. Because any Cauchy sequence converges uniformly, d is complete. Any function in I can be approximated by a piecewise linear function defined over $\mathbb{Q}$. Thus $C(\mathbb{I})$ is separable.

More generally, if X is a compact metric space and Y is a Polish space let $C(X, Y)$ be the space of continuous functions from X to Y with metric $d(f, g) = \sup\{|f(x) - g(x)| : x \in X\}$.

Other classical examples include the spaces l^p, l^∞ and L^p from functional analysis.

The next two lemmas will be useful in many results. If X is a metric space and $Y \subseteq X$, the diameter of Y is diam $(Y) = \sup\{d(x, y) : x, y \in Y\}$

Lemma 1.5 *Suppose X is a Polish space and $X_0 \supseteq X_1 \supseteq X_2 \supseteq \ldots$ are closed subsets of X such that* $\lim_{n \to \infty}$ diam $(X_n) = 0$. *Then there is $x \in X$ such that $\bigcap X_n = \{x\}$.*

Proof Choose $x_n \in X_n$. Since diam $(X_n) \to 0$, (x_n) is a Cauchy sequence. Let x be the limit of (x_n). Since each X_n is closed $x \in \bigcap X_n$. Since diam $(X_n) \to 0$, if $y \in \bigcap X_n$, then $x = y$.

Lemma 1.6 *If X is a Polish space, $U \subseteq X$ is open and $\epsilon > 0$, then there are open sets $U_0, U_1, U_2, \ldots$ such that $U = \bigcup U_n = \bigcup \overline{U}_n$ and* diam $(U_n) < \epsilon$ *for all n.*

Proof Let D be a countable dense set. Let $U_0, U_1, \ldots$ list all sets $B_{\frac{1}{n}}(d)$ such that $d \in D$, $\frac{1}{n} < \epsilon/2$ and $\overline{B_{\frac{1}{n}}(d)} \subseteq U$. Let $x \in U$. There is $n > 0$ such that $\frac{1}{n} < \epsilon$, $B_{\frac{1}{n}}(x) \subset U$. There is $d \in D \cap B_{\frac{1}{3n}}(x)$. Then $x \in B_{\frac{1}{3n}}(d)$ and $\overline{B_{\frac{1}{3n}}(d)} \subset U$. Thus $B_{\frac{1}{3n}}(d)$ is one of the U_i and $x \in \bigcup U_i$.

Baire Space and Cantor Space

If A is any countable set with the discrete topology and X is any countable set, then A^X is a Polish space. Two very important examples arise this way.

Definition 1.7 *Baire space* is the Polish space $\mathcal{N} = \mathbb{N}^{\mathbb{N}}$ and *Cantor space* is the Polish space $\mathcal{C} = 2^{\mathbb{N}}$.

An equivalent complete metric on $\mathbb{N}$ is $d(f, g) = \frac{1}{2^{n+1}}$ where n is least such that $f(n) \neq g(n)$.

Since the two point topological space $\{0, 1\}$ with the discrete topology is compact. By Tychonoff's Theorem $\mathcal{C}$ is compact.

Exercise 1.8 Show that $\mathcal{C}$ is homeomorphic to Cantor's "middle third" set.

Another subspace of $\mathcal{N}$ will play a key role later.

Example 1.9 *Let S_∞ be the group of all permutations of $\mathbb{N}$, viewed as a subspace of $\mathcal{N}$.*

If d is the metric on $\mathcal{N}$, then d is not complete on S_∞. For example let

$$f_n(i) = \begin{cases} i+1 & \text{if } i < n \\ 0 & \text{if } i = n \\ i & \text{otherwise} \end{cases}.$$

Then f_n is a Cauchy sequence in $\mathcal{N}$, but the limit is the function $n \mapsto n+1$ that is not surjective. Let $\widehat{d}(x, y) = d(x, y) + d(x^{-1}, y^{-1})$. It is easy to see that if (f_n) is a $\widehat{d}$-Cauchy sequence in S_∞, then (f_n) and (f_n^{-1}) are d-Cauchy sequences that converge in $\mathcal{N}$. One can then check that the elements the converge to must be inverses of each other and hence both in S_∞.

Exercise 1.10 A metric d on a group G is called left-invariant if $d(xy, xz) = d(y, z)$ for all $x, y, z \in G$. Show that the original metric d on S_∞ is left-invariant, but that there is no left-invariant complete metric on S_∞.

Exercise 1.11 Define $\phi : \mathcal{N} \to \mathcal{C}$ by

$$\phi(f) = \underbrace{00\ldots0}_{f(0)}\underbrace{11\ldots1}_{f(1)+1}\underbrace{00\ldots0}_{f(2)+1}1\ldots.$$

Show that ϕ is a continuous and one-to-one. What is the image of ϕ?

Exercise 1.12 We say that $x \in [0,1]$ is a dyadic-rational is $x = \frac{m}{2^n}$ for some $m, n \in \mathbb{N}$. Otherwise, we say x is a dyadic-irrational. Show that $\mathcal{N}$ is homeomorphic to the dyadic-irrationals (with the subspace topology). [Hint: let ϕ be as in Exercise 1.11 and map f to the dyadic-irrational with binary expansion $\phi(f)$.]

Exercise 1.13 † Show that

$$f \mapsto \frac{1}{1+f(0)+\frac{1}{1+f(1)+\frac{1}{1+f(2)+\frac{1}{1+\dots}}}}$$

is a homeomorphism between $\mathcal{N}$ and the irrational real numbers in $(0,1)$.

Because $\mathcal{N}$ will play a key role in our study of Polish spaces, we will look more carefully at its topology. First we notice that the topology has a very combinatorial/computational flavor.

If $\sigma \in \mathbb{N}^{<\omega}$, Let $N_\sigma = \{f \in \mathcal{N} : \sigma \subset f\}$. Then N_σ is an open neighborhood of f. It is easy to see that $\{N_\sigma : \sigma \in \mathbb{N}^{<\omega}\}$ is a basis for the topology. Notice that $\mathcal{N} \setminus N_\sigma = \bigcup\{N_\tau : \tau(i) \neq \sigma(i) \text{ for some } i \in \text{dom } \sigma\}$ is also open. Thus N_σ is clopen. It follows that the Baire Space is totally disconnected (i.e., any open set is the union of two disjoint open sets).

If $U \subseteq \mathcal{N}$ is open, there is $S \subseteq \mathbb{N}^{<\omega}$ such that $U = \bigcup_{\sigma \in S} N_\sigma$. Let $T = \{\sigma \in \mathbb{N}^{<\omega} : \forall \tau \subseteq \sigma\ \tau \notin S\}$. Note that if $\sigma \in T$ and $\tau \subseteq \sigma$, then $\tau \in T$. We call a set of sequences with this property a *tree*. We say that $f \in \mathcal{N}$ is a *path* through T if $(f(0), \dots, f(n)) \in T$ for $n = 0, 1, \dots$. We let

$$[T] = \{f \in \mathcal{N} : f \text{ is a path through } T\}.$$

Then $f \in [T]$ if and only if $\sigma \not\subset f$ for all $\sigma \in S$ if and only if $f \notin U$. We have proved the following characterizations of open and closed subsets of $\mathcal{N}$.

Lemma 1.14 *i) $U \subseteq \mathcal{N}$ is open if and only if there is $S \subseteq \mathbb{N}^{<\omega}$ such that $U = \bigcup_{\sigma \in S} N_\sigma$.*

ii) $F \subseteq \mathcal{N}$ is closed if and only if there is a tree $T \subseteq \mathbb{N}^{<\omega}$ such that $F = [T]$.

We can improve the characterization a little.

Definition 1.15 We say that a tree $T \subseteq \mathbb{N}^{<\omega}$ is *pruned* if for all $\sigma \in T$, there is $i \in \mathbb{N}$ with $\sigma\widehat{\ }i \in T$.

Equivalently, T is pruned if for all $\sigma \in T$, there is $f \in [T]$ with $\sigma \subset f$. If T is a tree, then $T' = \{\sigma \in T : \exists f \in [T]\ \sigma \subset f\}$. It is easy to see that T' is a pruned tree with $T \subseteq T'$. Thus every closed set F is the set of paths through a pruned tree.

If $f : \mathcal{N} \to \mathcal{N}$, then f is continuous if and only if for all x and $\sigma \subset f(x)$, there is a $\tau \subset x$ such that if $\tau \subset y$, then $\sigma \subset f(y)$. In other words, for all n

there is an m, such that the first n values of $f(x)$ are determined by the first m values of x. In §4 we will show how this brings in ideas from recursion theory.

Another key feature of the Baire space is that powers of the Baire space are homeomorphic to the Baire space. Thus there is no natural notion of dimension.

Lemma 1.16 *i) If $k > 0$, then $\mathcal{N}$ is homeomorphic to $\mathbb{N}^d \times \mathcal{N}^k$.*
ii) $\mathcal{N}$ is homeomorphic to $\mathcal{N}^{\mathbb{N}}$.

Proof If $\alpha = (n_1, \ldots, n_d, f_1, \ldots, f_k) \in \mathbb{N}^d \times \mathcal{N}^k$, let

$$\phi(f) = (n_1, \ldots, n_d, f_1(0), f_2(0), \ldots, f_k(0), f_1(1), \ldots, f_k(1), \ldots, f_1(n), \ldots, f_k(n), \ldots).$$

If $\beta = (f_0, f_1, \ldots) \in \mathcal{N}^N$, let

$$\psi(\beta) = (f_0(0), f_0(1), f_1(0), \ldots).$$

It is easy to see that ϕ and ψ are homeomorphisms.

A third important feature of the Baire space is that every Polish space is a continuous image of the Baire space. We first prove that every closed subset of $\mathcal{N}$ is a continuous image of $\mathcal{N}$.

Theorem 1.17 *If X is a Polish space, then there is a continuous surjective $\phi : \mathcal{N} \to X$.*

Proof Using Lemma 1.6 build a tree of sets $(U_\sigma : \sigma \in \mathbb{N}^{<\omega})$ such that:

i) $U_\emptyset = X$;

ii) U_σ is an open subset of X;

iii) diam $(U_\sigma) < \frac{1}{|\sigma|}$;

iv) $\overline{U_\tau} \subseteq U_\sigma$ for $\sigma \subset \tau$;

v) $U_\sigma = \bigcup_{i=0}^{\infty} U_{\sigma\frown i}$.

If $f \in \mathcal{N}$, then by 1.5 there is $\phi(f)$ such that

$$\phi(f) = \bigcap_{n=0}^{\infty} U_{f|n} = \bigcap_{n=0}^{\infty} \overline{U_{f|n}} = \{\phi(f)\}.$$

Suppose $x \in X$. We build $\sigma_0 \subset \sigma_1 \subset \ldots$ with $x \in U_{\sigma_i}$. Let $\sigma_0 = \emptyset$. Given σ_n with $x \in U_{\sigma_n}$, there is a j such that $x \in U_{\sigma_n \frown j}$. Let $\sigma_{n+1} = \sigma_n \widehat{\ } j$. If $f = \bigcup \sigma_n$, then $\phi(f) = x$. Thus ϕ is surjective.

Suppose $\phi(f) = x$. If $g|n = f|n$, then $\phi(g) \in U_{f|n}$ and $d(\phi(f), \phi(g)) < \frac{1}{n}$. Thus ϕ is continuous.

Indeed we have shown that there is an open, continuous, surjective $\phi : \mathcal{N} \to X$.

We will prove a refinement of this theorem. We need one lemma.

Recall that X is an F_σ*-set* if it a countable union of closed sets. If $O \subset X$ is open, then, by 1.6 there are open sets $U_0, U_1, \ldots$ such that $O = \bigcup \overline{U}_n$. Thus every open set is and F_σ-set. The union of countably many F_σ-sets is an F_σ-set. If $X = \bigcup A_i$ and $Y = \bigcup B_i$ are F_σ-sets, then $X \cap Y = \bigcup (A_i \cap B_j)$ is also an F_σ-sets.

Lemma 1.18 *Suppose X is a Polish space and $Y \subseteq X$ is an F_σ-set and $\epsilon > 0$. There are disjoint F_σ-sets $Y_0, Y_1, \ldots$ with* diam $(Y_i) < \epsilon$, $\overline{Y}_i \subseteq Y$ *and* $\bigcup Y_i = Y$.

Proof Let $Y = \bigcup C_n$ where C_n is closed. Replacing C_n by $C_0 \cup \ldots \cup C_n$ we may assume that $C_0 \subseteq C_1 \subseteq \ldots$. Thus Y is the disjoint union of the sets $C_0, C_1 \setminus C_0, C_2 \setminus C_1, \ldots$. Since $\overline{C_i \setminus C_{i+1}} \subseteq C_i \subseteq Y$, it suffices to show that each $C_i \setminus C_{i-1}$ is a disjoint union of F_σ-sets of diameter less than ϵ. Suppose $Y = F \cap O$ where F is closed and O is open. By Lemma 1.6, we can find $O_0, O_1, \ldots$ open sets with diam $(O_n) < \epsilon$ and $O = \bigcup O_n = \bigcup \overline{O_n}$. Let $Y_n = F \cap (O_n \setminus (O_0 \cap \ldots O_{n-1}))$. The Y_i are disjoint, $\overline{Y_i} \subseteq \overline{O_i} \subset O$, so $Y_i \subseteq Y$, and $\bigcup \overline{Y}_i = Y$.

Theorem 1.19 *If X is Polish, there is $F \subseteq \mathcal{N}$ closed and a continuous bijection $\phi : F \to X$.*

Proof Using the previous lemma, we build a tree $(X_\sigma : \sigma \in \mathbb{N}^{<\omega})$ of F_σ-sets such that

i) $X_\emptyset = X$;
ii) $X_\sigma = \bigcup_{i=0}^\infty X_{\sigma\frown i}$;
iii) $\overline{X_\tau} \subseteq X_\sigma$ if $\tau \subset \sigma$;
iv) diam $(X_\sigma) < \frac{1}{|\sigma|}$;
v) if $i \neq j$, then $X_{\sigma\frown i} \cap X_{\sigma\frown j} = \emptyset$.

If $f \in \mathcal{N}$, then $\bigcap X_{f|n}$ contains at most one point. Let

$$F = \left\{ f \in \mathcal{N} : \exists x \in X \ x \in \bigcap_{n=0}^{\infty} X_{f|n} \right\}.$$

Let $\phi : F \to X$ such that $\phi(f) = \bigcap X_{f|n}$. As above ϕ is continuous. By v) ϕ is one-to-one. For any $x \in X$ we can build a sequence $\sigma_0 \subset \sigma_1 \subset \ldots$ such that $x \in \bigcap X_{\sigma_n}$. We need only show that F is closed.

Suppose (f_n) is a Cauchy sequence in F. Suppose $f_n \to f \in \mathcal{N}$. We must show $f \in F$. For any n there is an m such that $f_i|n = f_m|n$ for $i > m$. But then $d(\phi(f_i), \phi(f_m)) < \frac{1}{n}$. Thus $\phi(f_n)$ is a Cauchy sequence. Suppose $\phi(f_n) \to x$. Then $x \in \bigcap \overline{X_{f|n}} = \bigcap X_{f_n}$, so $\phi(f) = x$ and $f \in F$.

Exercise 1.20 Prove that if X and Y are closed subsets of $\mathcal{N}$ with $X \subseteq Y$ then there is a continuous $f : Y \to X$ such that $f|X$ is the identity (we say that X is a *retraction of* Y). Use this to deduce 1.17 from 1.19.

Cantor–Bendixson analysis

We next show that the Continuum Hypothesis is true for Polish spaces, and closed subsets of Polish spaces.

Definition 1.21 Let X be a Polish space. We say that $P \subseteq X$ is *perfect* if X is a closed set with no isolated points.

Note that $\emptyset$ is perfect. Nonempty perfect sets have size $2^{\aleph_0}$.

Lemma 1.22 *If $P \subseteq X$ is a nonempty perfect set, then there is a continuous injection $f : \mathcal{C} \to P$. Indeed, there is a perfect $F \subseteq P$, homeomorphic to $\mathcal{C}$. In particular $|P| = 2^{\aleph_0}$.*

Proof We build a tree $(U_\sigma : \sigma \in 2^{<\omega})$ of nonempty open subsets of X such that:
i) $U_\emptyset = X$;
ii) $\overline{U_\tau} \subset U_\sigma$ for $\sigma \subset \tau$;
iii) $U_{\sigma\frown 0} \cap U_{\sigma\frown 1} = \emptyset$;
iv) diam $(U_\sigma) < \frac{1}{|\sigma|}$;
v) $U_\sigma \cap P \neq \emptyset$;
Suppose we are given U_σ with $U_\sigma \cap P \neq \emptyset$. Because P is perfect, we can find x_0 and $x_1 \in U_\sigma \cap P$ with $x_0 \neq x_1$. We can choose $U_{\sigma\frown 0}$ and $U_{\sigma\frown 1}$ disjoint open neighborhoods of x_0 and x_1, respectively such that $\overline{U_{\sigma\frown i}} \subset U_\sigma$ and diam $(U_{\sigma\frown i}) < \frac{1}{|\sigma+1|}$. This allows us to build the desired tree.
By Lemma 1.5, we can define $f : \mathcal{C} \to P$ such that

$$\{f(x)\} = \bigcap_{n=0}^{\infty} U_{x|n} = \bigcap_{n=0}^{\infty} \overline{U_{x|n}} = \bigcap_{n=0}^{\infty} \overline{U_{x|n}} \cap P.$$

It is easy to check that f is continuous and one-to-one.
Since f is continuous and $\mathcal{C}$ is compact, $F = f(\mathcal{C})$ is closed. By construction F is perfect. The map $f : \mathcal{C} \to F$ is open and hence a homeomorphism.

Exercise 1.23 Suppose $f : \mathcal{C} \to X$ is continuous and one-to-one. Prove that $f(C)$ is perfect.

Consider $\mathbb{Q}$ as a subspace of $\mathbb{R}$. As a topological space $\mathbb{Q}$ is closed and has no isolated points. Since $|\mathbb{Q}| = \aleph_0$, $\mathbb{Q}$ is not a Polish space.

We next analyze arbitrary closed subsets of Polish spaces. Let X be a Polish space. Let $U_0, U_1, \ldots$ be a countable basis for the open sets of X. If $F \subseteq X$ is closed, let F_0 be the isolated points of F. For each $x \in F_0$ we can find i_x such that $U_{i_x} \cap F = \{x\}$. Thus F_0 is countable and

$$F \setminus F_0 = F \setminus \bigcup_{x \in F_0} U_{i_x}$$

is closed.

Definition 1.24 If $F \subseteq X$ is closed, the *Cantor–Bendixson derivative* is

$$\Gamma(F) = \{x \in F : x \text{ is not an isolated point of } F\}.$$

For each countable ordinal $\alpha < \omega_1$, we define $\Gamma^\alpha(F)$ as follows:
i) $\Gamma^0(F) = F$;
ii) $\Gamma^{\alpha+1}(F) = \Gamma(\Gamma^\alpha(F))$;
iii) $\Gamma^\alpha(F) = \bigcap_{\beta<\alpha} \Gamma^\beta(F)$.

Lemma 1.25 *Suppose X is a Polish space and $F \subseteq X$ is closed.*

i) $\Gamma^\alpha(F)$ is closed for all $\alpha < \omega_1$;

ii) $|\Gamma^{\alpha+1}(F) \setminus \Gamma^\alpha(F)| \leq \aleph_0$;

iii) if $\Gamma(F) = F$, then F is perfect, and $\Gamma^\alpha(F) = F$ for all $\alpha < \omega_1$.

iv) there is an ordinal $\alpha < \omega_1$ such that $\Gamma^\alpha(F) = \Gamma^{\alpha+1}(F)$

Proof i)–iii) are clear. For iv), let $U_0, U_1, \ldots$ be a countable basis for X. If $\Gamma_{\alpha+1} \setminus \Gamma_\alpha \neq \emptyset$, we can find $n_\alpha \in \mathbb{N}$ such that U_{n_α} isolates a point of $\Gamma^\alpha(F)$. By construction U_{n_α} does not isolate a point of $\Gamma^\beta(F)$ for any $\beta < \alpha$. Thus $n_\alpha \neq n_\beta$ for any $\beta < \alpha$.

If there is no ordinal α with $\Gamma^\alpha(F) = \Gamma^{\alpha+1}(F)$, then $\alpha \mapsto n_\alpha$ is a one-to-one function from ω_1 into $\mathbb{N}$, a contradiction.

The *Cantor–Bendixson rank* of F, is the least ordinal α such that $\Gamma^\alpha(F) = \Gamma^{\alpha+1}(F)$.

Exercise 1.26 † Show that for all $\alpha < \omega_1$, there is a closed $F \subseteq \mathbb{R}$ with Cantor–Bendixson rank α.

Theorem 1.27 *If X is a Polish space and $F \subseteq X$ is closed, then $F = P \cup A$ where P is perfect (possibly empty), A is countable and $P \cap A = \emptyset$.*

Proof If $F \subseteq X$ is a closed set of Cantor–Bendixson rank $\alpha < \omega_1$, then $F = P \cap A$ where $P = \Gamma^\alpha(F)$ and $A = \bigcup_{\beta<\alpha} \Gamma^{\beta+1}(F) \setminus \Gamma^\beta(F)$. Clearly A is countable and $A \cap P = \emptyset$.

Corollary 1.28 *If X is a Polish space. and $F \subseteq X$ is an uncountable closed set then F contains a nonempty perfect set and $|F| = 2^{\aleph_0}$. Also, if $Y \subseteq X$ is an uncountable F_σ-set, then Y contains a perfect set.*

In particular every uncountable Polish space has cardinality $2^{\aleph_0}$.

Exercise 1.29 Show that there is an uncountable $A \subset \mathbb{R}$ such that no subset of A is perfect. [Hint: Build A be diagonalizing against all perfect sets. You will need to use a well-ordering of $\mathbb{R}$.]

Polish subspaces

Suppose X is a Polish space and $F \subseteq X$ is closed. If (x_n) is a Cauchy sequence with each $x_n \in F$, then $\lim x_n \in F$. Thus F is also a Polish space.

If $U \subset X$ is open, then Cauchy sequences in U, may not converge to elements of U. For example, $(0,1) \subset \mathbb{R}$ and $\frac{1}{n} \to 0 \notin (0,1)$. The next lemma shows that when U is open we are able to define a new complete metric on U compatible with the topology.

Lemma 1.30 *If X is a Polish space and $U \subseteq X$ is open, then U (with the subspace topology) is Polish.*

Proof Let d be a complete metric on X compatible with the topology, we may assume $d < 1$.

Let

$$\widehat{d}(x,y) = d(x,y) + \left| \frac{1}{d(x, X \setminus U)} - \frac{1}{d(y, X \setminus U)} \right|.$$

It is easy to see that $\widehat{d}(x,x)$ is a metric. Since $\widehat{d}(x,y) \geq d(x,y)$, every d-open, set is $\widehat{d}$-open. Suppose $x \in U$, $d(x, X \setminus U) = r > 0$ and $\epsilon > 0$. Choose $\delta > 0$ such that if $0 < \eta \leq \delta$, then $\eta + \frac{\eta}{r(r-\eta)} < \epsilon$. If $d(x,y) < \delta$, then $d(y, X \setminus U) > r - \delta$. Hence

$$\widehat{d}(x,y) \leq \delta + \left| \frac{1}{r} - \frac{1}{r+\delta} \right| \leq \delta + \left| \frac{-\delta}{r(r-\delta)} \right| < \epsilon.$$

Thus the $\widehat{d}$-ball of radius ϵ around x, contains the d-ball of radius δ. Hence every $\widehat{d}$-open subset is open. Thus $\widehat{d}$ is compatible with the subspace topology on U. We need only show $\widehat{d}$ is complete.

Suppose (x_n) is a $\widehat{d}$-Cauchy sequence. Then (x_n) is also a d-Cauchy sequence, so there is $x \in X$ such that $x_n \to x$. In addition for each n

$$\lim_{i,j \to \infty} \left| \frac{1}{d(x_i, X \setminus U)} - \frac{1}{d(x_j, X \setminus U)} \right| = 0.$$

Thus there is $r \in \mathbb{R}$ such that

$$\lim_{i \to \infty} \frac{1}{d(x_i, X \setminus U)} = r.$$

In particular, $\frac{1}{d(x_i, X \setminus U)}$ is bounded away from 0 and $d(x, X \setminus U) > 0$. Thus $x \in U$. Hence $\widehat{d}$ is a complete metric on U and U is a Polish space.

We can generalize this a bit further. Recall that $Y \subseteq X$ is a G_δ-set if Y is a countable intersection of open sets. The G_δ-sets are exactly the complements of F_σ-sets. Thus every open set is G_δ and every closed set is G_δ.

Corollary 1.31 *If X is a Polish space and $Y \subseteq X$ is G_δ, then Y is a Polish space.*

Proof Let $Y = \bigcap O_n$ where each O_n is open. Let d_n be a complete metric on O_n compatible with the topology. We may assume that $d_n < 1$. Let

$$\widehat{d}(x,y) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} d_n(x,y).$$

If (x_i) is a $\widehat{d}$-Cauchy sequence, then (x_i) is d_n-Cauchy for each n. Thus there is $x \in X$ such that each $x_i \to x$ in each O_n. Since each O_n is complete $x \in \bigcap O_n = Y$. Hence $\widehat{d}$ is complete.

Corollary 1.32 *If X is a Polish space and $Y \subseteq X$ is an uncountable G_δ-set, then Y contains a perfect set.*

Can we generalize Corollary 1.31 further? We already saw that $\mathbb{Q} \subseteq \mathbb{R}$ is not a Polish subspace. Since $\mathbb{Q}$ is countable it is F_σ. Thus we can not generalize this to F_σ-sets. Indeed the converse to the corollary is true.

Theorem 1.33 *If X is a Polish space, then $Y \subseteq X$ is a Polish subspace if and only if Y is a G_δ-set.*

Proof Suppose Y is a Polish subspace of X. Let d be a complete metric on Y compatible with the subspace topology. Let $U_0, U_1, \ldots$ be a basis of open subsets of X. If $x \in Y$ and $\epsilon > 0$, then for any open neighborhood V of X there is $U_n \subset V$ such that $x \in U_n$ and diam $(Y \cap U_n) < \epsilon$, where the diameter is computed with respect to d.

Let

$$A = \{x \in \overline{Y} : \forall \epsilon > 0 \exists n\ x \in U_n \wedge \text{diam } (Y \cap U_n) < \epsilon\}.$$

Then

$$A = \bigcap_{m=1}^{\infty} \bigcup \{U_n : \text{diam } (Y \cap U_n) < \frac{1}{m}\}$$

is a G_δ-set and $Y \subseteq A$. Suppose $x \in A$. For all $m > 0$, there is U_{n_m} such that $x \in U_{n_m}$ and diam $(Y \cap U_{n_m}) < \epsilon$. Since Y is dense in A, for each m we can find $y_m \in Y \cap U_{n_1} \cap \ldots \cap U_{n_m}$. Then $y_1, y_2, \ldots$ is a Cauchy sequence converging to x. Hence $x \in Y$. Thus $Y = A$ is a G_δ-set.

Corollary 1.34 *Every Polish space is homeomorphic to a G_δ-subset of $\mathbb{H}$.*

Proof By 1.4, if X is Polish space, then X is homeomorphic to a subspace Y of $\mathbb{H}$. By 1.33 Y is a G_δ-subset of X.

Spaces of $\mathcal{L}$-structures

We conclude this section with another important example of a Polish space.

Let $\mathcal{L}$ be a countable first-order language. Let $\text{Mod}(\mathcal{L})$ be the set of all $\mathcal{L}$-structures with universe $\mathbb{N}$. We will define two topologies on $\text{Mod}(\mathcal{L})$. Let $\{c_0, c_1, \ldots\}$ be a set of countably many distinct new constant symbols and let $\mathcal{L}^* = \mathcal{L} \cup \{c_0, c_1, \ldots\}$. If $\mathcal{M} \in \text{Mod}(\mathcal{L})$, then we can naturally view $\mathcal{M}$ as an $\mathcal{L}^*$-structure by interpreting the constant symbol c_i as i.

If ϕ is an $\mathcal{L}^*$-sentence, let $B_\phi = \{\mathcal{M} \in \text{Mod}(\phi) : \mathcal{M} \models \phi\}$. Let τ_0 be the topology with basic open sets $\{B_\phi : \phi \text{ a quantifier-free } \mathcal{L}^*\text{-formula}\}$ and let τ_1 be the topology with basic open sets $\{B_\phi : \phi \text{ an } \mathcal{L}^*\text{-formula}\}$. Clearly the topology τ_1-refines τ_0.

Theorem 1.35 $(\text{Mod}(\mathcal{L}), \tau_0)$ *and* $(\text{Mod}(\mathcal{L}), \tau_1)$ *are Polish spaces.*

We give one illustrative example to show that $(\text{Mod}(\mathcal{L}), \tau_0)$ is a Polish space. Suppose $\mathcal{L} = \{R, f, c\}$ where R is a binary relation symbol, f is a binary function symbol and c is a constant symbol. Let X be the Polish space $2^{\mathbb{N}^2} \times \mathbb{N}^{\mathbb{N}^2} \times \mathbb{N}$, with the product topology. If $\mathcal{M}$ is an $\mathcal{L}$-structure, let $R^{\mathcal{M}}$, $f^{\mathcal{M}}$ and $c^{\mathcal{M}}$ be the interpretation of the symbols of $\mathcal{L}$ in $\mathcal{M}$ and let $\chi_{R^{\mathcal{M}}} : \mathbb{N}^2 \to 2$ be the characteristic function of $R^{\mathcal{M}}$. The function $\mathcal{M} \mapsto (\chi_{R^{\mathcal{M}}}, f^{\mathcal{M}}, c^{\mathcal{M}})$ is a bijection between $\mathrm{Mod}(\mathcal{L})$ and X.

We will prove $(\mathrm{Mod}(\mathcal{L}), \tau_0)$ is Polish by showing that this map is a homeomorphism. Let $Y_0 = \{(g, h, n) \in X : g(i, j) = 1\}$, $Y_1 = \{(g, h, n) : h(i, j) = k\}$, $Y_3 = \{(g, h, n) \in X : n = m\}$. The inverse images of these sets are the basic clopen sets $B_{R(c_i,c_j)}$, $B_{f(c_i,c_j)=c_k}$ and $B_{c_m=c}$, respectively. It follows that this map is continuous. We need to show that if ϕ is quantifier-free, then the image of B_ϕ is clopen. This is an easy induction once we show it for atomic formulas. For formulas of the form $R(c_i, c_j)$ or $f(c_i, c_j) = c_k$, this is obvious. A little more care is needed to deal with formulas built up from terms. For example, let ϕ be the formula $f(c_0, f(c_1, c_2)) = c_3$. Then the image of B_ϕ is $Y = \{(g, h, n) : h(0, h(1, 2)) = 3\}$. Then

$$Y = \bigcup_{i \in \mathbb{N}} \{(g, h, n) : h(1, 2) = i \wedge h(0, i) = 3\}$$

is open and

$$\neg Y = \bigcup_{i \in \mathbb{N}} \bigcup_{j \neq 3} \{(g, h, n) : h(1, 2) = i \wedge h(0, i) = j\}$$

is open. Thus Y is clopen. This idea can be generalized to all atomic ϕ.

Exercise 1.36 Give a detailed proof that $(\mathrm{Mod}(\mathcal{L}), \tau_0)$ is a Polish space for any countable first order language $\mathcal{L}$.

Next, we consider $(\mathrm{Mod}(\mathcal{L}), \tau_1)$. Let S be all $\mathcal{L}^*$-sentences. Then 2^S with the product topology is a Polish space homeomorphic to the Cantor space. Let X be the set of all $f \in 2^S$ such that

i) $\{\phi \in S : f(\phi) = 1\}$ is consistent;

ii) for all ϕ we have $f(\phi) = 0 \leftrightarrow f(\neg\phi) = 1$;

iii) $f(c_i = c_j) = 0$ for $i \neq j$;

iv) for all ϕ, if $f(\exists v \phi(v)) = 1$, then $f(\phi(c_m)) = 1$ for some $m \in \mathbb{N}$.

Lemma 1.37 *X is G_δ-subset of 2^S.*

Proof Let $X_1 = \{f : \{\phi : \phi(f) = 1\}\}$ is consistent. Let I be the set of finite subsets of S that are inconsistent. Then

$$X_1 = \bigcap_{A \in I} \{f : f(\phi) = 0 \text{ for some } \phi \in A\}$$

and X_1 is closed.

Also

$$X_2 = \bigcap_{\phi \in S} \{f : f(\phi) = 0 \leftrightarrow f(\neg\phi) = 1\}$$

and

$$X_3 = \bigcap_{i \neq j} \{f : f(c_i = c_j) = 0\}$$

are closed.

Let F be the set of $\mathcal{L}^*$-formulas with one free-variable. Then

$$X_4 = \bigcap_{\phi \in F} \Big(\{f : f(\exists v \phi(v)) = 0\} \cup \bigcup_{n \in \mathbb{N}} \{f : f(\phi(c_n)) = 1\} \Big)$$

is G_δ. Since $X = X_1 \cap \ldots \cap X_4$, X is G_δ.

Thus X is a Polish subspace of 2^S.

If $\mathcal{M} \in \mathrm{Mod}(\mathcal{L})$, let $f_{\mathcal{M}}(\phi) = 1$ if $\mathcal{M} \models \phi$ and $f_{\mathcal{M}}(\phi) = 0$ if $\mathcal{M} \models \neg\phi$. It is easy to see that $f_{\mathcal{M}} \in X$. If $f \in X$, then Henkin's proof of Gödel's Completeness Theorem shows that there is an $\mathcal{L}$-structure $\mathcal{M}$ with universe $\mathbb{N}$ such that:

i) if R is an n-ary relation symbol, then $R^{\mathcal{M}} = \{(n_1, \ldots, n_m) : f(R(c_{n_1}, \ldots, c_{n_m}) = 1\}$;

ii) if g is an m-ary function symbol, then $g^{\mathcal{M}} : \mathbb{N}^m \to \mathbb{N}$ is the function where $g^{\mathcal{M}}(n_1, \ldots, n_m) = k$ if and only if $f(g(c_{n_1}, \ldots, c_{n_m}) = c_{n_k}) = 1$;

iii) if c is a constant symbol, then $c^{\mathcal{M}} = n$ if and only if $f(c = c_n) = 1$.

Thus $\mathcal{M} \mapsto f_{\mathcal{M}}$ is a bijection between $\mathrm{Mod}(\mathcal{L})$ and X. The image of B_ϕ is $\{f \in X : f(\phi) = 1\}$. Thus this map is a homeomorphism and $\mathrm{Mod}(\mathcal{L})$ is a Polish space.

Spaces of Compact Sets

We describe one more interesting example without giving proofs. For proofs see Kechris [6] 4.F.

Definition 1.38 Let X be a topological space. Let $K(X)$ be the collection of all compact subsets of X. The *Vietoris topology* on $K(X)$ is the smallest topology such that for each open $U \subseteq X$ the sets $\{A \in K(X) : A \subseteq U\}$ and $\{A \in K(X) : A \cap U \neq \emptyset\}$ are open.

Exercise 1.39 Suppose X is separable and $D \subseteq X$ is a countable dense set. Show that $\{A \subseteq D : A \text{ finite}\}$ is a dense subset of $K(X)$. Thus $K(X)$ is separable.

Definition 1.40 Suppose X is a metric space. We define the *Hausdorff metric* on $K(X)$ by

$$d_H(A, B) = \max\Big(\max_{a \in A} d(a, B), \max_{b \in B} d(b, A)\Big).$$

Exercise 1.41 Show that the Hausdorff metric on $K(X)$ is compatible with the Vietrois topology.

Theorem 1.42 *If d is a complete metric on X, then d_H is a complete metric on $K(X)$. In particular, if X is a Polish space, then so is $K(X)$.*

In 2.20 we show that $\{A \subseteq X : A \text{ is finite}\}$ is an F_σ subset of $K(X)$.

2 Borel Sets

Definition 2.1 If X is any set, a *σ-algebra* on X is a collection of subsets of X that is closed under complement and countable union. A *measure space* (X, Ω) is a set X equipped with a σ-algebra Ω.

If (X, Ω_X) and (Y, Ω_Y) are measure spaces, we say $f : X \to Y$ is a *measurable function* if $f^{-1}(A) \in \Omega_X$ for all $A \in \Omega_Y$. We say that (X, Ω_X) and (Y, Ω_Y) are *isomorphic* if and only if there is a measurable bijection with measurable inverse.

Definition 2.2 If X is a topological space, the class of *Borel sets* $\mathcal{B}(X)$ is the smallest σ-algebra containing the open sets.

If X and Y are topological spaces, we say that $f : X \to Y$ is *Borel measurable* if it is a measurable map between the measure spaces $(X, \mathcal{B}(X))$ and $(Y, \mathcal{B}(Y))$.

We say that a measure space (X, Ω) is a *standard Borel space* if there is a Polish space Y such that (X, Ω) is isomorphic to $(Y, \mathcal{B}(Y))$.

Lemma 2.3 *Suppose X and Y are topological spaces and $f : X \to Y$.*

i) f is Borel measurable if and only if the inverse image of every open set is Borel.

ii) If Y is separable, then f is Borel measurable if and only if the inverse image of every basic open set is Borel.

iii) If Y is separable and $f : X \to Y$ is Borel measurable, then the graph of f is Borel.

Proof If $f : X \to Y$ is Borel measurable, then the inverse image of every open set is Borel.

i) Let $\Omega = \{A \in \mathcal{B}(Y) : f^{-1}(A) \in \mathcal{B}(X)\}$. Suppose every open set is in Ω. If $A \in \Omega$, then $f^{-1}(Y \setminus A) = X \setminus f^{-1}(A)$ is Borel and $X \setminus A \in \Omega$. If $A_0, A_1, \ldots \in \Omega$, then $f^{-1}(\bigcup A_i) = \bigcup f^{-1}(A_i)$ is Borel and $\bigcup(A_i) \in \Omega$.

ii) Suppose O is open. There are basic open sets $U_0, U_1, \ldots$ such that $O = \bigcup U_i$. Then $f^{-1}(O) = \bigcup f^{-1}(U_i)$ is a countable union of Borel sets and hence Borel.

iii) Let $U_0, U_1, \ldots$ be a basis for the topology of Y. Then the graph of f is

$$\bigcap_{n=0}^{\infty} (\{(x, y) : y \notin U_n\} \cap \{(x, y) : x \in f^{-1}(U_n)\}.$$

Since each $f^{-1}(U_n)$ is Borel so is the graph of f.

By ii) any continuous $f : X \to Y$ is Borel measurable. We will see later that the converse of iii) is also true.

Since

$$\bigcap A_i = X \setminus \bigcup (X \setminus A_i),$$

any σ-algebra is also closed under countable intersections. Thus $\mathcal{B}(X)$ contains all of the open, closed, F_σ, and G_δ sets. We could generalize this further by taking $F_{\sigma\delta}$, intersections of F_σ-sets, $G_{\delta\sigma}$, unions of G_δ-sets, $F_{\sigma\delta\sigma}$, $G_{\delta\sigma\delta}\ldots$. There is a more useful way of describing these classes.

Definition 2.4 Let X be a metrizable space. For each $\alpha < \omega_1$ we define $\mathbf{\Sigma}^0_\alpha(X)$ and $\mathbf{\Pi}^0_\alpha(X) \subset \mathcal{P}(X)$ as follows:

$\mathbf{\Sigma}^0_1(X)$ is the collection of all open subsets of X;

$\mathbf{\Pi}^0_\alpha(X)$ is the collection of all sets $X \setminus A$ where $A \in \mathbf{\Sigma}^0_\alpha(X)$;

For $\alpha > 1$, $\mathbf{\Sigma}^0_\alpha(X)$ is the collection of all sets $X = \bigcup A_i$ where each $A_i \in \mathbf{\Pi}^0_{\beta_i}(X)$ for some $\beta_i < \alpha$.

We say that $A \in \mathbf{\Delta}^0_\alpha(X)$ if $A \in \mathbf{\Sigma}^0_\alpha(X)$ and $A \in \mathbf{\Pi}^0_\alpha(X)$.

When we are working in a single space we omit the X and write $\mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Pi}^0_\alpha$ instead of $\mathbf{\Sigma}^0_\alpha(X)$ and $\mathbf{\Pi}^0_\alpha(X)$.

Closed sets are $\mathbf{\Pi}^0_1$, F_σ-sets are $\mathbf{\Sigma}^0_2$, G_δ-sets are $\mathbf{\Pi}^0_2$,....

Lemma 2.5 *Suppose X is metrizable.*

i) $\mathbf{\Sigma}^0_\alpha \cup \mathbf{\Pi}^0_\alpha \subseteq \mathbf{\Delta}^0_{\alpha+1}$ for all $\alpha < \omega_1$.

ii) $\mathcal{B}(X) = \bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha$.

iii) If X is infinite, then $|\mathcal{B}(X)| = 2^{\aleph_0}$.

Proof In any metric space every open set is both F_σ and G_δ, thus $\mathbf{\Sigma}^0_1 \cup \mathbf{\Pi}^0_1 \subseteq \mathbf{\Delta}^1_0$. i) then follows easily by induction. An easy induction shows that any σ-algebra containing the open sets must contain $\mathbf{\Sigma}^0_\alpha$ for each $\alpha < \omega_1$.

iii) If $U_0, U_1, \ldots$ is a basis for the topology, then every open set is of the form $\bigcup_{n\in S} U_n$ for some $S \subseteq \mathbb{N}$, thus $|\mathbf{\Sigma}^0_\alpha| \le 2^{\aleph_0}$. Clearly $|\mathbf{\Pi}^0_\alpha| = |\mathbf{\Sigma}^0_\alpha|$. Suppose $\alpha < \omega_1$ and $|\mathbf{\Pi}^0_\beta| \le 2^{\aleph_0}$ for all $\beta < \alpha$. Then $|\bigcup_{\beta<\alpha} \mathbf{\Pi}^0_\beta| < \alpha$ and if $\mathcal{F}$ is the set of $f : \mathbb{N} \to \bigcup_{\beta<\alpha} \mathbf{\Pi}^0_\beta$, then $|\mathcal{F}| \le (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}$ and for any $A \in \mathbf{\Sigma}^0_\alpha$, there is $f \in \mathcal{F}$ such that $A = \bigcup f(n)$. Thus $|\mathbf{\Sigma}^0_\alpha| \le 2^{\aleph_0}$. Thus

$$|\mathcal{B}(X)| = \left| \bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha \right| \le \aleph_1 \times 2^{\aleph_0} = 2^{\aleph_0}.$$

If X is infinite, then every countable subset of X is $\mathbf{\Sigma}^0_2$. Hence $|\mathcal{B}(X)| = 2^{\aleph_0}$.

We state the basic properties of these classes.

Lemma 2.6 *i) $\mathbf{\Sigma}^0_\alpha$ is closed under countable unions and finite intersections.*

ii) $\mathbf{\Pi}^0_\alpha$ is closed under countable intersections and finite unions.

iii) $\mathbf{\Delta}^0_\alpha$ is closed under finite unions, finite intersections and complement.

iv) $\mathbf{\Sigma}^0_\alpha$, $\mathbf{\Pi}^0_\alpha$ and $\mathbf{\Delta}^0_\alpha$ are closed under continuous inverse images.

Proof We prove i) and ii) simultaneously by induction on α. We know that i) holds for the open sets. By taking complements, it is easy to see that if $\mathbf{\Sigma}^0_\alpha$ is closed under countable unions and finite intersections, then $\mathbf{\Pi}^0_\alpha$ is closed under countable intersections and finite unions.

Suppose $\alpha > 0$. $A_0, A_1, \ldots \in \mathbf{\Sigma}^0_\alpha$. Let $A_i = \bigcup_{j=0}^\infty B_{i,j}$ where each $B_{i,j} \in \mathbf{\Pi}^0_\beta$ for some $\beta < \alpha$. Then

$$\bigcup_{i=0}^\infty A_i = \bigcup_{i=0}^\infty \bigcup_{j=0}^\infty B_{i,j} \in \mathbf{\Sigma}^0_\alpha.$$

Suppose we have proved ii) for all $\beta < \alpha$. Then

$$A_0 \cap A_1 = \bigcup_{i=0}^{\infty} \bigcup_{j=0}^{\infty} (B_{0,i} \cap B_{0,j})$$

and each $B_{0,i} \cap B_{0,j}$ is $\mathbf{\Pi}^0_\beta$ for some $\beta < \alpha$. Thus $A_0 \cap A_1$ is $\mathbf{\Sigma}^0_\alpha$.

iii) is immediate from i) and ii).

iv) Suppose $f : X \to Y$ is continuous. We prove that if $A \subseteq Y$ is $\mathbf{\Sigma}^0_\alpha$ (respectively $\mathbf{\Pi}^0_\alpha$), then so is $f^{-1}(A)$. If $\alpha = 0$, this is clear. Since $f^{-1}(\bigcup A_i) = \bigcup f^{-1}(A)$ and $f^{-1}(Y \setminus A) = X \setminus f^{-1}(A)$, this follows easily by induction.

Corollary 2.7 *If $A \subseteq X \times Y$ is $\mathbf{\Sigma}^0_\alpha$ (respectively $\mathbf{\Pi}^0_\alpha$ or $\mathbf{\Delta}^0_\alpha$) and $a \in Y$, then $\{x \in X : (x, a) \in Y\}$ is $\mathbf{\Sigma}^0_\alpha$.*

Proof The map $x \mapsto (x, a)$ is continuous.

Exercise 2.8 Suppose X is a Polish space and Y is a subspace of X. a) Show that $\mathbf{\Sigma}^0_\alpha(Y) = \{Y \cap A : A \in \mathbf{\Sigma}^0_\alpha(X)\}$ and $\mathbf{\Pi}^0_\alpha(Y) = \{Y \cap A : A \in \mathbf{\Sigma}^0_\alpha(X)\}$.

b) This does not necessarily work for $\mathbf{\Delta}^0_\alpha$. Show that $\mathbf{\Delta}^0_2(\mathbb{Q}) \neq \{\mathbb{Q} \cap A : A \in \mathbf{\Delta}^0_2(\mathbb{Q})\}$.

Examples

We give several examples.

Example 2.9 *If $A \subseteq X$ is countable, then $A \in \mathbf{\Sigma}^0_2$.*

Point are closed, so every countable set is a countable union of closed sets.

Example 2.10 *Let $A = \{x \in \mathcal{N} : x$ is eventually constant$\}$. Then A is $\mathbf{\Sigma}^0_2$.*

$$x \in A \text{ if and only if } \exists m \forall n > m \ x(n) = x(n+1).$$

If $A_n = \{x : x(n) = x(n+1)\}$, then A_n is clopen and

$$A = \bigcup_{m=0}^{\infty} \bigcap_{n>m} A_n$$

is $\mathbf{\Sigma}^0_2$.

Example 2.11 *Let $A = \{x \in \mathcal{N} : x$ is a bijection$\}$. Then A is $\mathbf{\Pi}^0_2$.*

Let $A_0 = \{x : \forall n \forall m \ (n \neq m \to x(n) \neq x(m))\}$. Then

$$A_0 = \bigcap_{n=0}^{\infty} \bigcap_{n \neq m} \{x : x(n) \neq x(m)\}$$

is closed. Let $A_1 = \{x : \forall n \exists m\ x(m) = n\}$. Then

$$A_1 = \bigcap_{n=0}^{\infty} \bigcup_{m=0}^{\infty} \{x : x(m) = n\}$$

is $\mathbf{\Pi}_2^0$ and $A = A_0 \cap A_1$ is $\mathbf{\Pi}_2^0$.

As these examples make clear, existential quantification over $\mathbb{N}$ (or $\mathbb{Q}$ or any countable set) corresponds to taking countable union, while universal quantification over a countable set corresponds to taking a countable intersection.

Example 2.12 *For $x \in 2^{\mathbb{N}^2}$ we can view x as coding a binary relation R_x on $\mathbb{N}$, by $(i,j) \in R_f$ if and only if $x(i,j) = 1$. Then $LO = \{x : R_x$ is a linear order$\}$ is a $\mathbf{\Pi}_1^0$-set and $DLO = \{x \in LO : R_x$ is a dense linear order$\}$ is $\mathbf{\Pi}_2^0$.*

$x \in LO$ if and only if the following three conditions hold
$\forall n \forall m\ (x(n,m) = 0 \vee x(m,n) = 0)$
$\forall n \forall m\ (n = m \vee x(n,m) = 1 \vee x(m,n) = 1)$
$\forall n \forall m \forall k\ (x(n,m) = x(m,k) = 1 \rightarrow x(n,k) = 1.$
Thus LO is $\mathbf{\Pi}_1^0$. $x \in DLO$ if and only if $x \in LO$ and

$$\forall n \forall m\ (x(n,m) = 1 \rightarrow \exists k\ x(n,k) = x(k,m) = 1).$$

Thus DLO is $\mathbf{\Pi}_2^0$.

Example 2.13 *Let A be a countable set. $Tr_A = \{x \in 2^{A^{<\omega}} : x$ is a tree$\}$. Then Tr_A is $\mathbf{\Pi}_1^0$.*

The set $\{x \in Tr_2 : x$ has an infinite path$\}$ is also $\mathbf{\Pi}_1^0$.

$x \in Tr_A$ if and only if $\forall \sigma \forall \tau \subseteq \sigma\ (x(\sigma) = 1 \rightarrow x(\tau) = 1)$.
By König's Lemma, a binary tree T has an infinite path if and only if T is infinite. Thus $x \in WF_2$ if and only if $x \in Tr_2$ and

$$\forall n \exists \sigma \in 2^n\ x(\sigma) = 1$$

At first this looks $\mathbf{\Pi}_2^0$, but the existential quantifier is only over a finite set. Indeed

$$WF_2 = \bigcap_{n \in \mathbb{N}} \bigcup_{\sigma \in 2^n} \{x : x(\sigma) = 1\}$$

and $\bigcup_{\sigma \in 2^n} \{x : x(\sigma) = 1\}$ is a clopen set.

Example 2.14 *We say that $x \in \mathcal{C}$ is* normal *if*

$$\lim_{n \to \infty} \frac{1}{n+1} \sum_{i=0}^{n} x(i) = \frac{1}{2}.$$

Let $N = \{x \in \mathcal{C} : x \text{ is normal}\}$.

x is normal if and only if

$$\forall k > 0 \exists m \forall n \left(n > m \rightarrow \left| \frac{1}{2} - \frac{1}{n+1} \sum_{i=0}^{n} x(i) \right| < \frac{1}{k} \right).$$

If

$$A_{n,k} = \left\{ x \in \mathcal{C} : \left| \frac{1}{2} - \frac{1}{n+1} \sum_{i=0}^{n} x(i) \right| < \frac{1}{k} \right\},$$

then $A_{n,k}$ is clopen and

$$N = \bigcap_{k=1}^{\infty} \bigcup_{m=0}^{\infty} \bigcap_{n \geq m} A_{n,k}.$$

Hence N is a $\mathbf{\Pi}_3^0$-set.

Example 2.15 *Models of a first order theory.*

Suppose $\mathcal{L}$ is a first order language. Let $\mathcal{L}^*$, $\mathrm{Mod}(\mathcal{L})$, τ_0 and τ_1 be as in 1.35.

Suppose ϕ is an $\mathcal{L}^*_{\omega_1,\omega}$-sentence. Let $\mathrm{Mod}(\phi) = \{\mathcal{M} \in \mathrm{Mod}(\mathcal{L}) : \mathcal{M} \models \phi\}$. We claim that $\mathrm{Mod}(\phi)$ is a Borel subsets of $\mathrm{Mod}(\mathcal{L})$. It is enough to prove this for the weaker topology τ_0. In τ_0, if ϕ is quantifier-free then $\mathrm{Mod}(\phi)$ is clopen. The claim follows by induction since,

$$\mathrm{Mod}(\neg\phi) = \mathrm{Mod}(\mathcal{L}) \setminus Mod(\phi),$$

$$\mathrm{Mod}(\bigwedge_{i=1}^{\infty} \phi_i) = \bigcap_{i=0}^{\infty} \mathrm{Mod}(\phi_i)$$

and

$$\mathrm{Mod}(\exists v \ \phi(v)) = \bigcup_{n=0}^{\infty} \mathrm{Mod}(\phi(c_n)).$$

If T is a first order $\mathcal{L}$-theory, then

$$\mathrm{Mod}(T) = \bigcap_{\phi \in T} \mathrm{Mod}(\phi)$$

is Borel.

Exercise 2.16 Show that if ϕ is a first order $\mathcal{L}^*$-sentence, then $\mathrm{Mod}(\phi)$ is $\mathbf{\Sigma}_n^0$ for some n. (hint: prove that n depends only on the quantifier rank of ϕ.)

Conclude that if T is a first order theory, then $\mathrm{Mod}(T)$ is $\mathbf{\Pi}_\omega^0$.

In the topology τ_1, $\mathrm{Mod}(\phi)$ is clopen for all first order ϕ. Thus $\mathrm{Mod}(T)$ is closed.

Example 2.17 *Isomorphism classes of structures.*

Suppose $\mathcal{M} \in \mathrm{Mod}(\mathcal{L})$. There is $\phi_{\mathcal{M}} \in \mathcal{L}_{\omega_1,\omega}$, the *Scott sentence* of $\mathcal{M}$ (see [11] 2.4.15) such that if $\mathcal{M}_1$ is a countable $\mathcal{L}$-structure, then $\mathcal{M} \cong \mathcal{M}_1$ if and only if $\mathcal{M}_1 \models \phi_{\mathcal{M}}$. Thus

$$\{\mathcal{M}_1 \in \mathcal{M} : \mathcal{M}_1 \cong \mathcal{M}\} = \mathrm{Mod}(\phi_{\mathcal{M}})$$

is a Borel set.

Example 2.18 *Let $X = C(\mathbb{I}) \times I$ and let $D = \{(f,x) : f$ is differentiable at $x\}$. Then $D \in \mathbf{\Pi}^0_3$.*

f is differentiable at x if and only if
$\forall n \exists m\ \forall p, q \in \mathbb{Q} \cap [0,1]\ (|x-p| < \frac{1}{m} \wedge |x-q| < \frac{1}{m}) \rightarrow$
$|(f(p)-f(x))(q-x) - (f(q)-f(x))(p-x)| \leq \frac{1}{n}|(p-x)(q-x)|$.
The inner condition is closed in $C(\mathbb{I}) \times \mathbb{I}$ so this set is $\mathbf{\Pi}^0_3$.

Example 2.19 *If X is a Polish space, then $\{(A,B) \in K(X)^2 : A \subseteq B\}$ is $\mathbf{\Pi}^0_2$.*

If $a \in A \setminus B$, then there is a basic open set U such that $a \in U$ and $U \cap B = \emptyset$. Fix $U_0, U_1, \ldots$ a basis for X. Then $A \subseteq B$ if and only if

$$\forall n\ (U_n \cap B = \emptyset \rightarrow U_n \cap A = \emptyset).$$

This is a $\mathbf{\Pi}^0_2$ definition.

Example 2.20 *Suppose X is a Polish space. Then $\{A \subseteq X : A$ is finite$\}$ is an F_σ subset of $K(X)$.*

A is finite if and only there are basic open set $U_1, \ldots, U_n$ such that $A \subseteq U_1 \cup \ldots \cup U_n$ such that if V_i and V_j are disjoint basic open subsets of U_i, then $A \cap V_0 = \emptyset$ or $A \cap V_1 = \emptyset$.

Fix $U_0, U_1, \ldots$ a basis for the open sets. If $F \subseteq \mathbb{N}$ is finite, then

$$B_F = \{A : A \subseteq \bigcup_{i \in F} V_i\}$$

is open. Let $S_F = \{(i,j) \in \mathbb{N} : U_i$ and U_j are disjoint subsets of U_k for some $k \in F\}$. The set $C_{i,j} = \{A : A \cap U_i = \emptyset$ or $A \cap U_j = \emptyset\}$ is closed. Thus

$$\{A \in K(X) : A \text{ is finite }\} = \bigcup_{F \subseteq \mathbb{N} \text{ finite}} \left(B_F \cap \bigcap_{(i,j) \in S_F} C_{i,j}\right)$$

is $\mathbf{\Sigma}^0_2$.

Exercise 2.21 Show that $\{A \in K(X) : A$ is perfect$\}$ is $\mathbf{\Pi}^0_2$.

Changing the Topology

Suppose X is a Polish space, let τ be the topology of X. We will often prove interesting results about Borel sets $A \subseteq X$, by refining τ to a new topology τ_1 with the same Borel sets such that A is clopen in the new topology.

We start with one preparatory lemma.

Lemma 2.22 *Suppose X and Y are disjoint Polish spaces. The disjoint union $X \uplus Y$ is the space $X \cup Y$ where $U \subseteq X \cup Y$ is open if and only if $U \cap X$ and $U \cap Y$ are both open. Then $X \uplus Y$ is a Polish space.*

Proof Let d_X be a compatible metric on X and d_Y be a compatible metric on Y with $d_X < 1$ and $d_Y < 1$. Define $\widehat{d}$ on $X \uplus Y$ by

$$\widehat{d}(x,y) = \begin{cases} d_X(x,y) & \text{if } x,y \in X \\ d_Y(x,y) & \text{if } x,y \in Y \\ 2 & \text{otherwise.} \end{cases}$$

It is easy to see that X and Y are clopen in this topology and the open subsets of $X \uplus Y$ are unions of open subsets of X and open subsets of Y. Any Cauchy sequence must be eventually in either X or Y and converges in the original topology so this is a complete metric.

Lemma 2.23 *Let X be a Polish space with topology τ. Suppose $F \subseteq X$ is closed. There is a Polish topology τ_1 on X refining τ such that F is clopen in τ_1, and τ and τ_1 have the same Borel sets.*

Proof We know that $X \setminus F$ has a Polish topology and F has a Polish topology. Let τ_1 be the Polish topology on the disjoint union of $X \setminus F$ and F. Then F is open. The open subsets of τ_1 are either open in τ or intersections of τ open sets with F. In particular they are all Borel in τ. Thus the Borel sets of τ_1 are the Borel sets of τ.

Theorem 2.24 *Let X be a Polish space with topology τ. Suppose $A \subseteq X$ is Borel. There is a Polish topology τ^* on X such that A is clopen and τ^* has the same Borel sets as τ.*

Proof Let $\Omega = \{B \in \mathcal{B}(X)$: there is a Polish topology on X such that B is clopen. By the previous lemma, if B is open or closed, then $B \in \Omega$ and Ω is closed under complements.

Claim Ω is closed under countable intersections.

Suppose $A_0, A_1, \ldots \in \Omega$ and $B = \bigcap A_i$. Let τ_i be a Polish topology on X such that A_i is clopen in τ_i and τ and τ_i have the same Borel sets. The product $\prod(X, \tau_i)$ is a Polish space. Let $j : X \to \prod(X, \tau_i)$ be the diagonal embedding $j(x) = (x, x, x, \ldots)$. Let τ^* be the topology $j^{-1}(U)$ where U is an open subset in the product topology. Because $j(X)$ is a closed subset of the product, this is a Polish topology. A sub-basis for the topology τ^* can be obtained by taking inverse images of set $\{f : f(i) \in O_i\}$ where O_i is an open set in τ_i. Thus τ^* has a sub-basis of τ-Borel sets and every τ^*-Borel set is τ-Borel.

Since A_i is τ_i-clopen, it is also clopen in τ^*. Thus $\bigcap A_i$ is τ^*-closed. One further application of the previous lemma allows us to refine τ^* to τ^{**} keeping the same Borel sets but making $\bigcap A_i$ clopen.

Thus Ω is a σ-algebra, so $\Omega = \mathcal{B}(X)$.

We can use this observation to deduce several important results.

Theorem 2.25 (Perfect Set Theorem for Borel Sets) *If X is a Polish space and $B \subseteq X$ is an uncountable Borel set, then B contains a perfect set.*

Proof Let τ be the topology on X. We can refine the topology to τ_1 such that B is closed. Since B is uncountable, by 1.27 there is a nonempty τ_1-perfect $P \subseteq B$ and $f : \mathcal{C} \to P$ a homeomorphism. Since τ_1 refines τ_0, f is also continuous in the topology τ. Since $\mathcal{C}$ is compact, P is τ-closed. Since P has no isolated points in τ_1, this is still true in τ, so P is a perfect subsets of B.

Theorem 2.26 *If X is a Polish space and $B \subseteq X$ is Borel,*
i) there is $f : \mathcal{N} \to X$ continuous with $f(\mathcal{N}) = B$;
ii) there is a closed $F \subseteq \mathcal{N}$ and $g : F \to X$ continuous and one-to-one with $g(F) = B$;

Proof We refine the topology on X so that B is closed and X is still a Polish space. Then B with the subspace topology is Polish. By 1.17 we can find a continuous surjective $f : \mathcal{N} \to B$. f is still continuous with respect to the original topology of X. ii) is similar using 1.19.

We give one more application of this method.

Theorem 2.27 *If (X, τ) is Polish, Y is separable and $f : X \to Y$ is Borel measurable, then we can refine τ to τ^* with the same Borel sets such that f is continuous.*

Proof Let $U_0, U_1, \ldots$ be a countable basis for Y. Let τ^* be a Polish topology on X such that $f^{-1}(U_i)$ is open for all i and the τ^*-Borel sets are exactly the τ-Borel sets.

Exercise 2.28 Suppose X is a Polish space and $B \subseteq X \times X$ is Borel. Is it always possible to put a new Polish topology on X such that B is clopen in the new product topology on $X \times X$?

Borel Isomorphisms

Definition 2.29 If X and Y are Polish spaces, $A \in \mathcal{B}(X)$ and $B \in \mathcal{B}(Y)$, we say that $f : X \to Y$ is a *Borel isomorphism* if is a Borel measurable bijection with Borel measurable inverse.

Example 2.30 *If $A \subseteq X$ and $B \subseteq Y$ are countable and $|A| = |B|$, then any bijection $f : A \to B$ is a Borel isomorphism.*

In this case the inverse image of any open set is countable, and hence, F_σ.

Example 2.31 *The Cantor space $\mathcal{C}$ is Borel isomorphic to the closed unit interval $\mathbb{I}$.*

Let $C = \{x \in \mathcal{C} : x \text{ is eventually constant}\}$. Let $f : \mathcal{C} \setminus C \to \mathbb{I}$ by

$$f(x) = \sum_{n=0}^{\infty} \frac{x(n)}{2^{n+1}}.$$

Then f is a homeomorphism between $\mathcal{C} \setminus C$ and the dyadic-irrationals in $\mathbb{I}$. Since C is countable, we can also find a bijection g between C and the dyadic-rationals. Then

$$h(x) = \begin{cases} f(x) & \text{if } x \notin C \\ g(x) & \text{if } x \in C \end{cases}$$

is a Borel isomorphism between $\mathcal{C}$ and $\mathbb{I}$.

Corollary 2.32 *If X is a Polish space, there is a Borel $A \subseteq \mathcal{C}$ and a Borel isomorphism $f : X \to A$.*

Proof By 1.34 there is a Borel $B \subseteq \mathbb{H} = \mathbb{I}^{\mathbb{N}}$ and a homeomorphism $g : X \to B$. The Borel isomorphism between $\mathbb{I}$ and $\mathcal{C}$ induces a Borel isomorphism between $\mathbb{H}$ and $\mathcal{C}^{\mathbb{N}}$. But $\mathcal{C}^{\mathbb{N}}$ is homeomorphic to $\mathcal{C}$. Thus there is $h : \mathbb{H} \to \mathcal{C}$ a Borel isomorphism. Let $f = h \circ g$.

On the other hand if B is an uncountable Borel set, then B contains a perfect subset P that is homeomorphic to $\mathcal{C}$.

We will show that the Schröder-Bernstein Theorem holds for Borel isomorphisms. This will imply that any two uncountable Borel sets are Borel isomorphic.

Lemma 2.33 *Suppose X and Y are Polish spaces, $f : X \to Y$ is a Borel isomorphism between X and $f(X)$, and $g : Y \to X$ is a Borel isomorphism between Y and $g(Y)$. Then there is a Borel isomorphism between X and Y.*

Proof We follow the usual proof of the Schröder-Bernstein Theorem. We define $X = X_0 \supseteq X_1 \supseteq X_2 \ldots$ and $Y = Y_0 \supseteq X_1 \supseteq X_2 \ldots$ by $X_{n+1} = g(f(X_n))$ and $Y_{n+1} = f(g(Y_n))$. Since f^{-1} and g^{-1} are Borel measurable, each X_n and Y_n is Borel. Also, $X_\infty = \bigcup X_n$ and $X_\infty = \bigcup X_n$ are Borel.

Then $f|(X_n \setminus X_{n+1})$ is a bijection between $X_n \setminus X_{n+1}$ and $Y_{n+1} \setminus Y_{n+2}$ and $g|(Y_n \setminus Y_{n+1})$ is a bijection between $Y_n \setminus Y_{n+1}$ and $X_{n+1} \setminus X_{n+2}$. Also $f|X_\infty$ is a bijection between X_∞ and Y_∞.

Let $h : X \to Y$ be the function

$$h(x) = \begin{cases} f(x) & \text{if } x \in X_{2n} \setminus X_{2n+1} \text{ for some } n \text{ or } x \in X_\infty \\ g^{-1}(x) & \text{if } x \in X_{2n+1} \setminus X_{2n+2} \text{ for some } n. \end{cases}$$

Then $h : X \to Y$ is a Borel isomorphism.

Corollary 2.34 *i) If X is a Polish space and $A \subseteq X$ is an uncountable Borel set, then A is Borel isomorphic to $\mathcal{C}$.*

ii) Any two uncountable Polish spaces are Borel isomorphic.

iii) Any two uncountable standard Borel spaces are isomorphic.

Proof

i) If (X, τ) is a Polish space and $A \subseteq X$ is Borel, we can refine the topology of X making A clopen but not changing the Borel sets. Then A is a Polish space with the new subspace topology. We have shown that A is Borel isomorphic to a Borel subset of $\mathcal{C}$ and, by the Perfect Set Theorem, there is a Borel subset of A homeomorphic to $\mathcal{C}$. Thus there is a Borel isomorphism $f : A \to \mathcal{C}$. Since the new topology has the same Borel sets as the original topology, this is also a Borel isomorphism in the original topology.

ii) and iii) are clear from i).

Exercise 2.35 † Prove that if X and Y are Polish spaces, $A \subseteq X$ is Borel and $f : X \to Y$ is continuous, and $f|A$ is one-to-one, then $f(A)$ is Borel. Conclude that $f|A : A \to B$ is a Borel isomorphism. [This can be proved by the methods at hand, but we will give a very different proof later.]

The Borel Hierarchy

When constructing the Borel sets, do we really need $\mathbf{\Sigma}^0_\alpha$-sets for all $\alpha < \omega_1$? If X is countable and $Y \subset X$, then Y and $X \setminus Y$ are countable unions of points. Thus $Y \in \mathbf{\Delta}^0_2$. On the other hand, we will show that if X is an uncountable Polish space, then $\mathbf{\Sigma}^0_\alpha \neq \mathbf{\Sigma}^\beta$ for any $\alpha \neq \beta$.

If $U \subseteq Y \times X$ and $a \in Y$, we let $U_a = \{b \in X : (a, b) \in U\}$. In this way we think of U as a family of subsets of X parameterized by Y.

Definition 2.36 We say that $U \subset Y \times X$ is *universal*-$\mathbf{\Sigma}^0_\alpha$ if $U \in \mathbf{\Sigma}^0_\alpha(Y \times X)$, and if $A \in \mathbf{\Sigma}^0_\alpha(X)$, then $A = U_a$ for some $a \in A$.

We define $\mathbf{\Pi}^0_\alpha$ universal sets similarly.

Lemma 2.37 *If X is a separable metric space, then for all $1 < \alpha < \omega_1$ there is a $\mathbf{\Sigma}^0_\alpha$-universal set $U_\alpha \subseteq \mathcal{C} \times X$ and a $\mathbf{\Pi}^0_\alpha$-universal set $V_\alpha \subseteq \mathcal{C} \times X$.*

Proof Let $W_0, W_1, \ldots$ be a basis of open sets for X.

Let $U_1 = \{(f, x) : \exists n \in \mathbb{N}\ f(n) = 1 \wedge x \in W_n\}$. Since

$$U_1 = \bigcup_{n \in \mathbb{N}} \{(f, x) : f(n) = 1 \wedge x \in W_n\},$$

U_1 is open. If $A \subseteq X$ is open, define $f \in \mathcal{C}$ such that $f(n) = 1$ if and only if $W_n \subseteq A$. Then $x \in A$ if and only if $(f, x) \in U_0$. Thus U_1 is $\mathbf{\Sigma}^0_1$-universal.

If U_α is $\mathbf{\Sigma}^0_\alpha$-universal, then $V_\alpha = (\mathcal{C} \times X) \setminus U_\alpha$ is $\mathbf{\Pi}^0_\alpha$-universal.

Suppose V_β is $\mathbf{\Pi}^0_\alpha$-universal for all $\beta < \alpha$. Choose $\beta_0 \leq \beta_1 \leq \ldots$ a sequence of ordinals such that $\sup\{\beta_n + 1 : n = 0, \ldots\} = \alpha$.

Let $\widehat{U_\alpha} = \{(n, f, x) \in \mathbb{N} \times \mathcal{C} \times X : (f, x) \in V_{\beta_n}\}$. Then $\widehat{U_\alpha} = \bigcup_{n\in\mathbb{N}} \{n\} \times V_{\beta_n}$ is $\mathbf{\Sigma}^0_\alpha$.

Using the natural homeomorphism between $\mathcal{C}$ and $\mathcal{C}^{\mathbb{N}}$ we can identify every $f \in \mathcal{C}$ with $(f_0, f_1, f_2, \ldots) \in \mathcal{C}^{\mathbb{N}}$. Then $U_\alpha = \{(f, x) : \exists n\ (n, f_n, x) \in \widehat{U_\alpha}\}$ is $\mathbf{\Sigma}^0_\alpha$. If $A \subseteq X$ is $\mathbf{\Sigma}^0_\alpha$, then there are $B_0, B_1, \ldots$ such that B_n is $\mathbf{\Pi}^0_{\beta_n}$ and $A = \bigcup B_n$. Choose f_n such that $x \in B_n$ if and only if $(f_n, x) \in V_{\beta_n}$ and choose f coding $(f_0, f_1, f_2, \ldots)$. Then $(f, x) \in U_\alpha$ if and only if $x \in A$.

We can now prove that in an uncountable Polish space the Borel hierarchy is a strict hierarchy of ω_1-levels.

Corollary 2.38 *i)* $\mathbf{\Sigma}^0_\alpha(\mathcal{C}) \neq \mathbf{\Sigma}^0_\alpha(\mathcal{C})$ *for any* $\alpha < \omega_1$.

ii) If X is an uncountable Polish space, then $\mathbf{\Sigma}^0_\alpha(X) \neq \mathbf{\Sigma}^0_\alpha(X)$ for any $\alpha < \omega_1$. In particular, $\mathbf{\Sigma}^0_\alpha(X)$ is a proper subset of $\mathbf{\Delta}^0_{\alpha+1}(X)$.

Proof

i) Let $U \subseteq \mathcal{C} \times \mathcal{C}$ be $\mathbf{\Sigma}^0_\alpha$-universal. Let $Y = \{x : (x, x) \notin U_\alpha\}$. Clearly $Y \in \mathbf{\Pi}^0_\alpha$. If $Y \in \mathbf{\Sigma}^0_\alpha$, then there is $y \in X$ such that $x \in Y$ if and only if $(y, x) \in U_\alpha$. Then

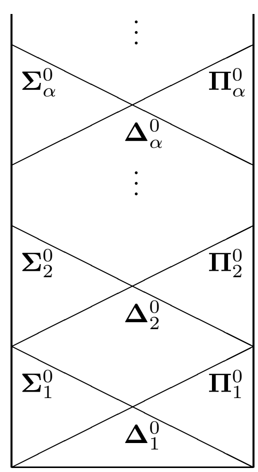
$$y \in Y \Leftrightarrow (y, y) \in U_\alpha \Leftrightarrow y \notin Y,$$

a contradiction.

ii) Suppose X is an uncountable Polish space. Then X contains a perfect set P homeomorphic to $\mathcal{C}$. If $\mathbf{\Sigma}^0_\alpha(X) = \mathbf{\Pi}^0_\alpha(X)$, then, by 2.8, $\mathbf{\Sigma}^0_\alpha(P) = \mathbf{\Sigma}^0_\alpha(P)$, contradicting i).

Since $\mathbf{\Sigma}^0_\alpha(X) \neq \mathbf{\Pi}^0_\alpha(X)$, there is $A \in \mathbf{\Pi}^0_\alpha(X) \setminus \mathbf{\Sigma}^0_\alpha(X)$.

This gives us the following picture of the Borel Hierarchy.



Definition 2.39 Let X and Y be Polish spaces, with $A \subseteq X$ and $B \subseteq Y$. We say that A is *Wadge-reducible* to B if there is a continuous $f : X \to Y$ such that $x \in A$ if and only if $f(x) \in B$ for all $x \in A$.

We write $A \leq_w B$ if A is Wadge-reducible to B.

Note that if $A \leq_w B$, then $X \setminus A \leq_w Y \setminus B$.

Example 2.40 *If $A \subseteq \mathcal{N}$ is open, then $A \leq_w \{x \in \mathcal{N} : \exists n\ x(n) = 1\}$.*

Let $A = \bigcup_n U_n$ where each U_n is a basic clopen set. Let $f : \mathcal{N} \to \mathcal{N}$ such that

$$(f(x))(n) = \begin{cases} 1 & \text{if } x \in U_n \\ 0 & \text{otherwise.} \end{cases}.$$

If $\sigma \in 2^{<\omega}$,

$$f^{-1}(N_\sigma) = \bigcap_{\sigma(n)=1} U_n \cap \bigcap_{\sigma(n)=0} X \setminus U_n$$

a clopen set. Thus f is continuous. Clearly $x \in A$ if and only if $\exists n\ f(n) = 1$.

Definition 2.41 For $\Gamma = \mathbf{\Sigma}^0_\alpha$ or $\mathbf{\Pi}^0_\alpha$ we say that $A \subseteq X$ is Γ-complete if $A \in \Gamma(X)$ and if $B \in \Gamma(X)$, then $B \leq_w A$.

Thus $\{x \in \mathcal{N} : \exists n\ x(n) = 1\}$ is $\mathbf{\Sigma}^0_1$-complete.

Let Γ be $\mathbf{\Sigma}^0_\alpha$ or $\mathbf{\Pi}^0_\alpha$. If $A \in \Gamma$ and $B \leq_w A$, then, by 2.6 iv), $B \in \Gamma$. Let $\check{\Gamma} = \{X \setminus A : A \in \Gamma$. Note that if A is Γ-complete, then $X \setminus A$ is $\check{\Gamma}$-complete.

Lemma 2.42 *If $A \subseteq X$ is Γ-complete, then $A \notin \check{\Gamma}$.*

Proof We know there is $B \in \Gamma \setminus \check{\Gamma}$. Since A is complete $B \leq_w A$. If $A \in \check{\Gamma}$, then $B \in \check{\Gamma}$, a contradiction.

Example 2.43 *The set $A = \{x \in \mathcal{N} : \exists n \forall m > n\ x(m) = 0\}$ of eventually zero sequences in $\mathbf{\Sigma}^0_2$-complete.*

Suppose B is $\mathbf{\Sigma}^0_2$. Suppose $B = \bigcup_n F_n$ where F_n is closed. Let $T_n \subseteq \mathbb{N}^{<\omega}$ be a tree such that $F_n = [T_n]$. We give a program to "compute" $(f(x))(m)$ from x and the sequence of trees run the following program until it outputs $(f(x))(m)$.

1) Let $n = i = s = 0$.

2) If $x|s \in T_i$, output $(f(x))(n) = 0$, set $s \leftarrow s + 1$; otherwise, output $(f(x))(n) = 1$, set $i \leftarrow i + 1$

3) Let $n = n + 1$

4) Go to 2)

The sequence $f(x)$ is an infinite sequence of 0s and 1s. If $x \in B$, there is a least m such that $x \in [T_m]$ we will eventually see that $x \notin [T_m]$ for $j \leq m$, and increment i until we get to m. Once i reaches m we will always have $x|s \in T_i$, so we will only output 0s. In this case $f(x) \in A$.

If $x \notin B$, then for each i we will at some point realize that $x \notin [T_i]$ and output a 1. Thus $f(x) \notin A$. Thus $B \leq_w A$.

It follows that A is $\mathbf{\Sigma}^0_2$ but not $\mathbf{\Pi}^0_2$.

Exercise 2.44 Show that $\{f \in \mathcal{N} : f \text{ is onto}\}$ is $\mathbf{\Pi}^0_2$-complete.

Exercise 2.45 Let $D = \{x \in \mathcal{N} : \lim\limits_{n\to\infty} x(n) = \infty\}$. Show that D is $\mathbf{\Pi}^0_3$-complete.

The Baire Property

We begin by recalling some basic ideas from analysis. Let X be a Polish space.

Definition 2.46 We say that $A \subseteq X$ is *nowhere dense set* , if whenever $U \subseteq X$ is open and nonempty, there is a nonempty open $V \subseteq U$ such that $A \cap U = \emptyset$.

We say that $B \subseteq X$ is *meager* if X is a countable union of nowhere dense sets.

Exercise 2.47 Show that Cantor's "middle third" set is nowhere dense.

Exercise 2.48 We say that $I \subseteq \mathcal{P}(X)$ is an *ideal* if i) $\emptyset \in I$, ii) if $A \in I$ and $B \subseteq A$, then $B \in I$, and iii) if $A, B \in I$, then $A \cup B \in I$. We say that an ideal I is a σ-ideal if $\bigcup_n A_n \in I$, whenever $A_0, A_1, \ldots \in I$.

a) Show that the nowhere dense sets form an ideal.
b) Show that the meager sets form a σ-ideal.

Exercise 2.49 a) Show that if A is nowhere dense, then $\overline{A}$ is nowhere dense.
b) Show that every meager set is contained in a meager F_σ-set.

Exercise 2.50 Show that if U is open, then $\overline{U} \setminus U$ is nowhere dense.

Lemma 2.51 *If F is closed, then $F \setminus \mathrm{intr}(F)$ is nowhere dense, where $\mathrm{intr}(F)$ is the interior of F.*

Proof Let V be open such that $V \cap (F \setminus \mathrm{intr}(F))$ is nonempty. Since $V \not\subseteq \mathrm{intr}(F)$, $V \setminus F$ is nonempty open and $(V \setminus F) \cap (F \setminus \mathrm{intr}(F)) = \emptyset$.

The next result is a classical fact from analysis.

Theorem 2.52 (Baire Category Theorem) *If X is a Polish space, then X is nonmeager.*

Proof Suppose $X = \bigcup_n A_n$ where each A_n is nowhere dense. Choose open sets $U_0 \subset U_1 \subset U_2 \subset \ldots$ such that $\overline{U_{n+1}} \subseteq U_n$, diam $(U_n) < \frac{1}{n+1}$ and $U_n \cap A_n = \emptyset$. Choose $x_n \in U_n$. Then (x_n) is a Cauchy sequence. Suppose $x_n \to x$. Then $x \in \bigcap_n U_n$. Thus $x \in X \setminus \bigcup_n A_n$ a contradiction.

Definition 2.53 For $A, B \subseteq X$ we define $A =_* B$ if and only if $A \triangle B$ is meager, where $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

Exercise 2.54 a) Show that $=_*$ is an equivalence relation.
b) Show that if $A =_* B$, then $X \setminus A =_* X \setminus B$.
c) Show that if $A_n =_* B_n$ for $n = 0, 1, \ldots$, then $\bigcup_n A_n =_* \bigcup_n B_n$.

Definition 2.55 Let $A \subseteq X$. We say that A has the *Baire property* if there is an open set U such that $A =_* U$.

Let BP$= \{A \subseteq X : A$ has the Baire property$\}$. Clearly every open set has the Baire property. If F is closed, then by 2.51 $F \setminus \mathrm{intr}(F)$ is nowhere dense. Thus every closed set has the Baire property. In fact BP is closed complements.

Lemma 2.56 *If A has the Baire property, then $X \setminus A$ has the Baire property.*

Proof Thus if U is open and $A =_* U$, then, by 2.54 b)

$$X \setminus A =_* X \setminus U =_* \mathrm{intr}(X \setminus U).$$

The later equality holding by 2.51.

By 2.54c), BP is closed under countable unions.

Corollary 2.57 BP *is a σ-algebra containing the Borel sets.*

In fact BP can contain many non Borel sets. For example if $F \subseteq \mathbb{R}$ is Cantor's "middle third" set then any $A \subseteq F$ is nowhere dense and in BP. Thus $|BP| = 2^{2^{\aleph_0}}$, while there are only $2^{\aleph_0}$ Borel sets.

Exercise 2.58 Show that if A has the Baire property, then there is a G_δ-set B and an F_σ-set C such that $B \subseteq A \subseteq F$ and $F \setminus G$ is meager

Exercise 2.59 Show that if A has the property of Baire, then either A is meager or there is σ such that $N_\sigma \setminus A$ is meager.

Do all sets have the Baire property?

Exercise 2.60 Use the axiom of choice to construct a subset of $\mathbb{R}$ without the Baire Property.

3 Effective Descriptive Set Theory: The Arithmetic Hierarchy

Several ideas from logic have has a big impact on descriptive set theory. In this chapter we will start to study the influence of recursion theory on descriptive set theory. At first it will look like an interesting but perhaps shallow analogy, but as we continue to develop these ideas in §and apply them in §8 we will eventually see that they lead to important new results that do not have classical proofs. Moschovakis [14], Kechris' portion of [12] and Mansfield and Weitkamp [10] are excellent reference for effective descriptive set theory.

Recursion Theory Review

We recall some of the basic ideas we will need from recursion theory. We assume that the reader has some intuitive idea what a "computer program" is. This could be a very precise notion like a Turing machine or an informal notion like Pascal program.

Definition 3.1 A partial function $f : \mathbb{N} \to \mathbb{N}$ is *partial recursive* if there is a computer program P such that P halts on input n if and only if $n \in \mathrm{dom}\ (f)$ and if P halts on input n, then the output is $f(n)$. We say that a set $A \subseteq \mathbb{N}$ is *recursive* if and only if its characteristic function is recursive.

We can code computer programs by integers so that each integer codes a program. Let P_e be the machine coded by e. Let ϕ_e be the partial recursive function computed by e. We write $\phi_e(n)\downarrow_s$ if P_e halts on input n by stage s and $\phi_e(n)\downarrow$ if P_e halts on input n at some stage. Our enumeration has the following features.

Fact 3.2 *i)* [universal function] *The function* $(e,n)\mapsto\phi_e(n)$ *is partial recursive.*
ii) The set $\{(e,n,s):\phi_e(n)\downarrow_s\}$ *is recursive.*
iii) [halting problem] *The set* $\{(e,n):\phi_e(n)\downarrow\}$ *is not recursive.*
iv) [parameterization lemma] *If* $F:\mathbb{N}^2\to\mathbb{N}$ *is partial recursive, there is a total recursive* $d:\mathbb{N}\to\mathbb{N}$ *such that*

$$\phi_{d(x)}(y)=F(x,y)$$

for all x,y.

Definition 3.3 We say that $A\subseteq\mathbb{N}$ is *recursively enumerable* if there is a partial recursive function $f:\mathbb{N}\to\mathbb{N}$ such that A is the image of f.

Fact 3.4 *The following are equivalent*
a) A *is recursively enumerable*
b) A *is the domain of a partial recursive function.*
c) $A=\emptyset$ *or* A *is the image of a total recursive function.*
d) there is a recursive B *such that* $A=\{n:\exists m\ (n,m)\in B\}$.

Fact 3.5 *a) If* A *and* B *are recursively enumerable, then so are* $A\cup B$ *and* $A\cap B$.
b) If $A\subseteq\mathbb{N}\times\mathbb{N}$ *is recursively enumerable so is* $\{n:\exists m\ (n,m)\in A\}$.
c) If A *is recursively enumerable and* $f:\mathbb{N}\to\mathbb{N}$ *is total recursive, then* $f^{-1}(A)$ *is recursively enumerable.*

Exercise 3.6 If you haven't seen them before prove the statements in the last Fact.

A program with oracle $x\in\mathcal{N}$ is a computer program which, in addition to the usual steps, is allowed at any stage to ask the value of $x(n)$.

We say that f is *partial recursive in* x if there is a program with oracle x computing f and say that $A\subseteq\mathbb{N}$ is recursive in x if the characteristic function of A is recursive in x. The facts above relativize to oracle computations. We write $\phi_e^x(n)$ for the value of the partial recursive function in x with oracle program P_e on input n. One additional fact is useful.

Fact 3.7 (Use Principle) *If* $\phi_e^x(n)\downarrow$, *then there is* m *such that if* $x|m=y|m$, *then* $\phi_e^y(n)=\phi_e^x(n)$.

Proof The computation of P_e with oracle x on input n makes only finitely many queries about x. Choose m greater than all of the queries made.

We may also consider programs with finitely many oracles.

For $x, y \in \mathcal{N}$ we say x is *Turing-reducible* to y and write $x \leq_T y$ if x is recursive in y. There is another useful reducibility that is the analog of Wadge-reducibility for $\mathbb{N}$.

Definition 3.8 We say A is *many-one reducible* to B if there is a total recursive f such that $n \in A$ if and only if $f(n) \in B$ for all $n \in \mathbb{N}$. We write $A \leq_m B$ if A is many-one reducible to B.

Clearly if $A \leq_m B$, then $A \leq_T B$.

There is one subtle fact that will eventually play a key role.

Theorem 3.9 (Recursion Theorem) *If $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is total recursive, there is an e such that $\phi_e(n) = \phi_{f(n)}$ for all n.*

Proof Let
$$F(x, y) = \begin{cases} \phi_{\phi_x(x)}(y) & \text{if } \phi_x(x) \downarrow \\ \uparrow & \text{if } \phi_x(x) \uparrow \end{cases}.$$
By the Parameterization Lemma, there is a total recursive d such that
$$\phi_{d(x)}(y) = F(x, y).$$
Let $\psi = f \circ d$. There is m such that $\psi = \phi_m$. Since ψ is total $\phi_{d(m)} = \phi_{\phi_m(m)}$. Let $e = d(m)$. Then
$$\phi_e = \phi_{d(m)} = \phi_{\phi_m(m)} = \phi_{\psi(m)} = \phi_{f(d(m))} = \phi_{f(e)}.$$

For those of you who haven't seen this before here is a sample of the many applications of the Recursion Theorem. Let
$$g(e, n) = \begin{cases} 1 & \text{if } e = n \\ 0 & \text{if } e \neq n \\ . \end{cases}$$
By the Parameterization Lemma, there is a total recursive f such that $\phi_{f(e)}(n) = g(e, n)$. By the Recursion Theorem there is an e such that $\phi_e(n) = 1$ if $n = e$ and $\phi_e(n) = 0$ if $n \neq e$. So this function "recognizes" its own code. The Recursion Theorem will be very useful in §7.

Computable Functions on $\mathcal{N}$

There is also a notion of computable function $f : \mathcal{N} \to \mathcal{N}$.

Definition 3.10 We say that $f : \mathcal{N} \to \mathcal{N}$ is *computable* if there is an oracle program P such that if $x \in \mathcal{N}$ and P is run with oracle x on input n, then P halts and outputs $(f(x))(n)$.

We say that $f : \mathcal{N} \to \mathcal{N}$ is computable from z if there is a two oracle program P such that if $x \in \mathcal{N}$ and P is run with oracles z and x on input n, then M halts and outputs $(f(x))(n)$.

Lemma 3.11 *$f : \mathcal{N} \to \mathcal{N}$ is continuous if and only if there is $z \in \mathcal{N}$ such that f is computable from z.*

Proof

($\Leftarrow$) Let P be the oracle program computing f from z. Suppose $f(x) = y$. By the Use Principle, for any m there is an n such that if $x|n = z|n$, then $f(z)|m = y|m$. Thus $N_{x|n} \subseteq f^{-1}N_{y|m}$ and f is continuous.

($\Rightarrow$) Let $X = \{(\tau, \sigma) : f^{-1}(N_\sigma) \subseteq N_\tau\}$. Since f is continuous, for all $x \in \mathcal{N}$ if $f(x) = y$, then for all n there is an m such that $(x|m, y|n) \in X$.

We claim that f is computable from X. Suppose we given oracles X and x and input n. We start searching X until we find $(\tau, \sigma) \in X$ such that $\tau \subset x$ and $|\sigma| > n$. Then $(f(x))(n) = \sigma(n)$.

The Arithmetic Hierarchy

For the next few sections we will restrict our attention to Polish spaces $X = \mathbb{N}^k \times \mathcal{N}^l$ where $k, l \geq 0$.[1] Of course if $k > 0$ and $l = 0$, X is homeomorphic to $\mathbb{N}$ while if $l > 0$, then X is homeomorphic to $\mathcal{N}$. (In [14] this theory is worked out for "recursively presented Polish spaces".)

Let $X = \mathbb{N}^k \times \mathcal{N}^l$. Let $S_X = \{(m_1, \ldots, m_k, \sigma_1, \ldots, \sigma_l) : m_i, \ldots, m_k \in \mathbb{N}, \sigma_1, \ldots, \sigma_l \in \mathbb{N}^{<\omega}\}$. For $\sigma = (m_i, \ldots, m_k, \sigma_1, \ldots, \sigma_l) \in S_X$, let

$$N_\sigma = \{(n_1, \ldots, n_k, f_1, \ldots, f_l) \in X : n_i = m_i \text{ if } i \leq k \text{ and } f_i \supset \sigma_i \text{ if } i = 1 \leq l\}.$$

Then $\{N_\sigma : \sigma \in S_X\}$ is a clopen basis for the topology on X. Of course S_X is a countable set and there is a recursive bijection $i \mapsto \sigma_i$ between $\mathbb{N}$ and S_X. Thus we can identify S_X with $\mathbb{N}$ and talk about things like recursive subsets of S_X and partial recursive functions $f : \mathbb{N} \to S_X$.

Definition 3.12 We say that $A \subseteq X$ is Σ^0_1 if there is a partial recursive $f : \mathbb{N} \to S_X$ such that $A = \bigcup_n N_{f(n)}$.

Note that here we are using a "lightface" Σ^0_1 rather than the "boldface" $\boldsymbol{\Sigma}^0_1$ that denotes the open subsets of X. Of course every Σ^0_1 set is open, but there are only countably many partial recursive $f : \mathbb{N} \to S_X$ thus there are only countably many Σ^0_1 sets. Thus $\Sigma^0_1 \subset \boldsymbol{\Sigma}^0_1$. Relativizing these notions we get all open sets.

Definition 3.13 If $x \in \mathcal{N}$ we say that $A \subseteq X$ is $\Sigma^0_1(x)$ if there is $f : \mathbb{N} \to S_X$ partial recursive in x such that $A = \bigcup_n N_{f(n)}$.

Lemma 3.14 $\boldsymbol{\Sigma}^0_1 = \bigcup_{x \in \mathcal{N}} \Sigma^0_1(x)$.

[1] At times we might also consider spaces $\mathbb{N}^k \times \mathcal{N}^l \times \mathcal{C}^m$ but everything is similar

We will tend to prove things only for Σ^0_1 sets. The relativization to $\Sigma^0_1(x)$ sets is usually straightforward.

For the two interesting examples $X = \mathbb{N}$ and $X = \mathcal{N}$ we get slightly more informative characterizations.

Lemma 3.15 *i) $A \subseteq X$ is Σ^0_1 if and only if there is a recursively enumerable $W \subseteq S_X$ such that $A = \bigcup_{\eta \in W} N_\eta$. In particular $A \subseteq \mathbb{N}$ is Σ^0_1 if and only if A is recursively enumerable.*

ii) $A \subseteq \mathcal{N}$ is Σ^0_1 if and only if there is a recursive $S \subseteq \mathbb{N}^{<\omega}$ such that $A = \bigcup_{\sigma \in S} N_\sigma$.

Proof

i) This is clear since the recursively enumerable sets are exactly the images of partial recursive functions.

ii) In this case $S_X = \mathbb{N}^{<\omega}$. Clearly if $S \subseteq \mathbb{N}^{<\omega}$ is recursive, there is $f : \mathbb{N} \to \mathbb{N}^{<\omega}$ partial recursive with image S and $\bigcup_{\sigma \in \mathbb{N}^{<\omega}} N_\sigma$ is Σ^0_1.

Suppose $A = \bigcup_n N_{f(n)}$ where f is partial recursive let $S = \{\sigma :$ there is $n \le |\sigma|$, the computation of $f(n)$ halts by stage $|\sigma|$ and $f(n) \subseteq \sigma$. It is easy to see that S is recursive. If $\sigma \in S$, then there is an n such that $f(n) \subseteq \sigma$, then $N_\sigma \subseteq N_{f(n)} \subseteq A$. On the hand if f halts on input n, there is $m \ge n, |f(n)|$ such that f halts by stage m. If $\tau \supset \sigma$ and $|\tau| \ge m$, then $\tau \in S$. Thus

$$\bigcup_{\sigma \in S} N_\sigma \supset \bigcup_{\tau \supset f(n), |\tau| = m} N_\tau = N_\sigma.$$

It follows that $A = \bigcup_{\sigma \in S} N_\sigma$.

We have natural analogs of the finite levels of the Borel hierarchy.

Definition 3.16 Let $X = \mathbb{N}^k \times \mathcal{N}^l$. We say that $A \subseteq X$ is Π^0_n if and only if $X \setminus A$ is Σ^0_n.

We say that $A \subseteq X$ is Σ^0_{n+1} if and only if there is $B \subseteq \mathbb{N} \times X$ in Π^0_n such that

$$x \in A \text{ if and only if } \exists n \ (n, x) \in B.$$

We say that A is Δ^0_n if it is both Σ^0_n and Π^0_n.

We say that $A \subseteq X$ is *arithmetic* if $A \in \Delta^0_n$ for some n.

Lemma 3.17 *$A \subseteq \mathcal{N}$ is Π^0_1 if and only if there is a recursive tree $T \subseteq \mathbb{N}^{<\omega}$ such that $A = [T]$.*

Proof If S is a recursive tree such that $X \setminus A = \bigcup_{\sigma \in S} N_\sigma$, let

$$T = \{\sigma \in \mathbb{N}^{<\omega} : \forall m \le |\sigma| \ \sigma|m \notin S\}.$$

Then T is recursive and, as in 1.14 $A = [T]$.

The next exercises shows that it is not always possible to find a recursive pruned tree T with $A = [T]$.

Exercise 3.18 a) Show that if T is a recursive pruned tree, then the left-most path through T is recursive.

b) Let $T = \{\sigma \in \mathbb{N}^{<\omega} :$ if $e < |\sigma|$ and $\phi_e(e)$ halts by stage $|\sigma|$, then $\phi_e(e)$ halts by stage $\sigma(e)\}$. Show that T is a recursive tree. Suppose $f \in [T]$. Show that $\phi_e(e)$ halts if and only if it halts by stage $f(e)$. Conclude that there are no recursive paths through T and, using a), that there is no recursive pruned subtree of T.

We show that Σ^0_n and Π^0_n have closure properties analogous to those proved in 2.6. The definition of computable function made in 3.10 makes sense for maps $f : X \to Y$ where both X and Y are of the form $\mathbb{N}^k \times \mathcal{N}^l$

Lemma 3.19 *i) Σ^0_n is closed under finite unions, finite intersections, and computable inverse images.*

ii) If $A \subseteq \mathbb{N} \times X \in \Sigma^0_n$, then $\{x \in X : \exists n\ (n, x) \in A\} \in \Sigma^0_n$.

iii) If $f : X \to \mathbb{N}$ is computable and $A \subseteq \mathbb{N} \times X$ is $\boldsymbol{\Sigma}^0_n$ then $\{x \in X : \forall m < f(x)\ (m, x) \in A\} \in \Sigma^0_n$.

iv) Similarly Π^0_n is closed under union, intersection, computable inverse images, $\forall n$ and $\exists n < f(x)$.

v) $\Sigma^0_n \subseteq \Delta^0_{n+1}$.

Proof We prove this for Σ^0_1 and leave the induction as an exercise.

i) Suppose W_0 and W_1 are recursively enumerable subsets of S_X and $A_i = \bigcup_{\eta \in W_i} N_\eta$. Replacing W_i by the recursively enumerable set $\{\nu : \exists \eta \in W_i\ \eta \subseteq \nu\}$ if necessary we may assume that if $\nu \in W_i$ and $\eta \supset \nu$, then $\eta \in W_i$. Then

$$A_0 \cup A_1 = \bigcup_{\eta \in W_0 \cup W_1} N_\eta$$

and

$$A_0 \cap A_1 = \bigcup_{\eta \in W_0 \cap W_1} N_\eta$$

and $W_0 \cup W_1$ and $W_0 \cap W_1$ are recursively enumerable. Thus $A_0 \cup A_1$ and $A_0 \cap A_1$ are Σ^0_1.

If $f : X \to Y$ is computable with program P_e, let

$$G = \{(\eta, \nu) \in S_X \times S_Y : x \in N_\eta \Rightarrow f(x) \in N_\nu\}.$$

Then $(\eta, \nu) \in G$ if and only if for all $m < |\nu|$ the program P_e using oracle η halts on input m and outputs $\nu(m)$.[2] Thus G is recursively enumerable. Suppose $A = \bigcup_{\nu \in W} N_\nu$ where W is recursively enumerable, let $V = \{\eta : \exists \nu\ (\nu \in W \wedge (\eta, \nu) \in G\}$. Then V is recursively enumerable and $f^{-1}(A) = \bigcup_{\eta \in V} N_\eta$.

ii) Suppose $A \subseteq \mathbb{N} \times X$ is Σ^0_1. There is a recursively enumerable $W \subseteq S_{\mathbb{N} \times X}$ such that $A = \bigcup_{\eta \in W} N_\eta$. Let $V = \{\nu \in S_X : \exists n\ (n, \nu) \in W\}$. Then V is

[2] We assume that if the computation makes any queries about numbers $i \geq |\eta|$, then the computation does not halt.

recursively enumerable and

$$\{x : \exists n\ (n,x) \in A\} = \bigcup_{\nu \in V} N_\nu.$$

iii) Suppose A and W are as in ii) and $f : X \to \mathbb{N}$ is computable by program P_e. Let $V = \{\nu \in S_X : \exists k\ P_e$ with oracle ν halts outputting k and $(m,\nu) \in W$ for all $m \leq k\}$. Then V is recursively enumerable and

$$\{x : \forall m < f(x)\ (m,x) \in A\} = \bigcup_{\nu \in V} N_\nu\}.$$

Exercise 3.20 Give the inductive steps to complete the proof of 3.19

We can make two interesting observations about universal sets. We state these results for Σ_n^0, but the analogous results hold for Π_n^0.

Proposition 3.21 *i) There is $U \subseteq \mathcal{N} \times X$ a Σ_n^0-set that is $\boldsymbol{\Sigma}_n^0$-universal.*
ii) There is $V \subseteq \mathbb{N} \times X$ a Σ_n^0-set that is Σ_n^0-universal.

Proof
i) Indeed the universal sets produced in 2.37 are Σ_n^0. Fix $f : \mathbb{N} \to S_X$ a recursive bijection. The set $U_1 = \{(x,y) : \exists n\ (x(n) = 1 \wedge y \in N_{f(n)})\}$ is Σ_1^0 and $\boldsymbol{\Sigma}_n^0$-universal.
If $U_n^* \subseteq \mathcal{N} \times \mathbb{N} \times X$ is Σ_n^0 and $\boldsymbol{\Sigma}_n^0$-universal for $\mathbb{N} \times X$, then

$$U_{n+1} = \{(x,y) : \exists n\ (x,n,y) \notin U_n^*\}$$

is Σ_{n+1}^0 and $\boldsymbol{\Sigma}_n^0$-universal.
ii) Let $V_1 = \{(n,x) : \exists m\ (\phi_n(m) \downarrow \wedge x \in N_{\phi_n(m)})\}$. Let $g : \mathbb{N} \times \mathbb{N} \to S_{\mathbb{N} \times X}$ be partial recursive such that $g(n,m) = (n, \phi_n(m))$, then

$$V_1 = \bigcup_{n,m} N_{g(n,m)}$$

is Σ_1^0 and $\boldsymbol{\Sigma}_1^0$-universal.
An induction as in i) extends this to all levels of the arithmetic hierarchy.

Corollary 3.22 *For any X there is $A \subseteq X$ such that A is Σ_n^0 but not Δ_n^0.*

Proof For $X = \mathbb{N}^k \times \mathcal{N}^l$ where $l > 0$ this follows as in §2 using 3.21 i). Suppose $U \subseteq \mathbb{N} \times \mathbb{N}$ is Σ_n^0 and universal Σ_n^0. Let $A = \{m : (m,m) \notin U\}$. If $U \in \Delta_n^0$, then $A \in \Sigma_n^0$ and $A = \{m : (i,m) \in U\}$ for some i. Then

$$i \in A \Leftrightarrow (i,i) \notin U \Leftrightarrow i \notin A,$$

a contradiction. Thus $U \in \Sigma_n^0 \setminus \Delta_n^0$. Using a recursive bijection $f : \mathbb{N}^2 \to \mathbb{N}^l$, shows that for all $X = N^l$, there is a Σ_n^0-set that is not Δ_n^0.

Let Γ be Π_n^0, Σ_n^0 or Δ_n^0 for $i = 0$ or 1. If $A, B \subseteq \mathbb{N}$, $B \in \Gamma$ and $A \leq_m B$, then $A \in \Gamma$.

We say that $A \subseteq \mathbb{N}$ is Γ-complete if $A \in \Gamma$ and $B \leq_w A$ for all $B \subseteq \mathbb{N}$ in Γ. Here are some well known examples from recursion theory.

Fact 3.23 *i)* $\{e : \mathrm{dom}\ (\phi_e) \neq \emptyset\}$ *is* Σ^0_1*-complete.*

ii) $\{e : \phi_e$ *is total*$\}$ *is* Π^0_2*-complete.*

iii) $\{e : \mathrm{dom}\ (\phi_e)$ *is infinite*$\}$ *is* Π^0_2*-complete.*

iv) If $U \subseteq \mathbb{N} \times \mathbb{N}$ *is* Γ*-universal, then* U *is* Γ*-complete.*

Exercise 3.24 Prove the statements in the last fact.

4 Analytic Sets

When studying the Borel sets, it is often useful to consider a larger class of sets.

Definition 4.1 Let X be a Polish space. We say that $A \subseteq X$ is *analytic* if there is a Polish space Y, $f : Y \to X$ continuous and $B \in \mathcal{B}(Y)$ such that $A = f(B)$ the image of B. We let $\mathbf{\Sigma}^1_1(X)$ denote the collection of all analytic subsets of X.

If no confusion arises we write $\mathbf{\Sigma}^1_1$ rather than $\mathbf{\Sigma}^1_1(X)$. The following lemma gives several alternative characterizations of analytic sets. In general if $X \times Y$ is a product space, we let π_X and π_Y denote the projections onto X and Y.

Lemma 4.2 *Let* X *be a Polish space. The following are equivalent:*

i) $A \in \mathbf{\Sigma}^1_1$*;*

ii) either $A = \emptyset$ *or there is* $f : \mathcal{N} \to X$ *continuous such that* $f(\mathcal{N}) = X$*;*

iii) there is $B \subseteq \mathcal{N} \times X$ *closed, such that* $A = \pi_X(B)$

iv) there is a Polish space Y *and* $B \subseteq Y \times X$ *Borel such that* $A = \pi_X(B)$*.*

Proof

i)$\Rightarrow$ ii) Since $A \in \mathbf{\Sigma}^1_1$, there is a Polish space Y, $f : Y \to X$ continuous, and $B \subseteq Y$ Borel such that $f(B) = A$. By 2.26 there is a continuous $g : \mathcal{N} \to Y$, such that $g(\mathcal{N}) = B$. Then $f \circ g$ is a continuous map from $\mathcal{N}$ to X whose image is A.

ii)$\Rightarrow$ iii) Suppose $f : \mathcal{N} \to X$ is continuous and $f(\mathcal{N}) = A$. Let $G(f) \subseteq \mathcal{N} \times X$ be the graph of f. Then $G(f)$ is closed and $\pi_X(G(f)) = A$.

iii)$\Rightarrow$ iv) and iv) $\Rightarrow$ i) are clear.

Exercise 4.3 a) Show that if $A \in \mathbf{\Sigma}^1_1(X)$, then there is $B \in \mathbf{\Pi}^0_2(\mathcal{C} \times X)$ such that $\pi(B) = A$.

b) Show that this cannot be improved to $\mathbf{\Sigma}^0_2(\mathcal{C} \times X)$.

Definition 4.4 Let X be a Polish space. We say that $A \subseteq X$ is $\mathbf{\Pi}^1_1(X)$ if $X \setminus A$ is $\mathbf{\Sigma}^1_1$. $\mathbf{\Pi}^1_1(X)$-sets are also called *coanalytic*.

We say $A \subseteq X$ is in $\mathbf{\Delta}^1_1(X)$ if $A \in \mathbf{\Pi}^1_1(X) \cap \mathbf{\Sigma}^1_1(X)$.

By 2.26 every Borel set is analytic. Since the complement of a Borel set is analytic, every Borel set is $\mathbf{\Delta}^1_1$. We will show below that there are analytic sets that are not Borel. First we prove some of the basic closure properties of analytic and coanalytic sets.

Lemma 4.5 *i) $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$ are closed under countable unions and intersections.*
ii) If $f : X \to Y$ is Borel measurable, and $A \in \mathbf{\Sigma}^1_1(X)$, then $f(A) \in \mathbf{\Sigma}^1_1(Y)$.
iii) $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$ are closed under Borel measurable inverse images.

Proof

i) Suppose $A_0, A_1, \ldots \in \mathbf{\Sigma}^1_1(X)$. Let $C_i \in \mathbf{\Pi}^0_1(\mathcal{N} \times X)$ such that $\pi(C_i) = A_i$. Then $\bigcup A_i = \pi(\bigcup C_i) \in \mathbf{\Sigma}^1_1$.

Using the homeomorphism between $\mathcal{N}$ and $\mathcal{N}^{\mathbb{N}}$ we can view $f \in \mathcal{N}$ as coding $(f_0, f_1, \ldots) \in \mathcal{N}^{\mathbb{N}}$. Let

$$C = \{(f, x) \in \mathcal{N} \times X : \forall n\ (f_n, x) \in C_i\}.$$

Then C is a closed subset of $\mathcal{N} \times X$ and $\bigcap A_i = \pi(C) \in \mathbf{\Sigma}^1_1$.

Since $\mathbf{\Sigma}^1_1$ is closed under countable unions and intersections, so is $\mathbf{\Pi}^1_1$.

ii) If $f : X \to Y$ is Borel measurable, then, by 2.3 iii), $G(f)$, the graph of f, is a Borel subset of $X \times Y$. Suppose $A \in \mathbf{\Sigma}^1_1(X)$, there is a closed $C \subseteq \mathcal{N} \times X$ such that $A = \pi_X(C)$. Let

$$D = \{(x, y, z) \in \mathcal{N} \times X \times Y : (x, y) \in C \text{ and } (y, z) \in G(f)\}.$$

Then D is Borel and $f(A) = \pi_Y(D) \in \mathbf{\Sigma}^1_1(Y)$.

iii) Suppose $f : X \to Y$ is Borel measurable, and $G(f)$ is the graph of X. If $A \in \mathbf{\Sigma}^1_1(Y)$, then

$$f^{-1}(A) = \{x : \exists y\ y \in A \wedge (x, y) \in G(f)\}.$$

If $C \subseteq \mathcal{N} \times Y$ is closed such that $\pi_Y(X) = A$, then

$$f^{-1}(A) = \pi_X(\{(x, y, z) \in \mathcal{N} \times X \times Y : (x, z) \in C \text{ and } (y, z) \in G(f)\}) \in \mathbf{\Sigma}^1_1(X).$$

If $A \in \mathbf{\Pi}^1_1(Y)$, then $f^{-1}(A) = X \setminus f^{-1}(Y \setminus A)$, so $f^{-1}(A) \in \mathbf{\Pi}^1_1(X)$.

Intuitively, $\mathbf{\Sigma}^1_1(X)$ is closed under $\wedge$, $\vee$, $\exists n \in \mathbb{N}$, $\forall n \in \mathbb{N}$ and $\exists x \in X$. While $\mathbf{\Pi}^1_1(X)$ is closed under $\wedge$, $\vee$, $\exists n \in \mathbb{N}$, $\forall n \in \mathbb{N}$ and $\forall x \in X$.

Examples

Example 4.6 *Let LO be as in 2.12 and let $WO = \{x \in LO : x \text{ is a well order}\}$. Then WO is $\mathbf{\Pi}^1_1$.*

A linear order is a well order if and only if there are no infinite descending chains. Thus

$$WO = \{x \in LO : \forall f : \mathbb{N} \to \mathbb{N}\ \exists n\ x(f(n), f(n+1)) = 0\}.$$

Example 4.7 *Let Tr be $Tr_{\mathbb{N}}$ as in 2.13, codes for subsets of $\mathbb{N}^{<\omega}$ that are trees. We say that $T \in Tr$ is* well-founded *if $[T] = \emptyset$. Let* WF $= \{x \in Tr : x$ *is well founded*$\}$. *Then* WF *is* $\mathbf{\Pi}^1_1$.

Let $s : (\mathbb{N}^{<\omega})^2 \to 2$ be such that $s(\sigma, \tau) = 1$ if and only if $\sigma \subset \tau$. Then

$$\mathrm{WF} = \{x \in Tr : \forall f \ (\forall n \ x(f(n)) = 1 \to \exists n \ s(f(n), f(n+1)) = 0\}.$$

Example 4.8 $\mathrm{Ism}(\mathcal{L}) = \{(\mathcal{M}_0, \mathcal{M}_1) \in \mathrm{Mod}(\mathcal{L}) : \mathcal{M}_0 \cong \mathcal{M}_1\}$ *is a* $\mathbf{\Sigma}^1_1$*-equivalence relation.*

For notational simplicity, we consider only the case $\mathcal{L} = \{R\}$ where R is a binary relation symbol, the general case is similar. Then

$$\mathcal{M}_0 \cong \mathcal{M}_1 \Leftrightarrow \exists f : \mathbb{N} \to \mathbb{N} \ \forall i, j \in \mathbb{N} \ R^{\mathcal{M}_0}(i, j) \leftrightarrow R^{\mathcal{M}_1}(f(i), f(j)).$$

Thus $\mathrm{Ism}(\mathcal{L})$ is a $\mathbf{\Sigma}^1_1$-equivalence relation.
By 2.17 every $\mathrm{Ism}(\mathcal{L})$ equivalence class is Borel.

Example 4.9 $D(\mathbb{I}) = \{f \in C(\mathbb{I}) : f$ *is differentiable*$\}$ *is* $\mathbf{\Pi}^1_1(C(\mathbb{I}))$.

We saw in 2.18 that $E = \{(f, x) \in C(\mathbb{I}) \times \mathbb{I} : f$ is differentiable at $x\}$ is $\mathbf{\Pi}^0_3$. Thus $D = \{f : \forall x \in I \ (f, x) \in E\}$ is $\mathbf{\Pi}^1_1$.

Example 4.10 *If X is a Polish space, then* $\{A \in K(X) : A$ *is uncountable*$\}$ *is* $\mathbf{\Sigma}^1_1$.

An uncountable closed set contains a perfect set. In 2.19 and 2.21 we saw that $\{(A, B) : A \subseteq B\}$ and $\{P \in K(X) : P$ is perfect$\}$ are Borel. But A is uncountable if and only if

$$\exists P \in K(X) \ (P \text{ is perfect and } P \subseteq A).$$

Thus the set of uncountable closed sets is $\mathbf{\Sigma}^1_1$.

Universal $\mathbf{\Sigma}^1_1$-sets

We now start to prove there is an analytic set that is not Borel.

Lemma 4.11 *There is* $U \in \mathbf{\Sigma}^1_1(\mathcal{C} \times X)$ *that is* $\mathbf{\Sigma}^1_1$*-universal.*

Proof By 2.37 there is a closed set $V \subseteq \mathcal{C} \times \mathcal{N} \times X$ such that if $A \subseteq \mathcal{N} \times X$ is closed then $A = V_a$ for some $a \in \mathcal{C}$. Let $U = \{(a, x) \in \mathcal{N} \times \mathbb{N} : \exists f \in \mathcal{N} \ (a, f, x) \in V\}$. Since U is the projection of a closed set, $U \in \mathbf{\Sigma}^1_1(X)$. If $A \in \mathbf{\Sigma}^1_1(X)$, there is a closed $B \subseteq \mathcal{N} \times X$ such that $\pi(B) = A$. There is $a \in \mathcal{C}$ such that $V_a = B$. Then $U_a = A$.

Corollary 4.12 *If X is an uncountable Polish space, then there is* $A \in \mathbf{\Sigma}^1_1(X)$ *that is not* $\mathbf{\Pi}^1_1$ *and hence not Borel.*

Proof We first prove this for $X = \mathcal{C}$. Let $A = \{a \in \mathcal{N} : (a, a) \notin U\}$. If $U \in \mathbf{\Pi}^1_1$, then $A \in \mathbf{\Sigma}^1_1$ and $A = U_a$ for some $a \in \mathcal{C}$. But then

$$a \in A \Leftrightarrow a \in U_a \Leftrightarrow (a, a) \in U \Leftrightarrow a \notin A,$$

a contradiction.

If X is an uncountable Polish space, there is $f : \mathcal{C} \to X$ be a Borel isomorphism. Let $A \subseteq \mathcal{C}$ be $\mathbf{\Sigma}^1_1$ but not $\mathbf{\Pi}^1_1$. By 4.5 ii) $f(A) \in \mathbf{\Sigma}^1_1(X)$. By 4.5 iii), if $f(A) \in \mathbf{\Pi}^1_1(X)$, then $A = f^{-1}(f(A)) \in \mathbf{\Pi}^1_1$, a contradiction.

The last proof illustrates an important point. Since any two uncountable Polish spaces are Borel isomorphic, if we are trying to prove something about Borel and analytic sets, it is often enough to prove it for one particular Polish space (like $\mathcal{N}$ or $\mathcal{C}$) and then deduce it for all Polish spaces.

The Separation Theorem

We noticed that every Borel set is $\mathbf{\Delta}^1_1$. We will show that the converse is true, proving Souslin's Theorem that $\mathcal{B}(X) = \mathbf{\Delta}^1_1(X)$.

Theorem 4.13 ($\mathbf{\Sigma}^1_1$-Separation Theorem) *Suppose X is a Polish space and $A, B \subseteq X$ are disjoint analytic sets. There is a Borel set $C \subseteq X$ such that $A \subseteq C$ and $B \cap C = \emptyset$.*

Proof Let $f, g : \mathcal{N} \to X$ be continuous functions such that $f(\mathcal{N}) = A$ and $g(\mathcal{N}) = B$. For $\sigma \in \mathbb{N}^{<\omega}$ let $A_\sigma = f(N_\sigma)$ and $B_\sigma = g(N_\sigma)$.

Let $\sigma \in \mathbb{N}^{<\omega}$. Suppose for all i, j, there is $C_{i,j}$ Borel such that $A_{\sigma^\frown i} \subseteq C_{i,j}$ and $B_{\sigma^\frown j} \cap C_{i,j} = \emptyset$. Then $C = \bigcup_i \bigcap_j C_{i,j}$ is a Borel set separating A_σ and B_σ.

Suppose A and B can not be separated by a Borel set. Using the observation above, we can inductively define $\emptyset = \sigma_0 \subset \sigma_1 \subset \ldots$ and $\emptyset = \tau_0 \subset \tau_1 \subset \ldots$ in $\mathbb{N}^{<\omega}$ such that $|\sigma_i| = |\tau_i| = i$ and A_{σ_i} and B_{τ_i} can not be separated by a Borel set. Let $x = \bigcup \sigma_i$ and $y = \bigcup \tau_i$. Then $f(x) \in A$ and $g(x) \in B$. Let U and V be disjoint open sets such that $f(x) \in U$ and $g(x) \in V$. By continuity, there is an n such that $f(N_{x|n}) \subseteq U$ and $g(N_{y|n}) \subseteq V$, but $x|n = \sigma_n$, $y|n = \tau_n$, so, $A_{\sigma_n} = f(N_{x|n})$ and $B_{\tau_n} = g(N_{y|n})$ are Borel separable, a contradiction.

Corollary 4.14 *If $A \in \mathbf{\Delta}^1_1(X)$, then A is Borel. Thus $\mathcal{B}(X) = \mathbf{\Delta}^1_1(X)$.*

Proof Since A and $X \setminus A$ are disjoint $\mathbf{\Sigma}^1_1$-sets. They are separated by a Borel set. The only set separating A and $X \setminus A$ is A.

We can now prove the converse to 2.3 iii).

Corollary 4.15 *Suppose X and Y are Polish spaces and $f : X \to Y$. The following are equivalent:*

i) f is Borel measurable;

ii) the graph of f is a Borel subset of $X \times Y$;

iii) the graph of f is an analytic subset of $X \times Y$.

Proof i) $\Rightarrow$ ii) is 2.3 iii), and ii) $\Rightarrow$ iii) is obvious.

Suppose the graph of f is analytic and $A \in \mathcal{B}(Y)$, then

$$\begin{aligned} x \in f^{-1}(A) &\Leftrightarrow \exists y\ (y \in A \wedge f(x) = y) \\ &\Leftrightarrow \forall y\ (f(x) = y \rightarrow y \in A). \end{aligned}$$

These are $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$, definitions of $f^{-1}(A)$, so $f^{-1}(A)$ is Borel.

The Perfect Set Theorem

Next we will show that the Perfect Set Theorem is still true for analytic sets. We need one preparatory lemma.

Lemma 4.16 *Let X be a Polish space. If $A \subseteq X$ is uncountable, then there are disjoint open sets U_0 and U_1 such that $U_i \cap A$ is uncountable for $i = 0, 1$.*

Proof Suppose not. For each $n > 1$ let $U_{n,0}, U_{n,1}, \dots$ be an open cover of X by open balls of radius $\frac{1}{n}$. Choose $x(n)$ such that $U_{n,x(n)} \cap A$ is uncountable. Let $A_n = A \setminus \overline{U}_{n,x(n)}$. If A_n is uncountable we can find an open set V disjoint from $U_{n,x(n)}$ such that $V \cap A$ is uncountable, thus A_n is uncountable. But

$$A \setminus (\bigcup_n A_n) = \bigcap_n \overline{U_n},$$

since diam $(U_n) \rightarrow 0$, there is at most one element in $A \setminus (\bigcup_n A_n)$ and hence A is countable, a contradiction.

Theorem 4.17 (Perfect Set Theorem of $\mathbf{\Sigma}^1_1$-sets) *If X is a Polish space and $A \subseteq X$ is analytic and uncountable, then X contains a perfect set.*

Proof Let $f : \mathcal{N} \rightarrow X$ be a continuous function with $f(\mathcal{N}) = X$. We build a function $\sigma \mapsto \tau_\sigma$ from $2^{<\omega}$ to $\mathbb{N}^{<\omega}$ such that:

i) $\tau_\emptyset = \emptyset$;

ii) if $\sigma_0 \subset \sigma_1$, then $\tau_{\sigma_0} \subset \tau_{\sigma_1}$;

iii) $f(N_{\tau_\sigma})$ is uncountable for all $\sigma \in 2^{<\omega}$.

iv) $f(N_{\tau_{\sigma\frown 0}}) \cap f(N_{\tau_{\sigma\frown 1}}) = \emptyset$ for all σ.

Let $\tau_\emptyset = \emptyset$. Suppose we have τ_σ such that $f(N_{\tau_\sigma})$ is uncountable.

Claim Suppose $V \subseteq \mathcal{N}$ is open and $f(V)$ is uncountable. There are W_1 and W_2 disjoint open subsets of V such that $f(W_i)$ is uncountable for $i = 0, 1$.

By the preceeding lemma there are U_0 and U_1 disjoint open subsets of X such that $f(V) \cap U_i$ is uncountable for $i = 0, 1$. Let $W_i = f^{-1}(U_i) \cap V$. Clearly $W_0 \cap W_1 = \emptyset$ and $f(W_i)$ is uncountable.

Thus given τ_σ with $f(N_{\tau_\sigma})$ uncountable, there are $\tau_{\sigma\frown 0}, \tau_{\sigma\frown 1} \supset \tau_\sigma$ such that $N_{\tau_{\sigma\frown 0}} \cap N_{\tau_{\sigma\frown 1}} = \emptyset$ and $f(N_{\tau_{\sigma\frown i}})$ is uncountable.

Let $g : \mathcal{C} \rightarrow \mathcal{N}$ by $g(x) = \bigcup_n \tau_{x|n}$. Then g is continuous and $f \circ g : \mathcal{C} \rightarrow X$ is one-to-one. Since $\mathcal{C}$ is compact, $f \circ g(\mathcal{C})$ is closed and uncountable and hence contains a perfect set.

A little extra care would allow us to conclude that $f \circ g(\mathcal{C})$ is perfect. We will give a second proof of this theorem in §6.

The Perfect Set Theorem for $\mathbf{\Sigma}^1_1$-sets is the best result of this kind that we can prove in ZFC. The next natural question is whether the Perfect Set Theorem is true for $\mathbf{\Pi}^1_1$-sets. Unfortunately, this depends on set theoretic assumptions. Let $\mathbb{L}$ be Gödel's constructible universe. If $x \subseteq \mathbb{N}$, let $\mathbb{L}(x)$ be the sets constructible from x.

Theorem 4.18 (Mansfield, Solovay) *The following are equivalent:*
i) every uncountable $\mathbf{\Pi}^1_1$-set contains a perfect subset;
ii) for all $x \subseteq \mathbb{N}$, $\aleph_1^{\mathbb{L}(x)}$ is countable;
iii) $\aleph_1^{\mathbb{V}}$ is an inaccessible cardinal in $\mathbb{L}$.

In particular if $\mathbb{V} = \mathbb{L}$, then there is $\mathbf{\Pi}^1_1$-set with no perfect subset. For proofs see [9] §41.

Baire Property

We will show that analytic sets have the Baire Property.

We begin by giving another normal form for $\mathbf{\Sigma}^1_1$-sets. Let X be a Polish space.

Definition 4.19 Suppose $B_\sigma \subseteq X$ for all $\sigma \in \mathbb{N}^{<\omega}$. We define

$$\mathcal{A}(\{B_\sigma\}) = \bigcup_{f \in \mathcal{N}} \bigcap_{n \in \mathbb{N}} B_\sigma.$$

We call $\mathcal{A}$ the *Souslin operation.*.

Exercise 4.20 a) Suppose B_σ is closed for all $\sigma \in \mathbb{N}^{<\omega}$. Show that $\mathcal{A}(\{B_\sigma\})$ is $\mathbf{\Sigma}^1_1$.

b) Suppose A is analytic and $f : \mathcal{N} \to X$ is continuous such that $f(\mathcal{N}) = A$. Let $B_\sigma = \overline{f(N_\sigma)}$. Show that $f(x) = \bigcap_n B_{x|n}$ and $A = \mathcal{A}(\{B_\sigma\})$.

Thus A is analytic if and only if $A = \mathcal{A}(\{B_\sigma\})$ for some family of closed sets.

We will assume that X is a topological space with a countable basis. Although we are primarily interested in Polish spaces. We will use the next two lemmas in a more general setting in §8.

Lemma 4.21 *Suppose $A \subseteq X$. There is $B \supseteq A$ such that B has the Baire Property and if $B' \supseteq A$ has the Baire Property, then $B \setminus B'$ is meager.*

Proof Let $U_0, U_1, \ldots$ be a basis for the topology on X. Let

$$A_1 = \{x \in X : \forall i \text{ if } x \in U_i, \text{ then } U_i \cap A \text{ is not meager}\}.$$

If $x \notin A_1$, there is an i such that $x \in U_i$ and $U_i \cap A$ is meager. If $y \in U_i$, then, since $U_i \cap A$ is meager, $y \notin A_1$. Thus $U_i \cap A_1 = \emptyset$ and A_1 is closed.

Then

$$A \setminus A_1 = \bigcup\{A \cap U_i : A \cap U_i \text{ is meager}\},$$

a meager set. Let $B = A \cup A_1$. Since $B = A_1 \cup (A \setminus A_1)$ is the union of a closed set and a meager set, B has the Baire Property.

Suppose $B' \supseteq A$ has Baire Property, then $C = B \setminus B'$ has the Baire Property. We must show that C is meager. If not then $U_i \setminus C$ is meager for some i. Then $U_i \cap A$ is meager. Since $U_i \cap C \neq \emptyset$ and $C \subseteq A_1$, there is $x \in U_i$ such that $x \in A_1$. Thus $U_i \cap A$ is not meager, a contradiction.

Theorem 4.22 *Suppose A_σ has the Baire Property for all $\sigma \in \mathbb{N}^{<\omega}$. Then $A = \mathcal{A}(\{A_\sigma\})$ has the Baire Property.*

Proof We may assume that $A_\sigma \subseteq A_\tau$ for $\tau \subseteq \sigma$; otherwise replace A_σ by $\bigcap_{\tau \subseteq \sigma} A_\tau$. For $\sigma \in \mathbb{N}^{<\omega}$ let

$$A^\sigma = \bigcup_{x \supset \sigma} \bigcap_{n \in \mathbb{N}} A_{x|n} \subseteq A_\sigma.$$

By 4.21 there is $B^\sigma \supseteq A^\sigma$ with the Baire Property such that if $B \supseteq A^\sigma$ has the Baire Property, then $B^\sigma \setminus B$ is meager. We may assume that $B^\sigma \subseteq A_\sigma$ and that $B^\sigma \subseteq B^\tau$ for $\tau \subseteq \sigma$, replacing B^σ by $\bigcap_{\tau \subseteq \sigma} B^\tau$ if necessary.

Let $C_\sigma = B^\sigma \setminus \bigcup_n B^{\sigma\frown n}$. Since $A^\sigma \subseteq \bigcup_n B^{\sigma\frown n}$, our choice of B^σs insures that C_σ is meager. Let $C = \bigcup_{\sigma \in \mathbb{N}^{<\omega}} C_\sigma$. Clearly C is meager.

Claim $B^\emptyset \setminus C \subseteq A$.

Let $b \in B^\emptyset \setminus C$. Since $b \notin C_\emptyset$, there is $x(0)$, such that $b \in B^{x(0)}$. Suppose we have $x(0), \ldots, x(n)$ such that $b \in B^{x(0),\ldots,x(n)}$. Since $b \notin C_{x(0),\ldots,x(n)}$, there is $x(n+1)$ such that $b \in B^{x(0),\ldots,x(n+1)}$. Continuing this way we construct $x \in \mathcal{N}$ such that

$$b \in \bigcap_n B^{x|n} \subseteq \bigcap_n A_{x|n} \subseteq A.$$

Thus $b \in A$.

Then $B^\emptyset \setminus A \subseteq C$. Hence $B^\emptyset \setminus A$ is meager. In particular $B^\sigma \setminus A$, and hence A, have the Baire Property.

Corollary 4.23 *If X is a Polish space, then every $\mathbf{\Sigma}^1_1$-set has the Baire Property.*

In §8 it will be useful to notice that our proof that the collection of sets with the Baire Property is closed under the Souslin operator works in any topological space with a countable basis (not just Polish spaces).

Exercise 4.24 a) Prove that if $A \subseteq \mathbb{R}^n$, then there is $B \supseteq A$ such that if $B' \subseteq A$ is Lebesgue measurable, then $B \setminus B'$ has measure zero. [Hint: If $\mu^*(A) < \infty$, choose $B \supseteq A$ measurable with $\mu(B) = \mu^*(A)$, where μ^* is Lebesgue outer measure. Otherwise write A as a union of sets with finite outer measure.]

b) Modify the proof of 4.22, using a), to prove the following theorem.

Theorem 4.25 *The collection of Lebesgue measurable subsets of $\mathbb{R}^n$ is closed under the Souslin operator. In particular every $\boldsymbol{\Sigma}^1_1$-set is Lebesgue measurable.*

We give another restatement.

Definition 4.26 Let C be the smallest σ-algebra containing the Borel sets and closed under the Souslin operator $\mathcal{A}$.

We have proved that every C-measurable set is Lebesgue measurable.

The Projective Hierarchy

The analytic and coanalytic sets form the first level of another hierarchy of subsets of a Polish space.

Definition 4.27 Let X be a Polish space. We say that $A \subseteq X$ is $\boldsymbol{\Sigma}^1_{n+1}(X)$ if there is $B \in \boldsymbol{\Pi}^1_n(X \times X)$ such that $A = \pi_X(B)$. We say that $A \subseteq X$ is $\boldsymbol{\Pi}^1_n(X)$ if $X \subseteq A$ is $\boldsymbol{\Sigma}^1_n$. We let $\boldsymbol{\Delta}^1_n(X) = \boldsymbol{\Sigma}^1_n(X) \cap \boldsymbol{\Pi}^1_n(X)$.

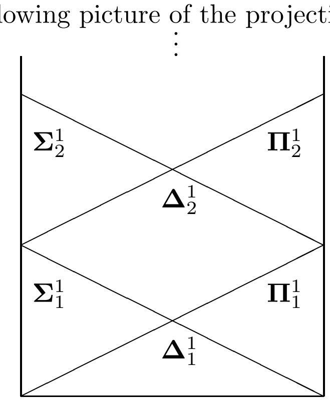
We say that $A \subset X$ is *projective* if it is $\boldsymbol{\Sigma}^1_n$ for some n.

Exercise 4.28 a) Prove that $\boldsymbol{\Sigma}^1_n$ is closed under countable unions, countable intersections, Borel measurable inverse images, and Borel measurable inverse images.

b) Show that for each n there is $U_n \in \boldsymbol{\Sigma}^1_n(\mathcal{C} \times \mathcal{C})$ that is $\boldsymbol{\Sigma}^1_n$-universal.

c) Show that if X is an uncountable Polish space, then for all n there is a $\boldsymbol{\Sigma}^1_n$ set that is not $\boldsymbol{\Pi}^1_n$.

Thus we have the following picture of the projective hierarchy.



We give several examples of higher level projective sets.

Example 4.29 *Let $MV = \{f \in C(\mathbb{I}) : f$ satisfies the mean value theorem$\}$. Then MV is $\boldsymbol{\Pi}^1_2$.*

$f \in MV$ if and only if

$$\forall x \forall y \left(x < y \rightarrow \exists z \left(f \text{ is differentiable at } z \text{ and } f'(z) = \frac{f(x) - f(y)}{x - y}\right)\right)$$

Two interesting example arise when studying $\mathbb{L}$. See [9] §41.

Example 4.30 *The set* $\{x \in \mathcal{N} : x \in \mathbb{L}\}$ *is* $\mathbf{\Sigma}^1_2$.

The idea of the proof is that there is a sentence Θ such that $ZF \vdash \Theta$ and $\mathbb{L}$ is absolute for transitive models of Θ. Using the Mostowski collapse

$x \in \mathbb{L}$ if and only if there is $\mathcal{M}$ a countable well-founded model of $\Theta + \mathbb{V} = \mathbb{L}$ with $x \in \mathcal{M}$.

This is a $\mathbf{\Sigma}^1_2$ definition of $\mathcal{N} \cap \mathbb{L}$.

Example 4.31 *If* $\mathbb{V} = \mathbb{L}$*, then there is a* $\mathbf{\Delta}^1_2$ *well-order of* $\mathcal{N}$ *of order type* ω_1*.*

Indeed the canonical well-ordering of $\mathbb{L}$ is $\mathbf{\Delta}^1_2$.

These example can be used to show that projective sets need not have nice regularity properties. We will use Fubini's Theorem, that a measurable $A \subseteq \mathbb{R}^2$ has positive measure if and only if $\{a : \{b : (a, b) \in A\}$ has positive measure$\}$ has positive measure.

Lemma 4.32 *If* R *is a well-ordering of* $\mathbb{R}$ *of order type* ω_1*, then* R *is not Lebesgue measurable.*

Proof Suppose R is Lebesgue measurable. We consider For each $x \in [0, 1]$, $R_x = \{y : yRx\}$ is countable and hence measure zero. By Fubini's Theorem, R has measure zero. We now exchange the order of integration. For each x, $R^x = \{y : xRy\}$ has a measure zero complement. Thus, by Fubini's Theorem, R has a measure zero complement, a contradiction.

Corollary 4.33 *If* $\mathbb{V} = \mathbb{L}$*, then there is a nonmeasurable* $\mathbf{\Delta}^1_2$*-set.*

Fubini's Theorem has a category analog.

Let X be a Polish space and suppose $A \subseteq X \times X$. For $x \in X$ let $A_x = \{y \in X : (x, y) \in A\}$.

Theorem 4.34 (Kuratowski-Ulam Theorem) *If* A *has the Baire property, then* A *is nonmeager if and only if* $\{a \in X : A_a$ *is nonmeager*$\}$ *is nonmeager.*

For a proof see [6] 8.41.

Exercise 4.35 Show that if $\mathbb{V} = \mathbb{L}$, then there is a $\mathbf{\Delta}^1_2$-set that does not have the Baire property.

On the other hand we (probably) can't prove in ZFC that there is a projective set where any of the regularity properties above fail.

Theorem 4.36 (Solovay) *If ZFC +* $\exists\kappa$ κ *inaccessible is consistent then so is ZFC + every uncountable projective set contains a perfect subset + every projective set is Lebesgue measurable and has the property of Baire.*

See [9] §42 for Solovay's proof. The same arguments also show that if ZFC + $\exists\kappa$ κ inaccessible is consistent, then so is ZF+every set of reals is Lebesgue measurable and has the Baire property.

By 4.18 the consistency of an inaccessible is needed to prove the consistency of every uncountable $\mathbf{\Pi}^1_1$-set containing a perfect subset. Shelah has shown that it is also needed to prove the consistency of all projective sets being measurable, but not to prove the consistency of all projective sets having the Baire property.

The Effective Projective Hierarchy

We also have effective analogs of the projective point classes. Let $X = \mathbb{N}^k \times \mathcal{N}^l$ for some $k, l \in \mathbb{N}$.

Definition 4.37 We say that $A \subseteq X$ is Σ^1_1 if there is a $B \subseteq \mathcal{N} \times X$ such that $B \in \Pi^0_1$ and $A = \{x : \exists y\ (y,x) \in B\}$.

We say $A \subseteq X$ is Π^1_n if $X \setminus A$ is Σ^1_n and we say that $A \subseteq X$ is Σ^1_{n+1} if there is a $B \subseteq \mathcal{N} \times X$ with $B \in \Pi^1_n$ such that $A = \{x : \exists y\ (y,x) \in B\}$.

We say A is Δ^1_n if it is both Σ^1_n and Π^1_n.

The next theorem summarizes a number of important properties of the classes Σ^1_n and Π^1_n. We leave the proofs as exercises.

Theorem 4.38 *i) The classes Σ^1_n and Π^1_n are closed under union, intersection, $\exists n \in \mathbb{N}$, $\forall n \in N$ and computable inverse images.*

ii) If $A \subseteq X \times \mathcal{N}$ is arithmetic, then $\{x : \exists y (x,y) \in A\}$ is Σ^1_n.

iii) There is $U \subseteq \mathcal{N} \times X$ a Σ^1_n-set that is $\boldsymbol{\Sigma}^1_n$-universal.

iv) There is $V \subseteq \mathbb{N} \times X$ a Σ^1_n-set that is Σ^1_n-universal.

v) $\Sigma^1_n \subset \Delta^1_{n+1}$, but $\Sigma^1_n \neq \Delta^1_n$.

vi) The set WF *of wellfounded trees is Π^1_1.*

Exercise 4.39 Prove 4.38.

5 Coanalytic Sets

In this section we will study the structure of $\boldsymbol{\Pi}^1_1$-sets. Because any two uncountable Polish spaces are Borel isomorphic, it will be no loss of generality to restrict our attention to the Baire space.

We begin by giving a normal form for $\boldsymbol{\Sigma}^1_1$ and $\boldsymbol{\Pi}^1_1$-sets.

Definition 5.1 We say that $T \subseteq \mathbb{N}^{<\omega} \times \mathbb{N}^{<\omega}$ is a tree if:

i) $|\sigma| = |\tau|$ for all $(\sigma, \tau) \in T$;

ii) if $(\sigma, \tau) \in T$ and $n \leq |\sigma|$, then $(\sigma|n, \tau|n) \in T$.

If $f, g \in \mathcal{N}$ we say that (f,g) is a path through T if $(f|n, g|n) \in T$ for all $n \in \mathbb{N}$. We let $[T]$ be the set of all paths through T.

Exercise 5.2 Show that $F \subseteq \mathcal{N} \times \mathcal{N}$ is closed if and only if there is a tree $T \subseteq \mathbb{N}^{<\omega} \times \mathbb{N}^{<\omega}$ such that $F = [T]$.

If $A \subseteq \mathcal{N}$ is $\boldsymbol{\Sigma}^1_1$, then there is $C \subseteq \mathcal{N} \times \mathcal{N}$ closed such that $A = \{x : \exists y\ (x,y) \in A\}$. Let T be a tree such that $[T] = C$. For each $x \in \mathcal{N}$, let

$$T(x) = \{\sigma \in \mathbb{N}^{<\omega} : (x|n, \sigma) \in T \text{ for some } n\}.$$

Then $T(x)$ is a tree and $x \in A$ if and only if there is $f \in [T(x)]$. Let Tr be as in 2.13.[3] Then $x \mapsto T(x)$ is a continuous map from $\mathcal{N}$ to Tr and $x \in A$ if and only if $T(x)$ is ill-founded (i.e., not well-founded).

[3] Although Tr was defined to be a subset of $2^{\mathbb{N}^{<\omega}}$, we will (by suitable coding) view it as a subset of $\mathcal{C}$ or $\mathcal{N}$.

Let IF be the set of ill-founded trees. We saw in §3 that WF the set of well-founded trees is $\mathbf{\Pi}^1_1$, thus IF is $\mathbf{\Sigma}^1_1$. We have just argued that IF is $\mathbf{\Sigma}^1_1$-complete. In particular, IF is $\mathbf{\Sigma}^1_1$ but not Borel.

If A is $\mathbf{\Pi}^1_1$, then $(\mathcal{N} \setminus A) \leq_w$ IF, so $A \leq_w$ WF. We summarize these results in the following theorem.

Theorem 5.3 (Normal form for $\mathbf{\Pi}^1_1$) *If $A \subseteq \mathcal{N}$ is $\mathbf{\Pi}^1_1$, then there is a tree $T \subseteq \mathbb{N}^{<\omega} \times \mathbb{N}^{<\omega}$ such that $x \in A$ if and only if $T(x) \in \mathrm{WF}$.*

Corollary 5.4 *If $A \in \mathbf{\Pi}^1_1$, then there is $f : \mathcal{N} \to Tr$ continuous such that $A = f^{-1}(\mathrm{WF})$. In otherwords,* WF *is $\mathbf{\Pi}^1_1$-complete. In particular* WF *is not $\mathbf{\Sigma}^1_1$.*

Ranks of Trees

Our analysis of $\mathbf{\Pi}^1_1$-sets starts with an analysis of trees.

Definition 5.5 If T is a tree, let $T' = \{\sigma \in T : \exists \tau \in T\ \sigma \subset \tau\}$ be the subtree of nonterminal nodes of T. For $\alpha < \omega_1$ define T^α as follows:

i) $T^0 = T$;

ii) $T^{\alpha+1} = (T^\alpha)'$;

iii) $T^\alpha = \bigcap_{\beta<\alpha} T^\beta$ for α a limit.

Lemma 5.6 *For any tree T there is an $\alpha < \omega_1$, such that $T^\alpha = T^\beta$ for all $\beta > \alpha$.*

Proof Clearly if $T^\alpha = T^{\alpha+1}$, then $T^\alpha = T^\beta$ for all $\beta > \alpha$. If $T^\alpha \neq T^{\alpha+1}$ there is $\sigma_\alpha \in T^\alpha \setminus T^{\alpha+1}$. If $\alpha \neq \beta$, then $\sigma_\alpha \neq \sigma_\beta$. Thus, since $\mathbb{N}^{<\omega}$ is countable, there is $\alpha < \omega_1$ such that $T^\alpha = T^{\alpha+1}$.

Definition 5.7 If $T \subseteq \mathbb{N}^{<\omega}$ is a tree, we define a rank $\rho_T : T \to \omega_1 \cup \{\infty\}$, by

i) if $\sigma \in T^\alpha \setminus T^{\alpha+1}$, then $\rho_T(\sigma) = \alpha$.

ii) if $\sigma \in \bigcap_{\alpha<\omega_1} T^\alpha$, then $\rho_T(\sigma) = \infty$.

If $T = \emptyset$ let $\rho(T) = -1$, otherwise let $\rho(T) = \sup\{\rho_T(\sigma) : \sigma \in T\}$.

In general, $\rho(T)$ is the least α such that $T^{\alpha+1} = \emptyset$, if there is such an α. When no confusion arises we drop the subscript T.

Lemma 5.8 *Let $T \subseteq \mathbb{N}^{<\omega}$ be a tree and let ρ be the rank of T.*

i) Suppose $\sigma, \tau \in T$ and $\sigma \subset \tau$. If $\rho(\tau) = \infty$, then $\rho(\sigma) = \infty$. If $\rho(\tau) < \infty$, then $\rho(\sigma) > \rho(\tau)$.

ii) If $\sigma \in T$ and $\rho(\tau) < \infty$ for all $\tau \in T$ with $\sigma \subset \tau$, then

$$\rho(\sigma) = \sup\{\rho(\sigma\widehat{\ }i) + 1 : \sigma\widehat{\ }i \in T\}.$$

iii) If $\rho(\sigma) = \infty$, then there is $f \in [T] \cap N_\sigma$;

iv) T is well-founded if and only if $\rho(T) < \infty$.

Proof

i) If $\sigma \subset \tau$ and $\tau \in T^\alpha$, then $\sigma \in T^{\alpha+1}$.

ii) By i) $\rho(\sigma) \geq \sup\{\rho(\sigma\widehat{\ }i) + 1 : \sigma\widehat{\ }i \in T\}$. On the other hand if $\alpha = \sup\{\rho(\sigma\widehat{\ }i) + 1 : \sigma\widehat{\ }i \in T\}$, then σ has no extensions in T^α so $\sigma \notin T^{\alpha+1}$. Thus $\rho(\sigma) = \sup\{\rho(\sigma\widehat{\ }i) + 1 : \sigma\widehat{\ }i \in T\}$.

iii) If $\rho(\sigma) = \infty$, then by ii) there is $\sigma\widehat{\ }i \in T$ with $\rho(\sigma\widehat{\ }i) = \infty$. This allows us to inductively build $f \in [T]$ with $f \supset \sigma$.

iv) Clear from i)–iii).

Exercise 5.9 a) Show that if $T \neq \emptyset$, then $\rho(T) = \rho_T(\emptyset)$.

b) Show that for all $\alpha < \omega_1$ there is a tree T with $\rho(T) = \alpha$.

c) If T is a tree and $\sigma \in \mathbb{N}^{<\omega}$, let $T_\sigma = \{\tau \in \mathbb{N}^{<\omega} : \sigma\widehat{\ }\tau \in T\}$. Show that T_σ is a tree and if $T \neq \emptyset$, then $\rho(T) = \sup_{n\in\mathbb{N}}(\rho(T_{\langle n\rangle}) + 1)$.

Definition 5.10 If S and T are trees we say that $f : S \to T$ is *order-preserving* if $f(\sigma) \subset f(\tau)$ for all $\sigma, \tau \in T$ with $\sigma \subset \tau$.

Lemma 5.11 *i) If $S, T \subseteq \mathbb{N}^{<\omega}$ are trees, then $\rho(S) \leq \rho(T)$ if and only if there is an order preserving $f : S \to T$.*

ii) If T is a well-founded tree, then $\rho(S) < \rho(T)$ if and only if $S = \emptyset$ and $T \neq \emptyset$ or there is $n \in \mathbb{N}$ and $f : S \to T_{\langle n\rangle}$ order preserving.

Proof

i) If $f : S \to T$ is order preserving, then an easy induction on rank shows that $\rho_S(\sigma) \leq \rho_T(f(\sigma))$ for all $\sigma \in S$. Thus $\rho(S) \leq \rho(T)$. For the converse, we build f by induction such that $\rho_S(\sigma) \leq \rho_T(f(\sigma))$ for all $\sigma \in S$. Let $f(\emptyset) = \emptyset$. Suppose we have defined $f(\sigma)$ with $\rho_S(\sigma) \leq \rho_T(f(\sigma))$ and $\sigma\widehat{\ }i \in T$. By 5.8 ii) and iii) there is $j \in \mathbb{N}$ such that $f(\sigma)\widehat{\ }j \in T$ and $\rho_T(f(\sigma)\widehat{\ }j) \geq \rho_S(\sigma\widehat{\ }i)$. Let $f(\sigma\widehat{\ }i) = f(\sigma)\widehat{\ }j$.

ii) If $f : S \to T_{\langle n\rangle}$ is order preserving. Then

$$\rho(S) \leq \rho(T_{\langle n\rangle}) = \rho_T(\langle n\rangle) < \rho_T(\emptyset) = \rho(T).$$

Conversely, if $\rho(S) < \rho(T)$ and $S \neq \emptyset$, then there is $n \in \mathbb{N}$ such that $\rho(S) \leq \rho(T_{\langle n\rangle})$ and by i) there is an order-preserving $f : S \to T_{\langle n\rangle}$.

If $\alpha < \omega_1$, let $\mathrm{WF}_\alpha = \{T \in \mathrm{WF} : \rho(T) < \alpha\}$. We will show that WF_α is Borel.

Lemma 5.12 WF_α *is Borel.*

Proof We prove this by induction on α. $\mathrm{WF}_0 = \{\emptyset\}$. For all α

$$\mathrm{WF}_{\alpha+1} = \bigcap_{n\in\mathbb{N}} \{T : \langle n\rangle \notin T \text{ or } T_{\langle n\rangle} \in \mathrm{WF}_\alpha\}.$$

Since $T \mapsto T_{\langle n\rangle}$ is continuous, by induction, WF_α is Borel. If α is a limit ordinal, then $\mathrm{WF}_\alpha = \bigcup_{\beta<\alpha} \mathrm{WF}_\beta$. Thus WF_α is Borel.

Ranks of $\mathbf{\Pi}^1_1$ sets

We can now see how $\mathbf{\Pi}^1_1$-sets are built up from Borel sets.

Theorem 5.13 *If $A \in \mathbf{\Pi}^1_1$, then A is the union of $\aleph_1$-Borel sets.*

Proof Suppose $f : \mathcal{N} \to Tr$ such that $x \in A$ if and only if $f(x) \in \mathrm{WF}$. Then $A = \bigcup_{\alpha<\omega_1} f^{-1}(\mathrm{WF}_\alpha)$ and each $f^{-1}(\mathrm{WF}_\alpha)$ is Borel.

This allows us to say something about the cardinality of $\mathbf{\Pi}^1_1$-sets.

Corollary 5.14 *If $A \in \mathbf{\Pi}^1_1$ and $|A| > \aleph_1$, then A contains a perfect set. In particular, $|A| \leq \aleph_1$ or $|A| = 2^{\aleph_0}$.*

Proof Let $A = \bigcup_{\alpha<\omega_1} A_\alpha$ where A_α is Borel. If any A_α is uncountable, then A contains a perfect set. Otherwise $|A| \leq \aleph_1$.

It is consistent with ZFC that there is a $\mathbf{\Pi}^1_1$-set that has cardinality $\aleph_1 < 2^{\aleph_0}$. For example, this is true in any model of ZFC where $\aleph_1^{\mathbb{L}} = \aleph_1^{\mathbb{V}} < 2^{\aleph_0}$.

We next examine the complexity of comparing ranks.

Lemma 5.15 *i) The set $\{(S,T) : \rho(S) \leq \rho(T)\}$ is Σ^1_1.*

ii) There is $R \in \Sigma^1_1(\mathcal{N} \times \mathcal{N})$ such that if $T \in \mathrm{WF}$, then $\{S : (S,T) \in R\} = \{S : \rho(S) < \rho(T)\}$.

Proof

i)

$$\rho(S) \leq \rho(T) \text{ if and only } \exists f : S \to T \text{ order-preserving.}$$

ii) For $T \in \mathrm{WF}$,

$$\rho(S) < \rho(T) \text{ if and only if } S = \emptyset \text{ and } T \neq \emptyset \text{ or } \exists n \in \mathbb{N}\, \exists f : S \to T_{\langle n \rangle} \text{ order-preserving.}$$

Both of these definitions are Σ^1_1.

Corollary 5.16 ($\mathbf{\Sigma}^1_1$-Bounding) *Suppose $A \subseteq \mathrm{WF}$ is $\mathbf{\Sigma}^1_1$. Then there is $\alpha < \omega_1$ such that $A \subseteq \mathrm{WF}_\alpha$.*

Proof Suppose not. Then

$$T \in \mathrm{WF} \Leftrightarrow \exists S\ (S \in A \wedge \rho(T) \leq \rho(S)\}$$

and WF is $\mathbf{\Sigma}^1_1$, a contradiction.

$\mathbf{\Sigma}^1_1$-Bounding gives us a different proof that $\mathbf{\Delta}^1_1$-sets are Borel. Suppose A is $\mathbf{\Delta}^1_1$. Since A is $\mathbf{\Pi}^1_1$ there is a tree $T \subseteq \mathbb{N}^{<\omega} \times \mathbb{N}^{<\omega}$ such that $x \in A$ if and only if $T(x) \in \mathrm{WF}$. Since A is $\mathbf{\Sigma}^1_1$, the set $\{T(x) : x \in A\}$ is a $\mathbf{\Sigma}^1_1$ subset of WF. By $\mathbf{\Sigma}^1_1$-Bounding there is $\alpha < \omega_1$ such that $T(x) \in \mathrm{WF}_\alpha$ for all $x \in A$. Thus $A \leq_w \mathrm{WF}_\alpha$ is Borel.

We will recast the work we just did in more modern language introduced by Moschovakis. This point of view is useful when one attempts to extend these ideas to higher levels of the projective hierarchy.

Definition 5.17 A *norm* on a set A is a function $\phi : A \to \mathbb{O}n$ where $\mathbb{O}n$ is the class of ordinals.

Suppose $A \in \mathbf{\Pi}^1_1$. We say that $\phi : A \to \mathbb{O}n$ is a $\mathbf{\Pi}^1_1$-*norm* if there are relations $\leq_\phi^{\mathbf{\Pi}^1_1} \in \mathbf{\Pi}^1_1(\mathcal{N} \times \mathcal{N})$ and $\leq_\phi^{\mathbf{\Sigma}^1_1} \in \mathbf{\Sigma}^1_1(\mathcal{N} \times \mathcal{N})$ such that if $y \in A$, then

$$\begin{aligned} x \in A \wedge \phi(x) \leq \phi(y) \quad &\Leftrightarrow \quad x \leq_\phi^{\mathbf{\Pi}^1_1} y \\ &\Leftrightarrow \quad x \leq_\phi^{\mathbf{\Sigma}^1_1} y. \end{aligned}$$

If A is $\mathbf{\Pi}^1_1$ and $f : \mathcal{N} \to Tr$ is continuous, let $\phi = \rho \circ f$.

Exercise 5.18 a) Show that ϕ is a $\mathbf{\Pi}^1_1$-norm on A.

For all $\alpha < \omega_1$, let $A_\alpha = \{x : \rho(x) < \alpha\}$.

b) Suppose $B \subseteq A_\alpha$ is $\mathbf{\Sigma}^1_1$. Show that $B \subseteq A_\alpha$ for some α.

c) Show that A is Borel if and only if $A = A_\alpha$ for all suitably large α.

Reduction and Separation

The following structural property of $\mathbf{\Pi}^1_1$-sets is a strong form of the $\mathbf{\Sigma}^1_1$-separation property.

Definition 5.19 A class of sets Γ has the *reduction property* if whenever $A, B \in \Gamma$ there are $A_0 \subseteq A$ and $B_0 \subseteq B$ such that $A_0, B_0 \in \Gamma$, $A_0 \cap B_0 = \emptyset$ and $A_0 \cup B_0 = A \cup B$.

Theorem 5.20 *$\mathbf{\Pi}^1_1$ has the reduction property.*

Proof Suppose $A, B \in \mathbf{\Pi}^1_1$. Let $f, g : \mathcal{N} \to Tr$ be continuous such that $A = f^{-1}(WF)$ and $B = g^{-1}(WF)$.

Let $A_0 = \{x \in A : \rho(f(x)) \leq \rho(g(x))\}$ and let $B_0 = \{x \in B : \rho(g(x)) < \rho(f(x))\}$. It is easy to see that $A_0 \subseteq A$, $B_0 \subseteq B$, $A_0 \cap B_0 = \emptyset$, and $A_0 \cup B_0 = A \cup B$. Notice that

$$x \in A_0 \text{ if and only if } x \in A \wedge \neg(\rho(g(x)) < \rho(f(x)))$$

and

$$x \in B_0 \text{ if and only if } \neg(\rho(f(x)) \leq \rho(g(x))).$$

By 5.15 A_0 and B_0 are $\mathbf{\Pi}^1_1$.

Definition 5.21 We say that Γ has the *separation property* if whenever $A, B \in \Gamma$ and $A \cap B = \emptyset$ there is $C \in \Gamma \cap \check{\Gamma}$ such that $A \subseteq C$ and $C \cap B = \emptyset$.

Lemma 5.22 *If $\check{\Gamma}$ has the reduction property, then Γ has the separation property.*

Proof Suppose $A, B \in \Gamma$ and $A \cap B = \emptyset$. Then $X \setminus A$, $X \setminus B \in \check{\Gamma}$ and $X \setminus A \cup X \setminus B = X$. Let $C \subseteq X \setminus B$, $D \subseteq X \setminus A$ such that $C, D \in \check{\Gamma}$, $C \cap D = \emptyset$ and $C \cup D = X$. Then $C = X \setminus D$ so $C \in \Gamma$. If $x \in A$, then $x \in (X \setminus B) \setminus (X \setminus A)$. Thus $x \in C$. Similarly if $x \in B$, then $x \in D = X \setminus C$. Thus C separates A and B.

This gives a different proof that $\boldsymbol{\Sigma}^1_1$ has the separation property. We next show that it is harder to separate $\boldsymbol{\Pi}^1_1$-sets.

Proposition 5.23 $\boldsymbol{\Pi}^1_1$ *does not have the separation property.*

Proof Let $U \subseteq \mathcal{N} \times \mathcal{N}$ be a universal $\boldsymbol{\Pi}^1_1$-set. If $x \in \mathcal{N}$ we think of x as coding $\langle x_0, x_1 \rangle \in \mathcal{N}^2$. Let $P = \{x : \langle x_0, x \rangle \in U\}$ and let $Q = \{x : \langle x_1, x \rangle \in U\}$. By reduction we can find P_0 and $Q_0 \in \boldsymbol{\Pi}^1_1$ such that $P_0 \cap Q_0 = \emptyset$ and $P_0 \cup Q_0 = P \cup Q$. Suppose, for contradiction, that C is a Borel set with $P_0 \subseteq C$ and $P_0 \cap C = \emptyset$. Suppose $C = U_a$ and $\mathcal{N} \setminus C = U_b$. Let $x = \langle b, a \rangle$.

Suppose $x \in C$, then $(a, x) \in U$ and, by the definition of Q, $x \in Q$. The only elements of Q that are in C are also in P. Thus $x \in P$. Using the definition of P, $(b, x) \in U$. Thus $x \notin C$.

Similarly

$$x \notin C \Rightarrow (b, x) \in U \Rightarrow x \in P \Rightarrow x \in Q \Rightarrow (a, x) \in U \Rightarrow x \in C$$

a contradiction.

Uniformization

Definition 5.24 Suppose $A \subseteq X \times Y$. We say that $B \subseteq A$ *uniformizes* A if and only if

i) $\pi_X(A) = \pi_X(B)$, and

ii) for all $x \in \pi_X(A)$ there is a unique $b \in Y$ such that $(x, b) \in B$.

In other words, B is the graph of a function $f : \pi_X(A) \to Y$ such that $(x, f(x)) \in A$ for all $x \in \pi_X(A)$.

The Axiom of Choice tells us that for every $A \subseteq X \times Y$, there is $B \subseteq A$ uniformizing A. We will be interested in trying to understand how complicated B is relative to A.

Definition 5.25 We say that Γ has the *uniformization property* if for all $A \in \Gamma(\mathcal{N} \times \mathcal{N})$, there is $B \in \Gamma$ a uniformization of A.

We first show that uniformization can be difficult.

Proposition 5.26 *There is a closed set $C \subseteq \mathcal{N} \times \mathcal{N}$ that can not be uniformized by a $\boldsymbol{\Sigma}^1_1$-set.*

Proof By 5.23 there are $\mathbf{\Pi}_1^1$-sets $A_0, A_1 \subseteq \mathcal{N}$ such that $A_0 \cap A_1 = \emptyset$ but there is no Borel set B with $A_0 \subseteq B$ and $A_1 \cap B = \emptyset$.

There are closed sets $C_0, C_1 \subseteq \mathcal{N}$ such that

$$X \setminus A_i = \{x : \exists y\ (x, y) \in C_i\}.$$

Without loss of generality we can take $C_i \subseteq \mathcal{N} \times N_{\langle i \rangle}$. Let $C = C_0 \cup C_1$. Suppose $B \in \mathbf{\Sigma}_1^1$ uniformizes C. Then B is the graph of a function $f : \mathcal{N} \to C$, and, by 4.15 f is Borel measurable. Let $B_i = f^{-1}(\mathcal{N} \times N_{\langle i \rangle})$. Then each B_i is a Borel set and $B_0 \cap B_1 = \emptyset$. If $x \in A_i$, then $x \in B_{1-i}$. Thus B_1 is a Borel set separating A_0 and A_1, a contradiction.

While Borel sets can not be uniformized by Borel sets, or even $\mathbf{\Sigma}_1^1$-sets, we will prove that any $\mathbf{\Pi}_1^1$-set can be uniformized by a $\mathbf{\Pi}_1^1$-set. As a warm-up we first prove a uniformization theorem for $\mathbf{\Pi}_1^1$-subsets of $\mathcal{N} \times \mathbb{N}$.

Theorem 5.27 (Kriesel's Uniformization Theorem) *Every $\mathbf{\Pi}_1^1$ subset of $\mathcal{N} \times \mathbb{N}$ can be uniformized by a $\mathbf{\Pi}_1^1$-set.*

Proof Let $A \subseteq X \times \mathbb{N}$ be $\mathbf{\Pi}_1^1$ and let $f : X \times \mathbb{N} \to Tr$ be continuous such that $x \in A$ if and only if $f(x) \in \mathrm{WF}$. Let

$$B = \{(x, n) \in A : \forall m \in \mathbb{N}\ \rho(f((x, m)) \not< \rho(f(x, n)) \text{ and}$$

$$\forall m < n\ \rho(f(x, m)) \not\leq \rho(f(x, n))\}.$$

Then B is $\mathbf{\Pi}_1^1$ and $(x, n) \in B$ if and only if $(x, n) \in A$, $\rho(x, n) = \inf_m \rho(x, m)$ and for all $m < n$, $\rho(x, m) > \rho(x, n)$. Clearly for all $x \in \pi(A)$ there is a unique n such that $(x, n) \in B$. Thus B uniformizes A.

We can do even better if $\pi(A)$ is Borel.

Corollary 5.28 (Selection) *Suppose $A \subseteq X \times \mathbb{N}$ is $\mathbf{\Pi}_1^1$ and $\pi(A)$ is Borel. Then A has a Borel-uniformization.*

Proof Let B be a $\mathbf{\Pi}_1^1$-uniformization of A. Then

$$(x, n) \notin B \Leftrightarrow \exists m \in \mathbb{N}\ (m \neq n \wedge (x, m) \in B).$$

This is a $\mathbf{\Pi}_1^1$-definition of $X \setminus B$. Thus B is Borel.

Theorem 5.29 (Kondo's Theorem) $\mathbf{\Pi}_1^1$ *has the uniformization property.*

Proof Let $A \subseteq \mathcal{N} \times \mathcal{N}$ by $\mathbf{\Pi}_1^1$. There is a tree T on $\mathbb{N}^{<\omega} \times \mathbb{N}^{<\omega} \times \mathbb{N}^{<\omega}$ such that $A = \{(x, y) : T(x, y) \in \mathrm{WF}\}$. Fix $\sigma_0, \sigma_1, \ldots$ an enumeration of $\mathbb{N}^{<\omega}$. We may assume that such that $\sigma_0 = \emptyset$, $|\sigma_i| \leq i$, and if $\sigma_i \subset \sigma_j$, then $i < j$.

We build a sequence of $\mathbf{\Pi}_1^1$-sets $A = A_0 \supseteq A_1 \supseteq A_2 \supseteq \ldots$ such that $(x, y) \in A_{n+1}$ if and only if

i) $(x, y) \in A_n$;

ii) if $(x, z) \in A_n$, then $y(n) \leq z(n)$;

iii) if $(x,z)\in A_n$ and $z(n)=y(n)$, then $\rho(T(x,y)_{\sigma_n})\le\rho(T(x,z)_{\sigma_n})$.

In other words, we first find $m_{x,n}$ minimal such that there is a z with $(x,z)\in A_n$ and $z(n)=m_{x,n}$, we then find $\alpha_{x,n}$ minimal such that there is $(x,z)\in A_n$ with $z(n)=m_{x,n}$ and $\rho(T(x,z)_{\sigma_n})=\alpha_{x,n}$. Then $(x,y)\in A_{n+1}$ if and only if $(x,y)\in A_n$, $y(n)=m_{x,n}$ and $\rho(T(x,y)_{\sigma_n})=\alpha_{x,n}$.

Let $B=\bigcap_n A_n$. We will show that B is a $\mathbf{\Pi}^1_1$-uniformization of A. Let π be the projection $(x,y)\mapsto x$. If $x\in\pi(A)$, then, by induction, $x\in\pi(A_n)$ for all n. Define $y_x\in\mathcal{N}$ by $y_x(n)=m_{x,n}$.

Claim 1 If $(x,y)\in B$, then $y=y_x$.

If $(x,y)\in A_n$, $y(n)=m_{x,n}$, thus $y=y_x$.

We need to show that $(x,y_x)\in B$ for all $x\in\pi(A)$. Fix $x\in\pi(A)$.

Claim 2 Suppose $\sigma_i,\sigma_j\in T(x,y_x)$ and $\sigma_i\subset\sigma_j$. Then $\alpha_{x,j}<\alpha_{x,i}$.

Choose $(x,z)\in A_{j+1}$. Since $i<j$, $(x,z)\in A_{i+1}$. Since $z|j+1=y|j+1$ and $|\sigma_j|\le j$, $\sigma_j\in T(x,z)$. Thus

$$\alpha_{x,i}=\rho(T(x,z)_{\sigma_i})>\rho(T(x,z)_{\sigma_j})=\alpha_{x,j},$$

as desired.

Claim 3 $(x,y_x)\in A$.

If $\sigma_{i_0}\subset\sigma_{i_1}\subset\dots$ is an infinite path through $T(x,y_x)$, then, by claim 2, $\alpha_{x,1}>\alpha_{x,2}>\dots$ a contradiction. Thus $T(x,y_x)$ is well founded and $(x,y_x)\in A$.

Claim 4 $(x,y_x)\in B$.

An induction on $T(x,y_x)$ shows that if $\sigma_n\in T(x,y_x)$, then $\rho(T(x,y_x)_{\sigma_n})\le\alpha_{x,n}$. By choice of $\alpha_{x,n}$, we have $\rho(T(x,y_x)_{\sigma_n})=\alpha_{x,n}$ and $(x,y_x)\in A_n$ for all n.

Thus B is the graph of a function uniformizing A.

Claim 5 B is $\mathbf{\Pi}^1_1$.

Define $R(x,y,n)$ by

$\exists z[(\forall k<n\ (y(k)=z(k)\wedge\rho(T(x,y)_{\sigma_k})=\rho(T(x,z)_{\sigma_k}))\wedge$
$(z(n)<y(n)\vee(z(n)=y(n)\wedge\rho(T(x,z)_{\sigma_n})<\rho(T(x,y)_{\sigma_n})$.

By 5.15 R is $\mathbf{\Sigma}^1_1$. Suppose $(x,y)\in A_n$. If $(x,y)\notin A_{n+1}$, then either $y(n)\neq m_{x,n}$ or $\rho(T(x,y)_{\sigma_n})\neq\alpha_{x,n}$. In either case $R(x,y,n)$ holds. On the other hand, suppose $R(x,y,n)$ and z witnesses the existential quantifier. Since $(x,y)\in A_n$, $(x,z)\in A_n$. It is easy to see that either $y(n)\neq m_{x,n}$ or $\rho(T(x,y)_{\sigma_n})\neq\alpha_{x,n}$. Thus $(x,y)\notin A_{n+1}$. Thus

$$(x,y)\in B\leftrightarrow(x,y)\in A\wedge\forall n\ \neg R(x,y,n)$$

and B is $\mathbf{\Pi}^1_1$.

Corollary 5.30 *$\mathbf{\Sigma}^1_2$ has the uniformization property.*

Proof Suppose $A\subseteq\mathcal{N}\times\mathcal{N}$ is $\mathbf{\Sigma}^1_2$. There is a $\mathbf{\Pi}^1_1$-set $B\subseteq\mathcal{N}^3$ such that $A=\{(x,y):\exists z\ (x,y,z)\in B\}$. By Kondo's Theorem, there is a $\mathbf{\Pi}^1_1$-set $\widehat{B}\subseteq B$ such that

$$\pi(A)=\{x:\exists y\exists z\ (x,y,z)\in B\}=\{x:\exists y\exists z\ (x,y,z)\in\widehat{B}\}$$

and for all $x \in \pi(A)$ there is a unique pair (y, z) such that $(x, y, z) \in \widehat{B}$. Let $\widehat{A} = \{(x, y) : \exists z\ (x, y, z) \in \widehat{B}\}$. Then $\pi(\widehat{A}) = \pi(A)$ and for all $x \in \pi(A)$ there is a unique y such that $(x, y) \in \widehat{B}$.

Exercise 5.31 a) Show that $\mathbf{\Sigma}^1_2$ has the reduction property.

b) Show that $\mathbf{\Pi}^1_2$ does not have the uniformization property.

c) Suppose there is a $\mathbf{\Delta}^1_2$ well-order of $\mathcal{N}$. Show that $\mathbf{\Sigma}^1_n$-uniformization holds for all $n \geq 2$. (In particular this is true if $\mathbb{V} = \mathbb{L}$).

While $\mathbf{\Sigma}^1_1$ does not have the uniformizaton property, the next two exercises show that we can get close.

Exercise 5.32 Let $\leq_{\rm lex}$ be the lexicographic order on $\mathcal{N}$. Suppose $F \subseteq \mathcal{N}$ is closed and nonempty. Show that there is $x \in F$ such that $x \leq_{\rm lex} y$ for all $y \in F$.

Exercise 5.33 [Von Neumann Uniformization] Suppose $A \subseteq \mathcal{N} \times \mathcal{N}$ is $\mathbf{\Sigma}^1_1$. Let $\mathcal{C}$ be the smallest σ-algebra with $\mathbf{\Sigma}^1_1 \subset \mathcal{C}$. There is $B \in \mathcal{C}$ uniformizing A. [Hint: There is a continuous $f : \mathcal{N} \to \mathcal{N} \times \mathcal{N}$ with $f(\mathcal{N}) = A$. Let $\pi(x, y) = x$. For $a \in A$, let $F_x = \{z \in \mathcal{N} : \pi(f(z)) = x\}$. Let $g : \pi(A) \to \mathcal{N}$ by $g(x)$ is the lexicographic least element of F_x. Show that g is an $\mathcal{C}$-measurable function. Let $B = \{(x, f(g(x))) : x \in \pi(A)\}$. Show that $B \in \mathcal{C}$ and B uniformizes A.]

Conclude from 4.23 that every $\mathbf{\Sigma}^1_1$-set has a $\mathcal{C}$-measurable uniformization, and hence a Lebesgue measurable uniformization.

Π^1_1-sets

Many of the proofs in this section work just as well for Π^1_1-sets. Here are statements of the effective versions.

Theorem 5.34 *i) If $A \subseteq X$ is Π^1_1, there is a computable $f : X \to Tr$ such that $x \in A$ if and only if $f(x) \in \mathrm{WF}$ for all $x \in X$.*

ii) Π^1_1 has the reduction property.

iii) Any two disjoint Σ^1_1 sets can be separated by a Δ^1_1-set.

iv) Any Π^1_1-subset of $\mathcal{N} \times \mathcal{N}$ can be uniformized by a Π^1_1-set.

v) If $A \subseteq \mathcal{N} \times \mathbb{N}$ is Π^1_1 and $\pi(A) = \mathcal{N}$, then A has a Δ^1_1-uniformization.

Further analysis of Π^1_1-sets will require looking at an effective version of "ordinals".

Recursive Ordinals

The set WF is Π^1_1. If $A \subseteq \mathcal{N}$ is Π^1_1, we know that $A \leq_w$ WF. We will show that the reduction f can be chosen computable. There is a recursive tree T, such that

$$x \in A \Leftrightarrow \forall y\ (x, y) \notin [T] \Leftrightarrow T(x) \in \mathrm{WF}.$$

The function $x \mapsto T(x)$ is computable.

A similar construction gives rise to a Π^1_1-complete (for $\leq_m$) subset of $\mathbb{N}$.

Let $O = \{e \in \mathbb{N} : \phi_e$ is the characteristic function of a Well-founded tree $T_e \subseteq \mathbb{N}^{<\omega}\}$. Then $e \in O$ if and only if

i) $\forall \sigma\ \phi_e(\sigma)\downarrow$

ii) $\forall \sigma \in \mathbb{N}^{<\omega}\ \forall \tau \in \mathbb{N}^{<\omega}\ ((\sigma \subseteq \tau \wedge \phi_e(\tau) = 1) \rightarrow \phi_e(\sigma) = 1)$.

iii) $\forall f : \mathbb{N} \rightarrow \mathbb{N}^{<\omega} \exists n\ (\phi_e(f(n)) = 0 \vee \phi_e(f(n+1)) = 0 \vee f(n) \not\subset f(n+1))$.

Conditions i) and ii) are Π^0_2 while iii) is Π^1_1. Thus O is Π^1_1.

Proposition 5.35 *O is Π^1_1-complete.*

Proof We will argue that $\mathbb{N} \setminus O$ is Σ^1_1-complete. Suppose $A \in \Sigma^1_1$. There is $B \subseteq \mathbb{N} \times \mathcal{N}$ in Π^0_1 such that $n \in A$ if and only if $\exists x\ (n, x) \in B$. There is a recursive tree $T \subseteq \mathbb{N} \times \mathbb{N}^{<\omega}$ such that $(n, x) \in A$ if and only if $(n, x|m) \in T$ for all $m \in \mathbb{N}$. There is a recursive $f : \mathbb{N} \rightarrow \mathbb{N}$ such that $\phi_f(n)$ is the characteristic function of $\{\sigma : (n, \sigma) \in T\}$. Then $\phi_f(n)$ is the characteristic function of a tree T_n and

$$n \in A \Leftrightarrow T_n \notin \mathrm{WF} \Leftrightarrow f(n) \notin O.$$

O will play a very important role in effective descriptive set theory. As a first example, we will show how once we know the complexity of a set, we can say find relatively simple elements of the set.

Lemma 5.36 *Suppose $T \subseteq \mathbb{N}^{<\omega}$ is a recursive tree. If $[T] \neq \emptyset$, there is $x \in [T]$ with $x \leq_T O$.*

Proof There is a recursive function f such that $\phi_{f(\sigma)}$ is the characteristic function of T_σ for all $\sigma \in \mathbb{N}^{<\omega}$. We build $\emptyset = \sigma_0 \subset \sigma_1 \ldots$ with $\sigma_i \in T$ such that $[T_{\sigma_i}] \neq \emptyset$. Given σ_i. Let $n \in \mathbb{N}$ be least such that $\sigma_i \widehat{\ } n \in T$ and $f(\sigma_i) \notin O$.

Corollary 5.37 (Kleene Basis Theorem) *If $A \subseteq \mathcal{N}$ is Σ^1_1 and nonempty, there is $x \in A$ with $x \leq_T O$.*

Proof There is a Π^0_1-set $B \subseteq \mathcal{N} \times \mathcal{N}$ such that $x \in A$ if and only if $\exists y\ (x, y) \in B$. By the previous lemma there is $(x, y) \in B$ with $(x, y) \leq_T O$. Clearly $x \leq_T O$.

Using the Uniformization Theorem, we can find definable elements of Π^1_1-sets.

Proposition 5.38 *If $A \subseteq \mathcal{N}$ is Π^1_1, there is $x \in A$ such that $x \in \Delta^1_2$.*

Proof Uniformizing $\{0\} \times A$, we find $x \in \mathcal{N}$ such that $B = \{(0, x)\}$ is Π^1_1. Then

$$\begin{aligned} x(n) = m \quad &\Leftrightarrow \exists y\ ((0, y) \in B \wedge y(n) = m) \\ &\Leftrightarrow \forall y\ ((0, y) \notin B \vee y(n) = m) \end{aligned}$$

The first definition is Σ^1_2, while the second is Π^1_2.

We next need to understand the possible heights of recursive trees.

Definition 5.39 We say that an ordinal α is *recursive* if there is a recursive set $A \subseteq \mathbb{N}$ and $\prec$ a recursive linear order of A such that $(A, \prec) \cong (\alpha, <)$.

Lemma 5.40 *a) If α is a recursive ordinal and $\beta < \alpha$, then β is a recursive ordinal.*

b) If α is a recursive ordinal, then $\alpha + 1$ is a recursive ordinal.

c) Suppose $f : \mathbb{N} \to \mathbb{N}$, $g : \mathbb{N} \to \mathbb{N}$ are recursive functions such that $P_{f(n)}$ is a program to compute the characteristic function of A_n, $P_{g(n)}$ is a program that computes the characterisitic function of $\prec_n$ a well-order of A_n and $(A_n, \prec_n)$ has order-type α_n. Then $\sup \alpha_n$ is a recursive ordinal.

Proof a) and b) are routine. For c) we show that $\sum A_n$ is a recursive well-order. Let $A = \{(n, m) : M_{f(n)}(m) = 1\}$ and let $(n, m) \prec (n', m')$ if and only if $n < n'$ or $n = n'$ and $m \prec_n m'$. Then $(A, \prec)$ is a recursive well-order. Let α be the order type of A. Then $\alpha_n \leq \alpha$ for all n. Since $\sup \alpha_n \leq \alpha$, $\sup \alpha_n$ is a recursive ordinal.

There are only countably many recursive well-orders. Thus there are only countably many recursive ordinals.

Definition 5.41 Let ω_1^{ck} be the least non-recursive ordinal. We call this ordinal the *Church–Kleene ordinal*.

More generally for any x we let ω_1^x be the least ordinal not recursive in x.

We need to be able to compare ordinals with trees.

Definition 5.42 For $\sigma, \tau \in \mathbb{N}^{<\omega}$ we say $\sigma \lhd \tau$ if $\tau \subset \sigma$ or there is an n such that $\sigma(n) \neq \tau(n)$, but $\sigma(m) = \tau(m)$ for all $m < n$. We call $\lhd$ the *Kleene–Brower order*.

Exercise 5.43 a) $\lhd$ is a linear order of $\mathbb{N}^{<\omega}$.

b) If $T \subseteq \mathbb{N}^{<\omega}$ is a tree, then T is well-founded if and only if $(T, \lhd)$ is a well-order.[Hint: If $\sigma_0, \sigma_1, \ldots$ is an infinite descending sequence in $(T, \lhd)$, define x inductively by $x(n) =$ least m such that $(x(0), \ldots, x(n-1), m) \lhd \sigma_i$ for some i. Prove that $x \in [T]$.]

c) Prove that $\omega_1^{\mathrm{ck}} = \sup\{\rho(T) : T \subseteq \mathbb{N}^{<\omega}$ a recursive well founded tree$\}$.

The proof of 5.15 actually shows the following.

Theorem 5.44 *i) The set $\{(S, T) : \rho(S) \leq \rho(T)\}$ is Σ_1^1.*

ii) There is $R \in \Sigma_1^1(\mathcal{N} \times \mathcal{N})$ such that if $T \in \mathrm{WF}$, then $\{S : (S, T) \in R\} = \{S : \rho(S) < \rho(T)\}$.

Corollary 5.45 (Effective Σ_1^1-Bounding) *i) If $A \subseteq O$ is Σ_1^1, then there is $\alpha < \omega_1^{\mathrm{ck}}$ such that $\rho(T) < \alpha$ for all $T \in A$.*

ii) If $A \subseteq \mathrm{WF}$ and $A \in \Sigma_1^1$. Then $\rho(T) < \omega_1^{\mathrm{ck}}$ for all $T \in A$.

Proof If either i) or ii) fails, then $O = \{e : \phi_e$ is the characteristic function of a recursive tree $\exists T\, \forall \sigma \in \mathbb{N}^{<\omega}((\sigma \in T \leftrightarrow \phi_e(\sigma) = 1)$ and $\exists S \in A\ \rho(T) \leq \rho(S))\}$ is Σ_1^1, a contradiction.

Exercise 5.46 Prove that if $A \subseteq \mathcal{N}$ is Δ_1^1, then A is $\mathbf{\Sigma}_\alpha^0$ for some $\alpha < \omega_1^{\mathrm{ck}}$.

6 Determinacy

In this section we will introduce another logical tool that sheds new light on Borel, analytic, and coanalytic sets, and is indispensable in the study of higher levels of the projective hierarchy.

Let X be any nonempty set and let $A \subseteq X^{\mathbb{N}}$. We define an infinite two player game $G(A)$. Players I and II alternate playing elements of X. Player I plays x_0, Player II replies with x_1, Player I then plays $x_3 \ldots$. A full play of the game looks like this.

$$\begin{array}{cc} \text{Player I} & \text{Player II} \\ x_0 & \\ & x_1 \\ x_2 & \\ & x_3 \\ x_4 & \\ & x_5 \\ \vdots & \vdots \end{array}$$

Together they play $x = (x_0, x_1, x_2, \ldots) \in X^{\mathbb{N}}$. Player I wins this play of the game if $x \in A$. Otherwise Player II wins.

Definition 6.1 A *strategy* for Player I is a function $\tau : X^{<\mathbb{N}} \to X$.

Player I uses the strategy by opening with $\tau(\emptyset)$. If Player II responds with x_0, then Player I replies $\tau(x_0)$. If Player II next plays, x_1, then Player II replies $\tau(x_0, x_1) \ldots$.

The full play looks like:

$$\begin{array}{cc} \text{Player I} & \text{Player II} \\ \tau(\emptyset) & \\ & x_0 \\ \tau(x_0) & \\ & x_1 \\ \tau(x_0, x_1) & \\ & x_2 \\ \tau(x_0, x_1, x_2) & \\ \vdots & \vdots \end{array}$$

Definition 6.2 We say that τ is a *winning strategy* for Player I if Player I wins any game played using the strategy τ, i.e., for any $x_0, x_1, x_2, \ldots \in X$, the sequence

$$\tau(\emptyset), x_0, \tau(x_0), x_1, \tau(x_0, x_1), x_2, \tau(x_0, x_1, x_2), \ldots$$

is in A.

There are analogous definitions of strategies and winning strategies for Player II.

Definition 6.3 We say that the game $G(A)$ is *determined* if either Player I or Player II has a winning strategy.

We first show that if A is not too complicated, then $G(A)$ is determined. We consider X with the discrete topology and $X^{\mathbb{N}}$ with the product topology.

Theorem 6.4 (Gale-Stewart Theorem) *If $A \subseteq X^{\mathbb{N}}$ is closed, then $G(A)$ is determined.*

Proof Let T be a tree such that $A = [T]$. Suppose Player II has no winning strategy. We will show that Player I has a winning strategy. Suppose $\sigma \in \mathbb{N}^{<\omega}$ and $|\sigma|$ is even. We consider the game $G_\sigma(A)$ where Players I and II alternate playing elements of $\mathbb{N}$ to build $x \in \mathcal{N}$ and Player I wins if $\sigma\widehat{\ }x \in A$.

Let $P = \{\sigma : |\sigma| \text{ is even and Player II has a winning strategy in } G_\sigma(A)\}$. If $\sigma \notin T$, then Player II has already won $G_\sigma(A)$. In particular, always playing 0 is a winning strategy for Player II. Thus $\mathbb{N}^{<\omega} \setminus T \subseteq P$.

Claim Suppose that for all $n \in \mathbb{N}$ there is $m \in \mathbb{N}$ such that $\sigma\widehat{\ }n\widehat{\ }m \in P$. Then $\sigma \in P$.

Player II has a winning strategy in $G_\sigma(A)$; namely if Player I plays n and Player II plays the least m such that Player II has a winning strategy in $G_{\sigma\widehat{\ }n\widehat{\ }m}$, and then uses the strategy in this game.

We describe a winning strategy for Player I. This strategy can be sumarized as "avoid losing postions".

Since Player II does not have a winning strategy $\emptyset \notin P$. Player I's strategy is to avoid P. If we are in position σ where $\sigma \notin P$ and $|\sigma|$ is even, then by the claim there is a least n such that $\sigma\widehat{\ }n\widehat{\ }m \notin P$ for all m. Player I plays n. No matter what m Player II now plays the new position is not in P. If Player I continues Playing playing this way they will play $x \in \mathcal{N}$ such that $x|2n \notin P$ for all n. In particular $x|2n \in T$ for all n. Thus $x \in [T]$ and this is a winning strategy for Player I.

Exercise 6.5 Show that if $A \subseteq X^{\mathbb{N}}$ is open, then $G(A)$ is determined

Exercise 6.6 Show that if $A, B \subseteq X^{\mathbb{N}}$, A is open and B is closed, then $G(A \cap B)$ is determined.

Exercise 6.7 Suppose $X_0, X_1, \ldots$ are discrete topological spaces. If $A \subseteq \prod X_i$ we can consider a modified game where Player I plays $x_0 \in X_0$, Player II plays $x_1 \in X_1$, Player I plays $x_2 \in X_2, \ldots$. Player I wins if $(x_0, x_1, \ldots) \in A$. Show that if A is closed this game is determined.

What other games are determined? Under the axiom of choice there are undetermined games.

Exercise 6.8 Use the axiom of choice to construct $A \subseteq \mathcal{N}$ such that no player has a winning strategy in $G(A)$. [Hint: Use AC to give a well-ordered enumeration of all strategies and diagonalize against them.]

Martin proved the determinacy of Borel games.

Theorem 6.9 (Borel Determinacy) *If $A \subseteq \mathcal{N}$ is Borel, then $G(A)$ is determined.*

For a proof see [6] II §20.

This is the best result provable in ZFC. The results of the next subsection, for example, show that if all analytic games are determined, then every uncountable $\mathbf{\Sigma}^1_2$-set contains a perfect subset and this is false if $\mathbb{V} = \mathbb{L}$.

For $\Gamma = \mathbf{\Sigma}^1_n$ or $\mathbf{\Pi}^1_n$ we let $\mathrm{Det}(\Gamma)$ be the assertion that if $A \in \Gamma$, then $G(A)$ is determined. *Projective determinacy* PD is the assertion that all projective games are determined.

Exercise 6.10 Show that $\mathrm{Det}(\mathbf{\Sigma}^1_n)$ if and only if $\mathrm{Det}(\mathbf{\Pi}^1_n)$.

The determinacy of projective games is intimately tied to the existence of large cardinals.

Theorem 6.11 (Martin/Harrington) *i) If there is a measurable cardinal, then Det($\mathbf{\Sigma}^1_1$) holds.*

ii) Det($\mathbf{\Sigma}^1_1$) holds if and only if $x^\#$ exists for all $x \in \mathcal{N}$.

For a proof see [9] Theorem 105.

More recently Martin and Steel [13] have found reasonable large cardinal hypotheses that imply PD.

Perfect Set Theorems

We first show how games can be used to prove perfect set theorems. Suppose $A \subseteq \mathcal{C}$. We define a game $G^*(A)$ where at stage i Player I plays $\sigma_i \in 2^{<\omega}$ and Player II plays $j_i \in \{0, 1\}$. Together they play

$$x = \sigma_0\widehat{\ }j_0\widehat{\ }\sigma_1\widehat{\ }j_1\sigma_2\widehat{\ }j_2 \ldots \in \mathcal{C}.$$

Player I wins if $x \in A$ and Player II wins if $x \notin A$.

Proposition 6.12 *If Player I has a winning strategy in $G^*(A)$, then A contains a perfect set.*

Proof Let τ be a winning strategy for Player I. Define $f : \mathcal{C} \to A$ by $f(x)$ is the play of the game where Player I uses τ and Player II plays $x(0), x(1), \ldots$. In other words

$$f(x) = \tau(\emptyset)\widehat{\ }x(0)\widehat{\ }\tau(x_0)\widehat{\ }x_1\widehat{\ }\tau(x(0), x(1))\widehat{\ }x_2\widehat{\ }\tau(x(0), x(1), x(2), x(3)) \ldots.$$

Clearly if $x|n = y|n$, then $f(x)|n = f(y)|n$. Thus f is continuous. Suppose $x \neq y$ and n is least such that $x(n) \neq y(n)$. Let

$$\mu = \tau(\emptyset)\widehat{\ }x(0)\widehat{\ }\tau(x(0))\widehat{\ }x_1 \ldots\widehat{\ }\tau(x(0), \ldots x(n-1)).$$

Then $f(x) \supset \mu\widehat{\ }x(n)$ and $f(y) \supset \mu\widehat{\ }y(n)$. Thus $f(x) \neq f(y)$. Thus f is continuous and one-to-one. Hence $f(\mathcal{C})$ is an uncountable closed subset of A.

Proposition 6.13 *If Player II has a winning strategy in $G^*(A)$, then A is countable.*

Proof Let τ be a winning strategy for Player II. Consider a position $p = (\sigma_0, j_0, \ldots, \sigma_n, j_n)$ where Player II has played using τ and it is Player I's turn to play. Suppose $x \in A$ and $x \supset \mu = \sigma_0 \widehat{\ } j_0, \ldots, \sigma_n \widehat{\ } j_n$. We say that x is *rejected* at p if for all σ_{n+1}, if $x \supseteq \mu \widehat{\ } \sigma_{n+1}$, then $x \not\supseteq \mu \widehat{\ } \sigma_{n+1} \widehat{\ } \tau(\sigma_0, \ldots, \sigma_{n+1})$. In other words, up to stage p, it looks like it is possible that we will eventually play x, but in fact no matter what Player I does at this stage, Player II will immediately make a play which ensures that we will not eventually play x.

Claim If $x \in A$, there is a position p such that x is rejected at p.

Suppose not. Consider the following play of the game. Since x is not rejected at the empty position. There is $\sigma_0 \subset x$ such that $x \supset \sigma_0 \widehat{\ } \tau(\sigma_0)$. Player I plays σ_0. Let p_n denote the position after Player II's nth move and let μ_n be the sequence $\sigma_0 \widehat{\ } \tau(\sigma(0)) \widehat{\ } \ldots \tau(\sigma_0, \ldots, \sigma_n)$. We assume by induction that $x \supset \mu_n$. Since x is not rejected at p_n, there is $\sigma_{n+1} \in 2^{<\omega}$ such that $x \supset \mu_n \widehat{\ } \sigma_{n+1} \widehat{\ } \tau(\sigma_0, \ldots, \sigma_{n+1})$. Player I plays σ_{n+1}. But then the final play of the game is $\bigcup \mu_n = x \in A$, contradicting the fact that τ is a winning strategy for Player II.

Claim There is at most one $x \in A$ rejected at p.

Suppose x is rejected at $p = (\sigma_0, \tau(\sigma_0), \ldots, \sigma_n, \tau(\sigma_0, \ldots, \sigma_n))$. Let $\mu = x|k$ be the portion of x we have decided by position p. We claim that knowing only p we can inductively determine the remaining values of x. Suppose we have determined $x(k), \ldots, x(m-1)$. If Player I plays $x(k), \ldots, x(m-1)$, the Player II must play $1 - x(m+1)$. Thus

$$x(m) = 1 - \tau(\sigma_0, \ldots, \sigma_n, \langle x(k), \ldots, x(m-1) \rangle).$$

Thus there is a unique element of A rejected at p.

Since every element of A is rejected at one of the countably many possible positions, A must be countable.

Corollary 6.14 *If A is uncountable and $G^*(A)$ is determined, then A contains a perfect set.*

Exercise 6.15 Let $A \subseteq X$. Prove the following without using determinacy.

a) If $|A| \leq \aleph_0$, then Player II has a winning strategy in $G^*(A)$.

b) If A contains a perfect set, then Player I has a winning strategy in $G^*(A)$.

We have only proved this for $A \subseteq \mathcal{C}$, but using the fact that any two uncountable standard Borel spaces are Borel isomorphic we see that it is true for any uncountable Polish space.

Corollary 6.16 *If PD holds, the any uncountable projective set contains a perfect subset.*

There is a technique of "unfolding" games, that allows us to show that if $\mathrm{Det}(\mathbf{\Sigma}^1_n)$ holds, then every uncountable $\mathbf{\Sigma}^1_{n+1}$ set contains a perfect subset. We will illustrate this idea by giving another proof of the perfect set theorem for $\mathbf{\Sigma}^1_1$-sets using only the determinacy of closed games.

Suppose $A \subseteq \mathcal{C}$ is $\mathbf{\Sigma}^1_1$. Let $B \subseteq \mathcal{C} \times \mathcal{N}$ such that $A = \{x : \exists y\ (x, y) \in B\}$. Consider the game $G^*_u(A)$ where at stage i Player I plays $\sigma_i \in 2^{<\omega}$ and $y(i) \in \mathbb{N}$ and Player II responds with $j_i \in 2$. Together they play

$$x = \sigma_0 \widehat{\ } j_0 \widehat{\ } \sigma_1 \widehat{\ } j_1 \widehat{\ } \ldots$$

and

$$y = (y(0), y(1), \ldots).$$

Player I wins if $(x, y) \in A$. By closed determinacy (or more correctly by 6.7), $G^*_u(A)$ is determined.

Lemma 6.17 *If Player I has a winning strategy in $G^*_u(A)$, then A contains a perfect subset.*

Proof As in 6.12 if τ is a winning strategy for Player I, there are continuous functions $f : \mathcal{C} \to \mathcal{C}$ and $g : \mathcal{C} \to \mathcal{N}$ such that if Player II plays $z(0), z(1), z(2), \ldots$ and Player I uses τ, then together they play $x = f(z) \in \mathcal{C}$ and $y = g(z) \in \mathcal{N}$ with $(x, y) \in B$. As in 6.12 f is one-to-one and $f(\mathcal{C})$ is an uncountable closed subset of A.

Lemma 6.18 *If Player II has a winning strategy in $G^*_u(A)$, then A is countable.*

Proof Suppose $x \in A$. Choose y such that $(x, y) \in B$. As in 6.13 there is a position p at which (x, y) is rejected. Let $\mu = (x(0), \ldots, x(k-1))$ be the portion of x forced by p. If Player I now play $(x(k), \ldots, x(m-1))$ and $y(n)$, then

$$x(m) = 1 - \tau(\sigma_0, y(0), \ldots, \sigma_{n-1}, y(n-1), \langle x(0), \ldots, x(m-1)\rangle, y(n)\rangle.$$

Indeed for each possible value of $y(n)$, there is at most one x rejected at p. Thus the set of x rejected at p is countable and A is countable.

Lemmas 6.17 and 6.18 together with the determinacy of closed games gives a second proof of the Perfect Set Theorem for $\mathbf{\Sigma}^1_1$.

In §7 we will examine this game again. At that time it will be useful to note that if x is rejected at p, then x is recursive in τ.

Banach-Mazur Games

We will show that, assuming Projective Determinacy, all projective sets have the Baire property. Unfolding this argument will prove in ZFC that all $\mathbf{\Sigma}^1_1$ sets have the Baire property (and hence all $\mathbf{\Pi}^1_1$-sets have the Baire property).

Let $A \subseteq \mathcal{N}$. Consider the *Banach–Mazur game* $G^{**}(A)$ where at stage i, Player I plays $\sigma_{2i} \in \mathbb{N}^{<\omega}$ and Player II plays $\sigma_{2i+1} \in \mathbb{N}^{<\omega}$ such that $\sigma_0 \subset \sigma_1 \subset \sigma_2 \subset \ldots$. The final play of the game is $x = \bigcup \sigma_n$ and Player I wins if $x \in A$.

Lemma 6.19 *Player II has a winning strategy in $G^{**}(A)$ if and only if A is meager.*

Proof

($\Leftarrow$) Suppose $A = \bigcup_n A_n$ where each A_i is nowhere dense. We informally describe a winning strategy for Player II. If, at stage i, Player I plays $\sigma_{2i} \in \mathbb{N}^{<\omega}$, then Player II plays $\sigma_{2i+1} \supset \sigma_{2i}$ such that $N_{\sigma_{2i+1}} \cap A_i = \emptyset$. Since each A_i is nowhere dense this is always possible. If $x = \bigcup \sigma_n$ is the final play of the game, then, for each i,

$$x \in N_{\sigma_{2i+1}} \subseteq \mathcal{N} \setminus A.$$

Thus this is a winning strategy for Player II.

($\Rightarrow$) Suppose τ is a winning strategy for Player II. Suppose $x \in A$. Let $p = (\sigma_0, \ldots, \sigma_{2m-1})$ be a position in the game where Player II has used τ. We say that x is rejected at p if and only if $x \supset \sigma_{2m-1}$ but for all $\sigma_{2m} \supset \sigma_{2m-1}$ if $x \supset \sigma_{2m-1}$, then $x \not\supset \tau(\sigma_0, \sigma_2, \ldots, \sigma_{2m})$.

Claim If $x \in A$, then there is a position $p = (\sigma_0, \ldots, \sigma_{2m-1})$ such that $x \supset \sigma_{2m-1}$ and x is rejected at p.

Suppose not. Because x is not rejected at $\emptyset$, there is σ_0 such that $x \supset \tau(\sigma_0)$. Inductively we build $\sigma_0, \sigma_2, \ldots$ such that

$$x \supset \tau(\sigma_0, \sigma_2, \ldots, \sigma_{2m})$$

for all m. But then if Player I plays $\sigma_0, \sigma_2, \ldots$ and Player II uses τ, then the eventually play $x \in A$ contradicting the claim that τ is a winning strategy for Player II.

Let $R_p = \{x \in A : \text{x is rejected at position } p\}$.

Claim R_p is nowhere dense.

Note that $R_p \subseteq N_{\sigma_{2m-1}}$. For all $\sigma \supset \sigma_{2m-1}$ let $\eta_\sigma = \tau(\sigma_0, \ldots, \sigma_{2m-1}, \sigma)$. Then $\eta_\sigma \supset \sigma$ and $R_p \cap N_{\eta_\sigma} = \emptyset$. Thus R_p is nowhere dense.

Thus $A = \bigcup_p R_p$ is meager.

Lemma 6.20 *If Player I has a winning strategy in $G^{**}(A)$, then there is $\eta \in \mathbb{N}^{<\omega}$ such that $N_\eta \setminus A$ is meager.*

Proof Let τ be Player I's winning strategy for $G^{**}(A)$. Suppose Player I's first move in $G^{**}(A)$ is η. We will show that $N_\eta \setminus A$ is meager, by showing that Player II has a winning strategy $\widehat{\tau}$ in $G^{**}(N_\eta \setminus A)$.

Let

$$\widehat{\tau}(\sigma_0, \sigma_2, \ldots, \sigma_{2m}) = \tau(\sigma_0, \sigma_2, \ldots, \sigma_{2m}).$$

In other words Player II plays $G^{**}(N_\eta \setminus A)$, by pretending to be Player I using τ in a game of $G^{**}(A)$.

If Player's I first move in $G^{**}(N_\eta \setminus A)$ is σ_0, Player II checks to see how Player I would reply if Player II played σ_0 in $G^{**}(A)$. The following picture describes the play of the games.

$G^{**}(N_\eta \setminus A)$		$G^{**}(A)$	
Player I	Player II	Player I	Player II
σ_0		η	
	$\tau(\sigma_0)$		σ_0
σ_1		$\tau(\sigma_0)$	
	$\tau(\sigma_0, \sigma_1)$		σ_1
σ_2		$\tau(\sigma_0, \sigma_1)$	
	$\tau(\sigma_0, \sigma_1, \sigma_2)$		σ_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$

If x is a play of $G^{**}(N_\eta \setminus A)$ where Player II uses the strategy $\widehat{\tau}$, then x is also a play of $G^{**}(A)$ where Player I uses τ. Thus $x \in N_\eta \cap A$ and Player II wins the play of $G^{**}(N_\eta \setminus A)$. Thus $\widehat{\tau}$ is a winning strategy for Player II and $N_\sigma \setminus A$ is meager.

Theorem 6.21 *Assuming Projective Determinacy all projective sets have the Baire property.*

Proof Let $A \subseteq \mathcal{N}$. If Player II has a winning strategy in $G^{**}(A)$, then by 6.20 A is meager.

Suppose Player I has a winning strategy, let $S = \{\sigma \in \mathbb{N}^{<\omega} : N_\sigma \setminus A$ is meager$\}$. By 6.19, S is nonempty. Let $U = \bigcup_{\sigma \in S} N_\sigma \setminus A$. Then

$$U \setminus A = \bigcup_{\sigma \in S} N_\sigma$$

is meager. It suffices to show that $A \setminus U$ is also meager.

Suppose $A \setminus U$ is nonmeager. Since the game $G^{**}(A \setminus U)$ is determined and, by 6.20, Player II does not have a winning strategy, Player I must have a winning strategy and, by 6.19, there is η such that $N_\eta \setminus (A \setminus U)$ is meager. But then $N_\eta \setminus A$ is meager and $\eta \in S$. But then $N_\eta \subseteq U$ and $N_\eta \setminus (A \setminus U) = N_\eta$, a contradiction.

Exercise 6.22 Give another proof that analytic sets have the Baire property using the determinacy of closed games and "unfolding" a Banach–Mazur game.

Further Results

Projective Determinacy can also be used to prove that all projective sets are Lebesgue measurable (see [6] §21 or [9] §43).

Can every projective set be uniformized by a projective set? If $\mathbb{V} = \mathbb{L}$, then we can use the $\mathbf{\Delta}^1_2$ well-ordering of $\mathcal{N}$ to show that they can. Moschovakis showed that Projective Determinacy also leads to an interesting answer.

Theorem 6.23 (Periodicity Theorems) *Assume Projective Determinacy.*

a) The classes with the reduction property are exactly $\mathbf{\Pi}^1_{2n+1}$ and $\mathbf{\Sigma}^1_{2n+2}$.

b) The classes with the uniformization property are exactly $\mathbf{\Pi}^1_{2n+1}$ and $\mathbf{\Sigma}^1_{2n+2}$.

One key idea is to use determinacy to build $\mathbf{\Pi}^1_{2n+1}$ prewellorderings. For proofs see [6] §39.

Another interesting class of games are the *Wadge games*. Suppose $A, B \subseteq \mathcal{N}$. Consider the game $G_w(A, B)$ where Player I plays $x(0), x(1), \ldots$, and Player II plays $y(0), y(1)$ with $x(i), y(i) \in \mathbb{N}$. Player II wins if $x \in A$ if and only if $x \in B$.

Lemma 6.24 *a) If A and B are Borel, then $G_w(A, B)$ is determined.*

b) Assuming Projective Determinacy if A and B are Projective, then $G_w(A, B)$ is determined.

c) If Player II has a winning strategy in $G_w(A, B)$, then $A \leq_w B$.

d) If Player I has a winning strategy in $G_w(A, B)$, then $B \leq_w (\mathcal{N} \setminus A)$.

Proof b) is clear. a) follows from Borel Determinacy.

c) Suppose Player II has a winning strategy. Let $f(x) = y$, where $y \in \mathcal{N}$ is Player II's plays using this strategy if Player I plays $x(0), x(1), \ldots$. Clearly f is continuous and $x \in A$ if and only if $f(x) \in B$. Thus $A \leq_w B$.

d) Suppose Player I has a winning strategy and $g(y) = x$ where x is Player I's play if Player II plays y and Player I uses the winning strategy. Then $y \in B$ if and only if $g(y) \notin A$. Thus $B \leq_w A$.

Corollary 6.25 *If $A \in \mathbf{\Sigma}^0_\alpha \setminus \mathbf{\Delta}^0_\alpha$, then A is $\mathbf{\Sigma}^0_\alpha$-complete.*

Proof Suppose $B \in \mathbf{\Sigma}^0_\alpha$ and $B \not\leq_w A$. Then Player II does not have a winning strategy in $G_w(B, A)$. By Borel Determinacy, Player I has a winning strategy. Thus $A \leq_w (\mathcal{N} \setminus B)$ and $A \in \mathbf{\Pi}^0_\alpha$, a contradiction.

Exercise 6.26 Show that under Projective Determinacy and non-Borel $\mathbf{\Sigma}^1_1$-set is $\mathbf{\Sigma}^1_1$-complete.

We write $A <_w B$ if $A \leq_w B$ but $B \not\leq_w A$.

Theorem 6.27 (Wadge, Martin) *There is no infinite sequence of Borel sets $A_0, A_1 \ldots$ with $A_{i+1} <_w A_i$ for all i.*

Similarly under Projective Determinacy, there is no infinite descending Wadge-chain of projective sets.

See [6] 21.15 for a proof.

We give one more application of determinacy as an Exercise.

Exercise 6.28 Let $\leq_T$ be Turing reducibility and $x \equiv_T y$ if $x \leq_T y$ and $x \leq_T y$. We say that $A \subseteq \mathcal{N}$ is *Turing-invariant* if whenever $x \in A$ and $y \equiv_T x$, then $y \in A$. If $z \in \mathcal{N}$, the cone $C_z = \{x \in \mathcal{N} : z \leq_T x\}$.

a) Show that if A is Turing-invariant and $C_z \subseteq A$, then there is no y with $C_y \cap A = \emptyset$.

b) Show that if A is Borel and Turing-invariant, then either A contains a cone or there is a cone disjoint from A. [Hint: Consider the game $G(A)$ where Player I plays $x(0), x(2), \ldots$, Player II plays $x(1), x(2), \ldots$ and Player I wins if

$x \in A$. Show that if τ is a winning strategy for Player I, then $C_\tau \subseteq A$, while if $\widehat{\tau}$ is a winning strategy for Player II, then $C_{\widehat{\tau}} \cap A = \emptyset$.]

c) Let Ω be the collection of Turing-invariant Borel subsets of $\mathcal{N}$. Show that Ω is σ-algebra.

d) Let $\mu : \Omega \to 2$ be $\mu(A) = 1$ if and only if A contains a cone. Show that μ is a σ-additive measure on Ω. μ is called the Martin-measure.

Assuming Projective Determinacy we can consider projective sets instead of Borel sets.

The axiom of choice tells us there are undetermined games. It is interesting to abandon the axiom of choice and consider ZF with the *Axiom of Determinacy* AD which asserts that all games are determined. While AD is refutable from ZFC, it is consistent with large cardinals that ZFC + $\mathbb{L}(\mathbb{R}) \models$ AD. ZF+AD has wild consequences. For example:

Theorem 6.29 (Solovay) *If* ZF + AD *then* $\aleph_1$ *and* $\aleph_2$ *are measurable cardinals, while* $\aleph_n$ *is singular of cofinality* ω *for* $3 \leq n < \omega$.

For a proof of the first assertion see [9] Theorem 103.

7 Hyperarithmetic Sets

Our first goal is to try to characterize the Δ^1_1-sets. In particular we will try to formulate the "light-faced" version of

$$\boldsymbol{\Delta}^1_1 = \text{ Borel.}$$

We begin by studying a method of coding Borel sets.

Borel Codes

Let $X = \mathbb{N}^k \times \mathcal{N}^l$. Let S_X be as in §3.

Definition 7.1 A *Borel code* for a subset of X is a pair $\langle T, l\rangle$ where $T \subseteq \mathbb{N}^{<\omega}$ is a well-founded tree and $l : T \to (\{0\} \times \{0,1\}) \cup (\{1\} \times S_X)$ such that:
i) if $l(\emptyset) = \langle 0, 0\rangle$, then $\sigma\widehat{\ }0 \in T$ and $\sigma\widehat{\ }n \notin T$ for all $n \geq 1$;
ii) if $l(\emptyset) = \langle 1, \eta\rangle$, then $\sigma\widehat{\ }n \notin T$ for all $n \in \mathbb{N}$.

Let BC be the set of all Borel codes. It is easy to see that BC is Π^1_1.

If $x = \langle T, l\rangle$ is a Borel code, we can define $B(x)$ the Borel set coded by x. If $\sigma \in T$, recall that $T_\sigma = \{\tau : \sigma\widehat{\ }\tau \in T\}$. We let $l_\sigma : T_\sigma \to \{0\} \times 2 \cup \{1\} \times S_X$ by $l_\sigma(\tau) = l(\sigma\widehat{\ }\tau)$. It is easy to see that $\langle T_\sigma, l_\sigma\rangle$ is also a Borel code.

Definition 7.2 We define $B(\langle T, l\rangle)$ inductively on the height of T.
i) $B(\langle\emptyset, \emptyset\rangle) = \emptyset$.
ii) If $l(\emptyset) = \langle 1, \eta\rangle$, then $B(\langle T, l\rangle) = N_\eta$.
iii) If $l(\emptyset) = \langle 0, 0\rangle$, then $B(\langle T, l\rangle) = X \setminus B(\langle T_{\langle 0\rangle}, l_{\langle 0\rangle}\rangle)$.

iv) If $l(\emptyset) = \langle 0, 1\rangle$, then

$$B(\langle T, l\rangle) = \bigcup_{\langle n\rangle \in T} B(\langle T_{\langle n\rangle}, l_{\langle n\rangle}\rangle).$$

Exercise 7.3 a) Show that if $x \in BC$, then $B(x)$ is a Borel set.
b) Show that if $A \subseteq X$ is Borel, then there is $x \in BC$ with $B(x) = A$.

Lemma 7.4 *There are $R \in \Sigma^1_1$ and $S \in \Pi^1_1$ such that if $x \in BC$ then*

$$y \in B(x) \Leftrightarrow (x, y) \in R \Leftrightarrow (x, y) \in S.$$

In particular $B(x) \in \Delta^1_1(x)$.

Proof We define a set A such that $(x, y, f) \in A$ if and only if x is a pair $\langle T, l\rangle$ where $T \subseteq \mathbb{N}^{<\omega}$ is a tree, $l : T \to \{0\} \times 2 \cup \{1\} \times S_X$ and $f : T \to 2$ such that for all $\sigma \in T$:

i) if $l(\emptyset) = \langle 1, \eta\rangle$, then $f(\sigma) = 1$ if and only if $y \in N_\eta$;
ii) if $l(\emptyset) = \langle 0, 0\rangle$, then $f(\sigma) = 1$ if and only if $f(\sigma \widehat{\ } 0) = 0$;
iii) if $l(\emptyset) = \langle 0, 1\rangle$, then $f(\sigma) = 1$ if and only if $f(\sigma \widehat{\ } n) = 1$ for some n.

An easy induction shows that if $x = \langle T, l\rangle$ is a Borel code then $(x, y, f) \in A$ if and only if f is the function

$$f(\sigma) = 1 \Leftrightarrow y \in B(\langle T_\sigma, l_\sigma\rangle).$$

It is easy to see that A is arithmetic and if $x \in BC$, then

$$\begin{aligned} y \in B(x) \quad &\Leftrightarrow \exists f \ ((x, y, f) \in A \wedge f(\emptyset) = 1) \\ &\Leftrightarrow \forall f \ ((x, y, f) \notin A \vee f(\emptyset) = 1) \end{aligned}$$

Corollary 7.5 *If $x \in BC$ is recursive, then $B(x)$ is Δ^1_1.*

Proof Let R and S be as in the previous lemma. Let $\phi_e = x$. Then

$$\begin{aligned} y \in B(x) \quad &\Leftrightarrow \exists z \ ((\forall n \ \phi_e(n) \downarrow= z(n)) \wedge R(z, y)) \\ &\Leftrightarrow \forall z \ ((\forall n \ \phi_e(n) \downarrow= z(n)) \to S(z, y)). \end{aligned}$$

The first condition is Σ^1_1 and the second is Π^1_1.

Recursively Coded Borel Sets

Our goal is to show that Δ^1_1 is exactly the collection of Borel sets with recursive codes. That will follow from the following two results and Σ^1_1-Bounding.

Theorem 7.6 *If $A \subseteq Y$ is a recursively coded Borel set and $f : X \to Y$ is computable, then $f^{-1}(A)$ is a recursively coded Borel set.*

Proposition 7.7 *If $\alpha < \omega_1^{\rm ck}$, then WF_α is a recursively coded Borel set.*

Corollary 7.8 *Suppose $A \subseteq X$. The following are equivalent:*
i) A is Δ^1_1;
ii) A is a recursively coded Borel set.

Proof We have already shown that every recursively coded Borel set is Δ^1_1. Suppose A is Δ^1_1. Since A is Π^1_1, there is a computable $f : X \to Tr$ such that $x \in A$ if and only if $f(x) \in \mathrm{WF}$. The set

$$f(A) = \{y : \exists x\ x \in A \wedge f(x) = y\}$$

is a Σ^1_1-subset of WF. By Σ^1_1-Bounding, there is $\alpha < \omega_1^{\rm ck}$ such that $f(A) \subseteq \mathrm{WF}_\alpha$. By 7.7 WF_α is recusively coded, and by 7.6 $A = f^{-1}(\mathrm{WF}_\alpha)$ is recursively coded.

For notational simplicity we will assume $X = \mathcal{N}$, but all our arguments generalize easily.

Let $BC_{\rm rec} = \{(e, x) : \phi^x_e$ is a total function and $\phi^x_e \in BC\}$. Then

$$(e, x) \in BC_{\rm rec} \Leftrightarrow \phi^x_e \text{ is total } \wedge \forall z\ (\forall n\ \phi^x_e(n) = z(n)) \to z \in BC).$$

Thus $BC_{\rm rec}$ is Π^1_1.

If $e \in BC_{\rm rec}$, then $B_{\rm rec}(e, x)$ is the Borel set coded by ϕ^x_e. A similar argument shows that there are $R_{\rm rec} \in \Sigma^1_1$ and $S_{\rm rec} \in \Pi^1_1$ such that if $(e, x) \in BC_{\rm rec}$ then

$$y \in B_{\rm rec}(e, x) \Leftrightarrow R_{\rm rec}(e, y) \Leftrightarrow S_{\rm rec}(e, x)$$

We say $e \in BC_{\rm rec}$ and $x \in B_{\rm rec}(e)$ if $(e, \emptyset) \in BC_{\rm rec}$ and $x \in B_{\rm rec}(e, \emptyset)$.

The proofs of both 7.6 and 7.7 will use the Recursion Theorem to do a transfinite induction.

We begin with the base case of the induction

Lemma 7.9 *There is a recursive function $F : \mathbb{N} \times S_Y \to \mathbb{N}$ such that if $f : X \to Y$ is computable and e is a code for the program computing f , then $B_{\rm rec}(F(e, i)) = f^{-1}(N_\eta)$.*

Proof For notational simplicity we assume $X = Y = \mathcal{N}$, this is no loss of generality. Let

$$W = \{\nu \in \mathbb{N}^{<\omega} : \forall m < |\eta|\ \exists s \leq |\nu|\ \phi^\nu_e(m) \downarrow_s = \eta(m)\}.$$

Then W is recursive and $f^{-1}(N_\eta) = \bigcup_{\nu \in W} N_\nu$. Let $\nu_0, \nu_1, \ldots$ be a recursive enumeration of $\mathbb{N}^{<\omega}$. Let $T = \{\emptyset\} \cup \{\langle n\rangle : \sigma_n \in W\}$ and let $l(\emptyset) = \langle 0, 1\rangle$, $l(\langle n\rangle) = \langle 1, \nu\rangle$. Then $x = \langle T, l\rangle$ is a recusive code. Given e and η we can easily compute $F(e, \eta) = i$ such that $\phi_i = x$.

Lemma 7.10 *i) There is a total recursive function $H_c : \mathbb{N} \to \mathbb{N}$ such that if $e \in BC$, then $B_{\rm rec}(H_c(e)) = \mathcal{N} \setminus B_{\rm rec}(e)$.*

ii) There is a total recursive function $H_u : \mathbb{N} \to \mathbb{N}$ such that if $\phi_e(n) \in BC_{\rm rec}$ for all n, then $B_{\rm rec}(H_u(e)) = \bigcup_n B_{\rm rec}(\phi_e(n))$.

Proof i) ϕ_e is a code for a pair $\langle T, l\rangle$. Let

$$T' = \{\emptyset\} \cup \{0\widehat{\ }\eta : \eta \in T\}$$

and $l'(\emptyset) = \langle 0, 0\rangle$, $l'(0\widehat{\ }\eta) = l(\eta)$. It is easy to find H_c such that $H_c(e)$ codes $\langle T', l'\rangle$ and that if $e \in BC_{\text{rec}}$, then $H_c(e)$ is a code for the complement.

ii) Suppose $\phi_e(n)$ code a pair $\langle T_n, l_n\rangle$. Let

$$T = \{\emptyset\} \cup \{n\widehat{\ }\sigma : \sigma \in T_n\}$$

and let $l(\emptyset) = \langle 0, 1\rangle$ and $l(n\widehat{\ }\sigma) = l_n(\sigma)$. It is easy to find H_u such that $H_u(e)$ codes $\langle T, l\rangle$. If each $\langle T_n, l_n\rangle$ is a Borel code, then $\langle T, l\rangle$ codes their union.

Theorem 7.6 follows from the next lemma.

Lemma 7.11 *If $x = \langle T, l\rangle$ is a recursive Borel code, there is a recursive function $G : \mathbb{N} \times T \to \mathbb{N}$ such that if $f : \mathcal{N} \to \mathcal{N}$ is a computed by program P_e, then $G(e, \sigma) \in BC_{\text{rec}}$ is a Borel code for $f^{-1}(B(\langle T_\sigma, l_\sigma\rangle)$ for all $\sigma \in T$.*

Proof

We define a recursive function $g : \mathbb{N} \times \mathbb{N} \times T \to \mathbb{N}$ as follows:

i) If $l(\sigma) = \langle 1, \eta\rangle$, then $g(i, e, \sigma) = F(e, \eta)$;

ii) If $l(\sigma) = \langle 0, 0\rangle$, then $g(i, e, \sigma) = H_c(\phi_i(e, \sigma\widehat{\ }0))$;

iii) Suppose $l(\sigma) = \langle 0, 1\rangle$. Choose j such that $\phi_j(n) = \phi_i(e, \sigma\widehat{\ }n)$. Then $g(i, e, \sigma) = H_u(j)$.

By the Recursion Theorem, there is $\widehat{i}$ such that $\phi_{\widehat{i}}(e, \sigma) = g(\widehat{i}, e, \sigma)$ for all e, σ. Let $G(e, \sigma) = \phi_{\widehat{i}}(e, \sigma)$.

We prove by induction on T, that $G(e, \sigma)$ is a code for $f^{-1}(B(\langle T_\sigma, l_\sigma\rangle)$. By i) this is clear if $l(\sigma) = \langle 1, \eta\rangle$. We assume the claim is true for all $\tau \supset \sigma$.

If $l(\sigma) = \langle 0, 0\rangle$, then

$$G(e, \sigma) = g(\widehat{i}, e, \sigma) = H_c(\phi_{\widehat{i}}(e, \sigma)) = H_c(G(e, \sigma\widehat{\ }0)).$$

By inducition, $H_c(G(e, \sigma\widehat{\ }n)$ is a code for

$$f^{-1}(B(\langle T_\sigma, l_\sigma\rangle) = X \setminus f^{-1}(B(\langle T_{\sigma\widehat{\ }0}, l_{\sigma\widehat{\ }0}\rangle).$$

If $l(\sigma) = \langle 0, 1\rangle$, then $G(e, \sigma\widehat{\ }n)$ is a Borel code for $A_n = f^{-1}(B(\langle T_\sigma\widehat{\ }n, l_{\sigma\widehat{\ }n}\rangle)$. We choose j such that $\phi_j(n)$ is a code for A_n and $G(e, \sigma) = H_u(j)$ is a code for $\bigcup A_n$.

Theorem 7.7 follows from the next lemma.

Lemma 7.12 *If T is a recursive well founded tree, then there is a recursive function $G : T \to BC_{\text{rec}}$, such that $B_{\text{rec}}(G(\sigma)) = \{S \in Tr : \rho(S) \leq \rho(T_\sigma)\}$.*

Proof For $\sigma \in \mathbb{N}^{<\omega}$ let $f_\sigma : Tr \to Tr$ be the computable function $S \mapsto S_\sigma$.

Note that $\rho(S) \leq \rho(T)$ if and only if for all $n \in \mathbb{N}$ there is $m \in \mathbb{N}$ such that $\rho(S_{\langle n \rangle}) \leq \rho(T_{\langle m \rangle})$. Thus

$$\{S \in Tr : \rho(S) \leq \rho(T_\sigma)\} = \bigcap_{n \in \mathbb{N}} \bigcup_{m \in N} f_{\langle n \rangle}^{-1}(\{S \in Tr : \rho(S) \leq \rho(T_{\sigma \frown m})\}).$$

Fix c such that $B_{\mathrm{rec}}(c) = \emptyset$. We define a recursive function $g : \mathbb{N} \times T \to \mathbb{N}$ as follows.

i) If $\sigma \notin T$, then $g(i, \sigma) = c$.

ii) Otherwise $g(i, \sigma)$ is a Borel code for

$$\bigcap_n \bigcup_m f_{\langle n \rangle}^{-1}(B_{\mathrm{rec}}(\phi_i(\sigma \widehat{\ } m))).$$

We can do this using the functions F, H_u and H_c above. Of course for some i, this may well be undefined.

By the Recursion Theorem there is $\widehat{i}$ such that $\phi_{\widehat{i}}(\sigma) = g(\widehat{i}, \sigma)$ for all σ.

An easy induction shows that $G = \phi_{\widehat{i}}$ is the desired function.

Hyperarithmetic Sets

Definition 7.13 We say $x \in \mathcal{N}$ is *hyperarithmetic* if $x \in \Delta_1^1$. We say that x is *hyperarithmetic in* y, and write $x \leq_{\mathrm{hyp}} y$ if $x \in \Delta_1^1(y)$.

We sometimes let HYP denote the hyperarithmetic elements of $\mathcal{N}$.

Exercise 7.14 i) Show that if $x \leq_{\mathrm{hyp}} y \leq_{\mathrm{hyp}} z$, then $x \leq_{\mathrm{hyp}} z$.

ii) Show that if $x \leq_T y$, then $x \leq_{\mathrm{hyp}} y$.

Lemma 7.15 *i)* $\{(x, y) : x \leq_{\mathrm{hyp}} y\}$ *is* Π_1^1. *In particular,* $\{x : x \in \Delta_1^1\}$ *is* Π_1^1.

Proof $x \leq_{\mathrm{hyp}} y$ if and only if $\exists e\ (BC_{\mathrm{rec}}(e, y) \wedge \forall n \forall m\ (x(n) = m \leftrightarrow (n, m) \in BC_{\mathrm{rec}}(e, y)$.

This definition is Π_1^1.

Theorem 7.16 *Suppose* $A \subseteq \mathcal{N} \times \mathcal{N}$ *is* Π_1^1. *Then* $B = \{x : \exists y \leq_{\mathrm{hyp}} x\ (x, y) \in A\}$ *is* Π_1^1.

Proof $x \in B$ if and only if

$\exists e \in \mathbb{N}\ \forall z \in \mathcal{N}(\phi_e = z \to (z \in BC \wedge\ (\forall n \forall m\ ((y(n) = m \to S((n, m), z)) \wedge (y(n) \neq m \to \neg R((n, m), z)))) \wedge (x, y) \in A)$

This definition is Π_1^1.

We next give a refinement of Kleene's Basis Theorem.

Lemma 7.17 *If* $\omega_1^{\mathrm{ck}} < \omega_1^x$, *then* $O \leq_{\mathrm{hyp}} x$.

Proof Clearly O is $\Pi^1_1(x)$. There is T recursive in x such that $T \in \mathrm{WF}$ and $\rho(T) > \omega_1^{\mathrm{ck}}$. Then

$$O = \{e : e \text{ codes a recursive tree } S \text{ and } \rho(S) < \rho(T)\}$$

is $\Sigma^1_1(x)$. Thus $O \leq_{\mathrm{hyp}} x$.

Theorem 7.18 (Gandy's Basis Theorem) *If $A \subseteq \mathcal{N}$ is Σ^1_1 and nonempty, there is $x \in A$ such that $x \leq_T O$, $x <_{\mathrm{hyp}} O$ and $\omega_1^{\mathrm{ck}} = \omega_1^x$.*

Proof Let $B = \{(x, y) : x \in A \wedge y \not\leq_{\mathrm{hyp}} x\}$. By 7.15 B is Σ^1_1. By Kleene's Basis Theorem, there is $(x, y) \in B$ with $(x, y) \leq_T O$. If $O \leq_{\mathrm{hyp}} x$, then $y \leq_T O \leq_{\mathrm{hyp}} x$, so $y \leq_{\mathrm{hyp}} x$, a contradiction. Thus $y \leq_{\mathrm{hyp}} x$, a contradiction. By the previous lemma $\omega_1^{\mathrm{ck}} = \omega_1^x$.

The Effective Perfect Set Theorem

The following theorem is very important.

Theorem 7.19 (Harrison) *Let $A \subseteq \mathcal{N}$ be Σ^1_1. If A is countable, then every element of A is hyperarithmetic. In particular, if A contains a nonhyperarithmetic element, then A contains a perfect set.*

We delay the proof to the end of the section and look at some important corallaries.

Corollary 7.20 *Suppose $A \subseteq \mathcal{N} \times \mathcal{N}$ is Δ^1_1 and $\{y : (x, y) \in A\}$ is countable for all $x \in \mathcal{N}$. Then*
i) the projection $\pi(A) = \{x : \exists y\ (x, y) \in A\}$ is Δ^1_1 and
ii) A has a Δ^1_1-uniformization

Proof
i) Clearly $\pi(A)$ is Σ^1_1, but by Harrison's Theorem

$$\exists y\ (x, y) \in A \leftrightarrow \exists y \leq_{\mathrm{hyp}} x\ (x, y) \in A.$$

The later condition is Π^1_1.
ii) Let

$$A^* = \{(x, e) : e \in BC_{\mathrm{rec}}(x) \wedge \forall y(y = B_{\mathrm{rec}}(e, x) \rightarrow (x, y) \in A\}.$$

Then A^* is Π^1_1 and has a Π^1_1 uniformization B. But

$$(x, e) \notin B \Leftrightarrow x \notin \pi(A) \vee \exists i \neq e\ (x, i) \in B.$$

Thus B is Δ^1_1. Let

$$C = \{(x, y) : \exists e\ (x, e) \in B \wedge y = B_{\mathrm{rec}}(e, x)\}.$$

Then C is a Δ^1_1-uniformization of A.

Relativizing these corollaries lead to interesting results about Borel sets.

Corollary 7.21 *Suppose $A \subseteq \mathcal{N} \times \mathcal{N}$ is a Borel set such that every section is countable. Then X the projection of X is Borel and X can be uniformized by a Borel set.*

Corollary 7.22 *Suppose $f : \mathcal{N} \to \mathcal{N}$ is continuous, A is Borel and $f|A$ is one-to-one. Then $f(A)$ is Borel.*

Proof $f(A)$ is the projection of $\{(x, y) : x \in A \wedge f(x) = y\}$ and sections are singletons.

These results all have classical proofs, but in §8 we will give an example of an effective proof of a result where no classical proof is known.

Suppose $A \subseteq \mathcal{N}$ is Π^0_1. We consider the game $G(A)$ where Player I and II alternate playing $x(i) \in \mathbb{N}$ and Player I wins if $x \in A$.

Theorem 7.23 *If Player II has a winning strategy in $G(A)$, then Player II has a hyperarithmetic winning strategy.*

Proof Let T be a recursive tree such that $A = [T]$. Suppose Player II does not have a hyperarithmetic winning strategy. We will show that Player I has a winning strategy. Suppose $\sigma \in \mathbb{N}^{<\omega}$ and $|\sigma|$ is even. We consider the game $G_\sigma(A)$ where Players I and II alternate playing elements of $\mathbb{N}$ to build $x \in \mathcal{N}$ and Player I wins if $\sigma\widehat{\ }x \in A$.

Let $P = \{\sigma : |\sigma|$ is even and Player II has a hyperarithmetic winning strategy in $G_\sigma(A)\}$. If $\sigma \notin T$, then Player II has already won. In particular, always playing 0 is a hyperarithmetic winning strategy for Player II. Thus $\mathbb{N}^{<\omega} \setminus T \subseteq P$.

Claim Suppose that for all $n \in \mathbb{N}$ there is $m \in \mathbb{N}$ such that $\sigma\widehat{\ }n\widehat{\ }m \in P$. Then $\sigma \in P$.

Let $B = \{(n, m, e) : e$ is a hyperarithmetic code for τ and $\forall y$ if we play $G_{\sigma\widehat{\ }n\widehat{\ }m}(A)$ where Player I plays y and Player II plays using σ, then the result is in $A\}$. The set B is Π^1_1. and $\forall n \exists m \exists e (n, m, e) \in B$. By selection there is a Δ^1_1-function $f : \mathbb{N} \to \mathbb{N}^2$ such that $(n, f(n)) \in B$ for all $n \in \mathbb{N}$. Player II has a hyperarithmetic winning strategy in $G_\sigma(A)$; namely if Player I plays n and $f(n) = (m, e)$, then Player II plays m, and then uses the strategy coded by e.

We describe a winning strategy for Player I.

Since Player II does not have a hyperarithmetic winning strategy $\emptyset \notin P$. Player I's strategy is to avoid P. If we are in position σ where $\sigma \notin P$ and $|\sigma|$ is even, then by the claim there is a least n such that $\sigma\widehat{\ }n\widehat{\ }m \notin P$ forall m. Player I plays n. No matter what m Player II now plays the new postion is not in P. If Player I continues Playing playing this way they will play $x \in \mathcal{N}$ such that $x|2n \notin P$ for all n. In particular $x|2n \in T$ for all n. Thus $x \in [T]$ and this is a winning strategy for Player I.

Exercise 7.24 Suppose A is Π^0_1 and Player I has a winning strategy in $G(A)$. Then Player I has a winning strategy hyperaritmetic in O.

Proof of 7.19 Suppose A is Σ^1_1. We consider the unfolded game $G^*_u(A)$ from §5. This is a closed game and an argument similar to the one above shows that if Player II has a winning strategy, then there is a hyperarithmetic winning strategy τ. Then A is countable. The proof of 6.18 shows that every $x \in A$ is rejected at some position and $x \leq_T \tau$. Thus x is hyperarithmetic and $A \subseteq \mathrm{HYP}$.

Exercise 7.25 Show that if A is Σ^1_1 and uncountable, then there is a continuous injection $f : \mathcal{C} \to A$ with f computable in x for some $x \leq_{\mathrm{hyp}} O$.

Further Unifomization Results

We will use hyperarithmetic theory to prove several other classical unifomization results. We begin with a variant of Corollary 7.21.

Theorem 7.26 *Suppose $A \subseteq \mathcal{N} \times \mathcal{N}$ is a Borel set with countable sections. Then there are Borel measurable functions $f_0, f_1, \ldots$ with disjoint graphs such that A is the union of the graphs.*

Before proving this we need one lemma about hyperarithmetic sets. If $x \in \mathbb{N}^{\mathbb{N}^2}$ we identify x with $(x_0, x_1, \ldots)$ in $\mathcal{N}^{\mathbb{N}}$ where $x_n(m) = x(n, m)$.

Lemma 7.27 *Suppose A is a Δ^1_1-subset of* HYP. *There is a hyperarithmetic $x \in \mathbb{N}^{\mathbb{N}^2}$ such that $A \subseteq \{x_0, x_1, \ldots\}$.*

Proof Let

$B = \{(x, i) \in \mathcal{N} \times \mathbb{N} : x \in A \wedge i \in BC_{\mathrm{rec}} \wedge \forall n \forall m \, (x(n) = m \leftrightarrow (n, m) \in B_{\mathrm{rec}}(i))\}.$

Then B is Π^1_1 and $\pi(B) = A$. By selection, there is a Δ^1_1 function $s : A \to \mathbb{N}$, uniformizing B.

Let $C = \{i : \exists x \in A \ s(x) = i\}$. Clearly C is Σ^1_1. Since

$$i \in C \leftrightarrow \exists x \in \mathrm{HYP} \ (x \in A \wedge s(x) = i),$$

C is Δ^1_1. Let

$$x(i, n) = \begin{cases} 0 & \text{if } i \notin C \\ m & \text{if } i \in D \wedge (n, m) \in B_{\mathrm{rec}}(i) \end{cases}.$$

Then $A \subseteq \{x_0, x_1, \ldots\}$.

Exercise 7.28 Show that the same is true if A is Σ^1_1. [Hint: First show that any Σ^1_1 subset of HYP is contained in an Δ^1_1 subset of HYP.]

Proof of Theorem 7.26 By relativizing, we assume that A is Δ^1_1. Suppose $A \subseteq \mathcal{N} \times \mathcal{N}$ has countable sections. By the Effective Perfect Set Theorem, for any x the set $A_x = \{y : (x, y) \in X\}$ is a $\Delta^1_1(x)$ subset of $\{y : y \leq_{\mathrm{hyp}} x\}$. By relativising the lemma, there is $y \in \mathbb{N}^{\mathbb{N}^2}$ such that $y \leq_{\mathrm{hyp}} x$ and $A_x \subseteq \{y_0, y_1, \ldots\}$.

Let $B = \{(x, j) \in \mathcal{N} \times \mathbb{N} : j \in BC_{\mathrm{rec}}(x) \wedge \forall z \ ((x, z) \in A \to \exists n \forall m \forall k \ (z(m) = k) \leftrightarrow (n, m, k) \in B_{\mathrm{rec}}(x, j)\}$. Then B is Π^1_1 and $\pi(B) = \pi(A)$. By 7.21 $\pi(A)$ is Δ^1_1. Thus, by selection, there is a Δ^1_1 function $t : \pi(A) \to \mathbb{N}$ uniformizing B.

Let $g_n : \pi(A) \to \mathcal{N}$ be such that $g_n(x) = y$ if and only if

$$y(i) = j \Leftrightarrow (n, i, j) \in B_{\text{rec}}(x, t(x)).$$

Since g_n has a Δ^1_1-graph, it is Borel measurable and A is contained in the union of the graphs of the g_n.

Let

$$C_n = \left\{ x \in \pi(A) : (x, g_n(x)) \in A \wedge \bigwedge_{i=0}^{n-1} g_n(x) \neq g_i(x) \right\}$$

and let $f_n = g_n | C_n$. Each g_n is Borel measurable and A is the disjoint union of the graphs of the g_n.

We next uniformize Borel sets with compact sections. We begin with a lemma that compares two different measures of the complexity of a set. Suppose a set A is Δ^1_1 and open. There is no reason to believe A is Σ^0_1. For example if $W \subset \mathbb{N}$ is any Δ^1_1 set, then $A = \{x \in \mathcal{N} : x(0) \in W\}$ is Δ^1_1 and open, but need not be Σ^0_1. The next lemma shows while A is not Σ^0_1, A will be $\Sigma^0_1(x)$ for some $x \in \text{HYP}$.

Lemma 7.29 *Suppose $A \subseteq \mathcal{N}$ is Δ^1_1 and open. Then there is a hyperarithmetic $S \subseteq \mathbb{N}^{<\omega}$ such that $A = \bigcup_{\sigma \in S} N_\sigma$.*

Proof Let $S = \{\sigma : \forall x \; x \supset \sigma \to x \in A\}$. Then S is Π^1_1 and $A = \bigcup_{\sigma \in S} N_\sigma$. There is a computable $f : \mathbb{N}^{<\omega} \to Tr$ such that

$$x \in S \text{ if and only if } f(x) \in \text{WF}.$$

Let

$$B = \{\tau : \forall x \in A \exists \sigma \; \rho(f(\tau)) \not\leq \rho(f(\sigma)) \wedge \sigma \subset x\}.$$

Note that B is Π^1_1 and $\mathbb{N}^{<\omega} \setminus S \subseteq B$. If $B = \mathbb{N}^{<\omega} \setminus S$, then S is Δ^1_1, as desired. If not, there is $\tau \in S \cap B$. Then $S_0 = \{\sigma : f(\sigma) < f(\tau)\}$ is Δ^1_1 and

$$A = \bigcup_{\sigma \in S_0} N_\sigma \subseteq \bigcup_{\sigma \in S} N_\sigma = A.$$

If A is Δ^1_1 and closed, then there is a Δ^1_1-set $S \subseteq \mathbb{N}^{<\omega}$ with $\mathcal{N} \setminus A = \bigcup_{\sigma \in S} N_\sigma$. Let $T = \{\sigma \in \mathbb{N}^{<\omega} : \forall \tau \subseteq \sigma \; \tau \notin S\}$. Then T is a hyperarithmetic tree and $A = [T]$. If A is compact, we can go a bit further.

Lemma 7.30 *If A is Δ^1_1 and compact, then there is a finite branching hyperarithmetic tree T such that $A = [T]$. More generally, if A is a compact Σ^1_1-set, F is a closed Δ^1_1-set, and $A \subseteq F$, then there is a finite branching hyperarithmetic tree T such that $A \subseteq [T] \subseteq F$.*

Proof By the remarks above, there is a hyperarithmetic tree T such that $F=[T]$.

Let $B=\{(\sigma,C):\sigma\in\mathbb{N}^{<\omega}\wedge C\subset\mathbb{N}$ finite $\wedge\,\forall x\,((x\in A\wedge\sigma\subset x)\rightarrow$
$(\bigwedge_{i\in C}\sigma\widehat{\ }i\in T\wedge\bigvee_{i\in C}x\supset\sigma\widehat{\ }i)$.

Then B is Π^1_1. If $N_\sigma\cap A=\emptyset$, then $(\sigma,C)\in B$ for all finite $C\subset\mathbb{N}$. If $N_\sigma\cap A\neq\emptyset$, then, by compactness, there is a finite set $C\subset\mathbb{N}$ such that

$$A\cap N_\sigma=\bigcup_{i\in C}A\cap N_{\sigma\widehat{\ }i}.$$

Clearly $(\sigma,C)\in B$. By 5.28 there is a Δ^1_1-function f such that if $\sigma\in\mathbb{N}^{<\omega}$, then $(\sigma,f(\sigma))\in B$. Let

$$T_1=\{\sigma\in T:\bigwedge_{i<|\sigma|}\sigma(i)\in f(\sigma|i)\}.$$

Clearly $T_1\subseteq T$ is Δ^1_1 and finite branching. By choice of B, if $x\in A$, then $x\in[T_1]$.

Corollary 7.31 *If $A\subseteq\mathcal{N}$ is Δ^1_1, compact and nonempty, then there is $x\in A$ such that $x\in$ HYP.*

Proof There is a hyperarithmetic finite branching tree T such that $A=[T]$. By König's Lemma, if $\{\tau\in T:\sigma\subseteq\tau\}$ is infinite, then there is $x\in N_\sigma\cap A$. Let

$$T_2=\{\sigma\in T:\forall n>|\sigma|\;\exists\tau\;\sigma\subset\tau\wedge\tau\in T\wedge|\tau|=n\}.$$

Then T_2 is hyperarithmetic (indeed T_2 is arithmetic in T) and T_2 is pruned. Inductively define $x\in\mathcal{N}$ such $x(n)$ is least such that $\langle x(0),x(1),\dots,x(n)\rangle\in T_2$. Then x is recursive in T_2 and hence, hyperarithmetic.

Corollary 7.32 (Novikov) *If $A\subseteq\mathcal{N}\times\mathcal{N}$ is Δ^1_1 and all sections $A_x=\{y:(x,y)\in A\}$ are compact, then $\pi(A)$ is Δ^1_1 and there is a Δ^1_1 uniformization of X.*

In particular, any Borel $A\subseteq\mathcal{N}\times\mathcal{N}$ with compact sections has a Borel uniformization.

Proof Clearly $\pi(A)$ is Σ^1_1. By relativizing the previous corollary, we see that

$$x\in\pi(A)\Leftrightarrow\exists y\leq_{\rm hyp}x\;(x,y)\in A.$$

Hence $\pi(A)$ is Δ^1_1.

Let $B=\{(x,e):x\notin\pi(A)\vee(e,x)\in BC_{\rm rec}$ and if $y\in\mathcal{N}$ is coded by (e,x) then $(x,y)\in A\}$. Then B is Π^1_1 and by 5.28 there is a Δ^1_1 function f unifomizing B. The set $C=\{(x,y):x\in\pi(A)\wedge(x,f(x))$ codes $y\}$ is a Δ^1_1-uniformization of A.

We will prove one further generalization. We first need one topological result.

Exercise 7.33 Suppose $A \subseteq \mathcal{N}$ is closed, $f : \mathcal{N} \to \mathcal{N}$ is continuous and $f(A) \subseteq \bigcup F_n$ where each F_n is closed. There is $\sigma \in \mathbb{N}^{<\omega}$ and $n \in \mathbb{N}$ such that $A \cap N_\sigma \neq \emptyset$ and $f(A \cap N_\sigma) \subseteq F_n$. [Hint: Suppose not. Build $\sigma_0 \subset \sigma_1 \subset \ldots$ such that $A \cap N_{\sigma_i} \neq \emptyset$ and $f(N_{\sigma_{i+1}} \cap F_i) = \emptyset$. Consider $x = \bigcup \sigma_i$ to obtain a contradiction.]

Definition 7.34 We say that A is a K_σ set if it is a countable union of compact sets.

Since $\mathbb{R}^n$ is locally compact, in $\mathbb{R}^n$ every F_σ-set is a K_σ-set.

Lemma 7.35 *Suppose A is Δ^1_1 and a K_σ-set. Then there is $x \in A \cap \mathrm{HYP}$.*

Proof There is a Π^0_1-set $B \subseteq \mathcal{N} \times \mathcal{N}$ such that A is the projection of B. By the previous exercise, there is a basic open set N such

$$A_1 = \{x : \exists y\ (x, y) \in N \times B\}$$

is Σ^1_1 and contained in a closed subset of F of A. Then

$$\overline{A_1} = \{x : \forall \sigma\ (\sigma \subset x \to \exists y\ (y \in A_1 \wedge \sigma \subset y))\}$$

the closure of A_1 is also Σ^1_1 and contained in A.

Let $B = \{(x, \sigma) : x \notin A \wedge \sigma \subset x \wedge \forall y\ (y \supset \sigma \to y \notin A_1\}$. Then B is Π^1_1 and for all $x \notin A$ there is a σ such that $(x, \sigma) \in B$. By 5.28 there is a Δ^1_1 function f such that $(x, f(x)) \in B$ for all $x \notin A$.

Let $W_0 = \{f(x) : x \notin A\}$, let $W_1 = \{\sigma : \forall y \supset \sigma\ y \notin A_1\}$. Then W_0 is Σ^1_1, W_1 is Π^1_1 and $W_0 \subseteq W_1$. By Σ^1_1-separation, there is a Δ^1_1-set W such that $W_0 \subseteq W \subseteq W_1$. Let

$$T = \{\sigma : \forall \tau \subseteq \sigma : \tau \notin W\}.$$

Then T is a Δ^1_1 tree. Since $W \subseteq W_1$, $A_1 \subseteq [T]$. Since $W_0 \subseteq W$, $[T] \subseteq A$.

By 7.30 there is a finite branching tree $T_1 \in \Delta^1_1$ such that $T_1 \subset T$ and

$$\overline{A}_1 \subseteq [T_1] \subseteq [T] \subseteq A.$$

As in 7.31 there is $x \in [T_1] \cap \mathrm{HYP}$.

Corollary 7.36 (Aresenin, Kunugui) *If $A \subseteq \mathcal{N} \times \mathcal{N}$ is Borel and every section if K_σ, then $\pi(A)$ is Borel and A has a Borel uniformization.*

These proofs are a little unsatisfactory as we have only proved the uniform-results for $\mathcal{N} \times \mathcal{N}$ or more generally recursively presented Polish spaces (like $\mathbb{R}^n$). Since "compact" and "F_σ" are not preserced by Borel isomorphisms we can not immediately transfer these results to arbitrary Polish spaces. In fact these results are true in general (see [6] 35.46).

Exercise 7.37 a) Modify the proof of the Effective Perfect Set Theorem, using the Banach–Mazur game, to prove that if $A \subseteq \mathcal{N}$ is a nonmeager Δ^1_1-set, then there is a hyperarithmetic $x \in A$.

b) Prove that any Borel set with nonmeager sections can be uniformized by a Borel set.

Part II

Borel Equivalence Relations

The second half of these notes will be concerned with the descriptive set theory of equivalence relations.

Part of our interest in Descriptive Set Theory is motivated by Vaught's Conjecture. Suppose $\mathcal{L}$ is a countable language and T is an $\mathcal{L}$-theory. Let $I(T, \aleph_0)$ be the number of isomorphism classes of countable models of T.

Vaught's Conjecture If $I(T, \aleph_0) > \aleph_0$, then $I(T, \aleph_0) = 2^{\aleph_0}$.

Of course if the Continuum Hypothesis is true, Vaught's Conjecture is true. But perhaps it is provable in ZFC (though at the moment there is a manuscript with a plausible counterexample due to Robin Knight).

We have seen before that $\mathrm{Mod}(T)$ is a Polish space and $\cong$ is a $\mathbf{\Sigma}^1_1$-equivalence relation on $\mathrm{Mod}(T)$. A first hope would be to deduce Vaught's Conjecture from a perfect set theorem for $\mathbf{\Sigma}^1_1$-equivalence relations. This won't work. For example, consider the following equivalence relation on Tr.

$$T \sim S \Leftrightarrow \rho(S) = \rho(T)$$

Then $\sim$ is Σ^1_1. There is one equivalence class for all the ill-founded trees and then one for each possible value of ρ. Thus $\sim$ has exactly $\aleph_1$-equivalence classes.

We will see in §8, that while there is a perfect set theorem for $\mathbf{\Pi}^1_1$ (and hence Borel) equivalence relations and a weaker perfect set theorem for$\mathbf{\Sigma}^1_1$-equivalence relations.

The rest of the notes will be concerned with two special cases of $\mathbf{\Sigma}^1_1$-equivalence relations:

i) Borel Equivalence relations

ii) Orbit Equivalence realations, suppose G is a Polish group, X is a Borel set in a Polish space and $\mu : G \times X \to X$ is a continuous action of G on A. Let E_G be the equivalence relation xE_Gy if and only if there is $g \in G$ such that $gx = y$.

It is easy to see that the orbit equivalence relations E_G are $\mathbf{\Sigma}^1_1$. Of particular interest is the case where S_∞ acts on $\mathrm{Mod}(T)$. In this case E_G is the isomorphism equivalence relation on $\mathrm{Mod}(T)$.

The study of these equivalence relations is also tied up with the study of the dynamics of group actions and these ideas will also play a key role.

8 $\mathbf{\Pi}^1_1$-Equivalence Relations

Vaught's Conjecture would be true if it were true that every $\mathbf{\Sigma}^1_1$-equivalence relation with uncountably many classes has a perfect set of inequivalent elements. But the example above shows this is false. On the other hand, Silver proved that $\mathbf{\Pi}^1_1$-equivalence relations, and in particular Borel equivalence relations, are better behaved.

Theorem 8.1 (Silver's Theorem) *If X is a Polish space and E is a $\mathbf{\Pi}^1_1$-equivalence relation with uncountably many classes, then there is a nonempty perfect set P of inequivalent elements. In particular, if there are uncountably many classes, then there are $2^{\aleph_0}$ classes.*

Silver's original proof used heavy set-theoretic machinery. We will describe a proof given by Harrington that uses the effective descriptive set theory developed in §7.

To warm up we prove the following result that illustrates a key idea of Harrington's proof.

Proposition 8.2 *Suppose E is an equivalence relation on $\mathcal{N}$ and there is a nonempty open set U such that $E \cap (U \times U)$ is meager. Then there is a nonempty perfect set P of E-inequivalent elements.*

Proof

Let $E \cap U \times U = \bigcup A_n$ where each A_n is nowhere dense.

We build $(U_\sigma : \sigma \in 2^{<\omega})$ nonempty basic clopen sets such that:

i) $U_\emptyset \subseteq U$;

ii) $U_\sigma \subseteq U_\tau$ for $\sigma \subseteq \tau$;

iii) diam $U_\sigma < \frac{1}{|\sigma|}$;

iv) if $|\sigma| = |\tau| = n$ and $\sigma \neq \tau$, then $E \cap (U_\sigma \cap U_\tau) \cap (A_0 \cup \ldots \cup A_{n-1}) = \emptyset$.

For $f \in \mathcal{C}$, let $x_f = \bigcap U_{f|n}$. By contstruction if $f \neq g$, then $x_f \not\!\!E\, x_g$. Thus there is a perfect set of E-inequivalent elements.

We choose $U_\emptyset$ an nonempty basic clopen subset of U.

Suppose we have constructed U_σ for all σ with $|\sigma| = n$ satisfying i)–iv). Let $\{(\sigma_i, \tau_i) : i = 1, \ldots k\}$ list all pairs of distinct sequences of length $n+1$. If $|\sigma| = n+1$ we inductively define U^i_σ for $i = 0, \ldots k$.

Let $U^0_\sigma = U_{\sigma|n}$.

If $\sigma \neq \sigma_i$ and $\sigma \neq \tau_i$, then $U^{i+1}_\sigma = U^i_\sigma$. Otherwise, since $A_0 \cup \ldots \cup A_n + 1$ is nowhere dense in $U \times U$. We can find basic clopen $U^{i+1}_{\sigma_i} \subseteq U^i_{\sigma_i}$ and $U^{i+1}_{\tau_i} \subseteq U^i_{\tau_i}$ such that

$$E \cap \left(U^{i+1}_{\sigma_i} \times U^{i+1}_{\tau_i}\right) = \emptyset.$$

Choose U_σ a basic closed subset of U^k_σ of diameter less that $\frac{1}{|\sigma|}$.

While the argument above will be the model for our proof of Silver's theorem, there are some significant obstacles. First and foremost, if E is a $\mathbf{\Pi}^1_1$-equivalence relation, there is no reason to believe that there is an open set U such that $E \cap (U \times U)$ is meager. Harrington's insight was to change the topology so that that this is true.

Gandy Topology

Definition 8.3 The *Gandy topology* on $\mathcal{N}$ is the smallest topology in which every Σ^1_1-set is open.

We let τ_G denote the Gandy topology on $\mathcal{N}$.

Since there are only countably many Σ^1_1-sets, the Gandy topology has a countable basis. As we will be considering meager and comeager sets in the Gandy topology, we first note that the Baire Category Theorem is still true for τ_G.

Proposition 8.4 *If $U \subseteq \mathcal{N}$ is nonempty ant τ_G-open, then U is not τ_G-meager.*

Proof

Suppose $A_0, A_1, \ldots$ are τ_G-nowhere dense subsets of U. It is easy to construct $U = U_0 \supseteq U_1 \supseteq U_2 \supseteq \ldots$ a sequence of nonempty Σ^1_1-sets such that $U_n \cap A_n = \emptyset$. To prove the Lemma we need only do this in such a way that $\bigcap U_n \neq \emptyset$. This will require a bit more work.

At stage s of the construction we will have:

i) nonempty Σ^1_1-sets $U = U_0 \supseteq U_1 \supseteq U_2 \ldots \supseteq U_s$ such that $U_i \cap A_i = \emptyset$ for all $i \leq s$;

ii) recursive trees $T_0, T_1, \ldots T_s$ such that $U_i = \{x : \exists y\ (x, y) \in [T_i]\}$;

iii) sequences $\sigma_0 \subset \sigma_1 \ldots \subset \sigma_s$ such that $U_i \subset N_{\sigma_i}$ for all i;

iv) sequences η^i_j for $i \leq j \leq s$ such that $\eta^i_i \subset \eta^i_{i+1} \subset \eta^i_s$ and there is (x, y) such that $\sigma_s \subset x$, $\eta^i_s \subset y$ and $(x, y) \in [T_i]$ for all $i \leq s$.

Suppose we have done this. Let $x = \bigcup_{n \in \mathbb{N}} \sigma_n$ and $y_i = \bigcup_{n \geq i} \eta^i_n$. Then $(x, y_i) \in [T_i]$ for all i. Hence $x \in \bigcap U_n$ and $x \notin \bigcup A_n$.

At stage 0 we let $U_0 = U$ and $\sigma_0 = \eta^0_0 = \emptyset$.

At stage $s+1$ let

$$W = \{x \in U_s : x \supset \sigma_s \wedge \exists y_0 \ldots \exists y_s \bigwedge_{i=0}^{s} (x, y_i) \in [T_i] \wedge \bigwedge_{i \leq s} y_i \supset \eta^i_s\}.$$

Then W is a nonempty Σ^1_1-subset of U_s. Since A_{s+1} is τ_G-nowhere dense, there is $V \subset W$ a nonempty Σ^1_1-set such that $V \cap A_{s+1} = \emptyset$. Let $v \in V$. Choose $\sigma_{s+1} \supset \sigma_s$ such that $v \supset s$. Let $U_{s+1} = V \cap N_{\sigma_{s+1}}$. Let T_{s+1} be a recursive tree such that U_{s+1} is the projection of T_{s+1}. For $i \leq s+1$ choose z_i such that $(v, z_i) \in [T_i]$ and $\eta^i_s \subset z_i$ and let $\eta^{s+1}_{s+1} = \emptyset$. These choices satisfy i)–iv).

We will use the fact proved in §3 that in any topological space with a countable basis, the Baire Property is preserved by the Souslin operation.

Let τ^n_G denote the Gandy topology on $\mathcal{N}^n$. Since there is a computable bijection between $\mathcal{N}^n$ and $\mathcal{N}$, $(\mathcal{N}, \tau_G)$ and $(\mathcal{N}^n, \tau^n_G)$ are homeomorphic topological spaces. We have to be a little bit careful here since, for example, τ^2_G is not the τ_G-product topology. We let $\tau^{k,l}_G$ denote the product of $(\mathcal{N}^k, \tau^k_G)$ and $(\mathcal{N}^l, \tau^l_G)$. The topology τ^{k+l}_G refines the topology $\tau^{k,l}_G$.

Exercise 8.5 Modify the proof of 8.4 to show each topology $\tau_G^{k,l}$ also satisfies the Baire category theorem.

We will need the following technical lemma. Let $A \subseteq \mathcal{N}^2$ and let $A^* = \{(x,y,z) \in \mathcal{N}^3 : (x,z) \in A\}$.

Lemma 8.6 *If A is $\tau_G^{1,1}$-nowhere dense, then A^* is $\tau_G^{2,1}$-nowhere dense.*

Proof Suppose $B \subseteq \mathcal{N}^2$ and $C \subseteq \mathcal{N}$ are Σ_1^1-sets. Let $B_1 = \{x : \exists y\ (x,y) \in B\}$. Then B_1 is Σ_1^1. Since A is $\tau_G^{1,1}$-nowhere dense, there are $B_2 \subseteq B_1$ and $C_2 \subseteq C$ nonempty such that $A \cap (B_2 \times C_2) = \emptyset$. Let $B' = \{(x,y) \in B : x \in B_2\}$. Then B' is nonempty and $(B' \times C_2) \cap A^* = \emptyset$.

Harrington's Proof

We will prove Silver's Theorem for Π_1^1-equivalence relations. The proof will easily relativize to $\mathbf{\Pi}_1^1$-equivalence relations.

Suppose E is a Π_1^1-equivalence relation on $\mathcal{N}$ with uncountably many equivalence classes. We say that $A \subseteq \mathcal{N}$ is *E-small* if xEy whenever $x, y \in A$. Let

$$U = \{x : \text{ there is no } E\text{-small } \Sigma_1^1\text{-set } A \text{ with } x \in A\}.$$

Since there are only countably many Σ_1^1-sets, U is non-empty.

Lemma 8.7 *If $x \notin U$, then there is an E-small Δ_1^1-set A with $x \in A$.*

Proof There is an E-small Σ_1^1-set B such that $x \in B$ and

$$yEx \Leftrightarrow \forall z\ (z \in B \rightarrow zEy).$$

Thus the E-class of x is Π_1^1. By Σ_1^1-separation there is a Δ_1^1 set A such that

$$B \subseteq A \subseteq \{y : yEx\}.$$

Hence A is an E-small Δ_1^1-set containing x.

Corollary 8.8 *U is Σ_1^1.*

Proof $x \in U$ if and only if

$$\forall e\ ((e \in BC_{\rm rec} \wedge x \in B_{\rm rec}(e)) \rightarrow \exists y \exists z\ (y, z \in B_{\rm rec}(e) \wedge x \not{E} y).$$

This is a Π_1^1-definition of U.

We now show the connection to our "warm up" argument.

Lemma 8.9 *$E \cap (U \times U)$ is $\tau_G^{1,1}$-meager.*

Proof

We first argue that E has the Baire property in the $\tau_G^{1,1}$-topology.

Claim If $A \subseteq \mathcal{N}$ is $\boldsymbol{\Sigma}_1^1$, then there are basic open sets $(B_\sigma : \sigma \in \mathbb{N}^{<\omega})$ such that $A = \mathcal{A}(B_\sigma)$.

Let $T \subseteq \mathbb{N}^{<\omega} \times \mathbb{N}^{<\omega}$ be a tree such that $A = \{x : \exists y\ (x, y) \in [T]\}$. Let $B_\sigma = \{x : (x||\sigma|, \sigma) \in T\}$. Then

$$A = \bigcup_{y \in \mathcal{N}} \bigcap_{n \in \mathbb{N}} B_{y|n} = \mathcal{A}(B_\sigma).$$

The basic open sets of $\mathcal{N} \times \mathcal{N}$ are open in the topology $\tau_G^{1,1}$, and hence have the Baire Property, in this topology. Since the Souslin operator preserves the Baire Property, every $\boldsymbol{\Sigma}_1^1$ subset of $\mathcal{N} \times \mathcal{N}$ has the Baire Property in the $\tau_G^{1,1}$-topology.

Suppose for purposes of contradiction that $E \cap (U \times U)$ is $\tau_G^{1,1}$-nonmeager. Since E has the Baire property there are nonempty Σ_1^1-sets $A, B \subseteq U$, such that E is $\tau_G^{1,1}$-comeager in $A \times B$.

Let $A_1 = \{(x_0, x_1) \in A \times A : x_0 \not\!E\ x_1\}$. Since $A \subseteq U$ is a nonempty Σ_1^1-set, A is not E-small. Thus A_1 is a nonempty Σ_1^1-set.

Let $C_i = \{(x_0, x_1, y) : (x_0, x_1) \in A_1, y \in B, x_i \not\!E y\}$, for $i = 0, 1$.

Claim C_i is $\tau_G^{2,1}$-meager.

Since E is $\tau_G^{1,1}$-comeager in $A \times B$, there are $D_0, D_1, \ldots$ $\tau_G^{1,1}$-nowhere dense, such that

$$\bigcup D_n = \{(x, y) \in A \times B : x \not\!E y\}.$$

By Lemma 8.6, $D_n' = \{(x_0, x_1, y) : (x_0, y) \in D_n\}$ is $\tau_G^{2,1}$-nowhere dense, and $C_i \subseteq \bigcup D_n'$ is $\tau_G^{2,1}$-meager.

Since $\tau_G^{2,1}$ satisfies the Baire Category Theorem, There is

$$(x_0, x_1, y) \in (A_1 \times B) \setminus (C_0 \cup C_1).$$

But then $x_0 E y$, $x_1 E y$ and $x_0 \not\!E x_1$, a contradiction.

We now proceed as in our "warm up" to construct a perfect set of E-inequivalent elements. We need to exercise a little care—as in the proof of the Baire Category Theorem—to ensure that $\bigcap_n U_{f|n}$ are nonempty.

Let $A_0, A_1, \ldots$ be $\tau_G^{1,1}$-nowhere dense such that $\bigcup A_i = E \cap (U \times U)$. Let $T \subseteq \mathbb{N}^{<\omega} \times \mathbb{N}^{<\omega}$ be a tree such that $U = \{x : \exists y\ (x, y) \in [T]\}$.

We construct a family $(U_\sigma : \sigma \in 2^{<\omega})$ of nonempty Σ_1^1-sets such that:

i) $U_\emptyset \subseteq U$;

ii) $U_\sigma \subseteq U_\tau$ for $\sigma \subseteq \tau$;

iii) if $|\sigma| = |\tau| = n$ and $\sigma \neq \tau$, then $(U_\sigma \times U_\tau) \cap (A_0 \cup \ldots \cup A_{n-1}) = \emptyset$.

As in the proof of the Baire Category Theorem for τ_G, we also need to take extra measures to insure that $\bigcap U_{f|n} \neq \emptyset$. We construct $(T^\sigma : \sigma \in 2^{<\omega})$, $(\mu_\sigma : \sigma \in 2^{<\omega})$, and $(\eta_\tau^\sigma : \sigma \subseteq \tau \in 2^{<\omega})$ such that:

iv) each $T^\sigma \subset \mathbb{N}^{<\omega} \times \mathbb{N}^{<\omega}$ is a recursive tree such $U_\sigma = \{x : \exists y\ (x,y) \in [T^\sigma]\}$ and $T^\sigma \supseteq T^\tau$ for $\sigma \subseteq \tau$;

v) $\mu_\sigma \in \mathbb{N}^{<\omega}$, $\mu_\sigma \subset \mu_\tau$ for $\sigma \subset \tau$, and $U_\sigma \subseteq N_{\mu_\sigma}$;

vi) $\eta^\sigma_\tau \in \mathbb{N}^{<\omega}$, $\eta^\sigma_{\tau_0} \subset \eta^\sigma_{\tau_1}$ if $\sigma \subseteq \tau_0 \subset \tau_1$, and

$$\exists x \exists y\ x \supset \mu_\tau \wedge y \supset \eta^\sigma_\tau \wedge (x,y) \in [T^\sigma].$$

Exercise 8.10 Finish the proof of Silver's Theorem by showing:

a) if E is a Π^1_1-equivalence relation with uncountably many classes, then, taking U as above we can find $(U_\sigma : \sigma \in 2^{<\omega})$, $(T^\sigma : \sigma \in 2^{<\omega})$, $(\mu_\sigma : \sigma \in 2^{<\omega})$, and $(\eta^\sigma_\tau : \sigma \subseteq \tau \in 2^{<\omega})$ satisfying i)–vi).

b) if we let $x_f = \bigcup \mu_{f|n}$, then $P = \{x_f : f \in \mathcal{C}\}$ is a perfect set of E-inequivalent elements.

Harrington's original proof used forcing rather than the category argument given above. We sketch the main idea.

Exercise 8.11 Let $\mathcal{P} = \{A : A \in \Sigma^1_1, A \neq \emptyset\}$.

a) If $G \subseteq \mathcal{P}$ is sufficiently generic, then there is $x \in \mathcal{N}$ such that

$$\{x\} = \bigcap \{A : A \in G\}.$$

[Hint: This just the Baire Category Theorem for τ_G.]

b) For $x \in \mathcal{N}$ let $x = \langle x_0, x_1 \rangle \in \mathcal{N}^2$. If b is sufficiently generic, the so are b_0 and b_1.

c) Let U be as in the proof above. If (a,b) are sufficiently $\mathcal{P} \times \mathcal{P}$ generic below (U,U), then $a \not{E} b$.

d) There is a perfect set of mutually sufficiently $\mathcal{P} \times \mathcal{P}$-generic elements below (U,U).

e) Conclude Silver's Theorem.

$\mathbf{\Sigma}^1_1$-Equivalence Relations

While Silver's Theorem can not be generalized to $\mathbf{\Sigma}^1_1$-equivalence relations, Burgess showed that it can be used to prove the following result.

Theorem 8.12 (Burgess' Theorem) *If X is a Polish space and E is a $\mathbf{\Sigma}^1_1$-equivalence relation with at least $\aleph_2$ equivalence classes, then there is a perfect set of inequivalent elements.*

Suppose E is a $\mathbf{\Sigma}^1_1$-equivalence relation. There is a continuous function $f : X \times X \to Tr$ such that xEy if and only if $f(x,y) \notin \mathrm{WF}$.

For $\alpha < \omega_1$, let $E_\alpha = \{(x,y) : \rho(f(x,y) \geq \alpha\}$. Then E_α is Borel, $E_\alpha \supseteq E_\beta$ for $\alpha < \beta$, $E_\alpha = \bigcap_{\beta<\alpha} E_\beta$ for α a limit ordinal, and $E = \bigcap\limits_{\alpha<\omega_1} E_\alpha$.

Let $A = \{\alpha < \omega_1 : E_\alpha$ is an equivalence relation$\}$.

Lemma 8.13 *A is a closed unbounded subset of ω_1.*

Proof Suppose $\alpha_0 < \alpha_1 < \ldots$, $\alpha_i \in A$ and $\alpha = \sup \alpha_i$. Since each E_{α_i} is reflexive, symmetric and transitive so is E_α. Thus A is closed.

Claim 1 For all $\alpha < \omega_1$, there is $\beta < \omega_1$, such that if $x \not\!E_\alpha y$, then $y \not\!E_\beta x$ for all $x, y \in X$.

Since E is an equivalence relation and $E_\alpha \supseteq E$, if $x \not\!E_\alpha y$, then $y \not\!E x$. Let $B = \{f(y,x) : x \not\!E_\alpha y\}$. Then B is a $\mathbf{\Sigma}^1_1$ subset of WF. Thus by $\mathbf{\Sigma}^1_1$-bounding, there is $\beta < \omega_1$, such that if $T \in B$, then $\rho(T) < \beta$. Thus if $x \not\!E_\alpha y$, then $y \not\!E_\beta x$.

Claim 2 For all $\alpha < \omega_1$, there is $\beta < \omega_1$, such that if $xE_\alpha y$, $yE_\alpha z$ and $x \not\!E_\alpha z$, then $x \not\!E_\beta y$ or $y \not\!E_\beta z$.

Let $C = \{T : \exists x, y, z\; xE_\alpha y \wedge yE_\alpha z \wedge x \not\!E_\alpha z \wedge \rho(T) \leq f(x,y) \wedge \rho(T) \leq f(y,z)\}$.

Then C is a $\mathbf{\Sigma}^1_1$-set of well-founded trees. Thus there is $\beta < \omega_1$ such that $\rho(T) < \beta$ for all $T \in C$. If $xE_\alpha y$, $yE_\alpha z$ and $x \not\!E_\alpha z$, then either $f(x,y) \in C$ or $f(y,z) \in C$. Thus either $x \not\!E_\beta y$ or $y \not\!E_\beta z$.

Let $g, h : \omega_1 \to \omega_1$ such that $g(\alpha)$ is the least $\beta < \omega_1$ such claim 1 holds and $h(\alpha)$ is the least $\beta < \omega_1$ such that claim 2 holds.

Given $\alpha < \omega_1$, build $\alpha_0 < \alpha_1 < \ldots < \omega_1$ such that $\alpha_0 = \alpha$ and $\alpha_{i+1} > h(\alpha_i), g(\alpha_i)$. Let $\beta = \sup \alpha_i$.

Clearly $xE_\beta x$ for all x (since this is true of E).

If $x \not\!E_\beta y$, then $x \not\!E_{\alpha_i} y$ for some i, thus $y \not\!E_{\alpha_{i+1}} x$. Thus $y \not\!E_\beta x$.

Suppose $xE_\beta y$ and $yE_\beta z$. We claim $xE_\beta z$. Suppose not. Then $x \not\!E_{\alpha_i} z$ for some i. But then $x \not\!E_{\alpha_{i+1}} y$ or $y \not\!E_{\alpha_{i+1}} z$. Hence $x \not\!E_\beta y$ or $y \not\!E_\beta z$, a contradiction.

Thus for all $\alpha < \omega_1$, there is $\beta \geq \alpha$ with $\beta \in A$. Thus A is unbounded.

We can inductively define $\phi : \omega_1 \to A$ such that:
i) $\phi(0) \in A$;
ii) $\phi(\alpha + 1) > \phi(\alpha)$ for all $\alpha < \omega_1$;
iii) $\phi(\alpha) = \sup_{\beta<\alpha} \phi(\beta)$ for $\alpha < \omega_1$ a limit ordinal.

Corollary 8.14 *If E is a $\mathbf{\Sigma}^1_1$-equivalence relation, there is a sequence $(E_\alpha : \alpha < \omega_1)$ of Borel equivalence relations such that:*
i) $E_\alpha \supseteq E_\beta$ for $\alpha < \beta$;
ii) $E_\alpha = \bigcap_{\beta<\alpha} E_\beta$, for α a limit ordinal;
iii) $E = \bigcap_{\alpha<\omega_1} E_\alpha$.

Proof Let $E'_\alpha = E_{\phi(\alpha)}$. Then the sequence E'_α has the desired properties.

We are now ready to prove Burgess' Theorem. Suppose E is a $\mathbf{\Sigma}^1_1$-equivalence relation with at least $\aleph_2$ equivalence classes. Let E_α be a sequence of Borel equivalence relations such that $E_\beta \subseteq E_\alpha$ for $\alpha < \beta$ and $E = \bigcap E_\alpha$. By Silver's Theorem, if any E_α has uncountably many classes, then there is a perfect set of E-inequivalent elements. Thus we will assume that each E_α has only countably many classes.

Lemma 8.15 *Suppose $A \subseteq \mathcal{N}$ contains at least $\aleph_2$ E-classes. Then there is $\alpha < \omega_1$ and $a \in A$ such that $\{x \in A : xE_\alpha a\}$ and $\{x \in A : x \not\!E_\alpha\ a\}$ each contain at least $\aleph_2$ elements. In particular there is $b \in A$ such that $b \not\!Ea$ and $\{x \in A : x \not\!E_\alpha\ b\}$ also contains at least $\aleph_2$ E-classes*

Proof Since each E_α has only countably many equivalence classes. For each $\alpha < \omega_1$ we can find $a_\alpha \in A$ such that $\{x \in A : xE_\alpha a_\alpha\}$ represents at least $\aleph_2$ E-inequivalent elements. If the lemma is false, then $\{x \in A : x \not\!E_\alpha\ a_\alpha\}$ represents at most $\aleph_1$-inequivalent elements. Thus

$$B = \bigcup_{\alpha<\omega_1} \{x \in A : x \not\!E_\alpha a_\alpha\}$$

represents at most $\aleph_1$-equivalence classes. But if $x, y \in A \setminus B$, then $xE_\alpha y$ for all $\alpha < \omega_1$ and xEy. Thus A represents at most $\aleph_1$ E-classes, a contradiction.

To finish the proof of Burgess' Theorem we use 8.15 to build $(U_\sigma : \sigma \in 2^{<\omega})$ such that:

i) U_σ is a nonempty $\mathbf{\Sigma}^1_1$-set with $U_\sigma \subseteq U_\tau$ for $\sigma \subseteq \tau$ such that U_σ represents at least $\aleph_2$ E-classes;

ii) if $x \in U_{\sigma\frown 0}$ and $y \in U_{\sigma\frown 1}$, then $x \not\!Ey$;

If $f, g \in \mathcal{C}$, $x \in \bigcap U_{f|n}$ and $y \in \bigcap U_{g|n}$, and m is least such that $f(m) \neq g(m)$ then $x \not\!E_{\alpha_m}\ y$ and $x \not\!E\ y$. This would suffice if we knew that the $\bigcap U_{f|n} \neq \emptyset$ for $f \in \mathcal{C}$. We can insure this as in the proof of Silver's Theorem.

We also build $(T^\sigma : \sigma \in 2^{<\omega})$, $(\mu_\sigma : \sigma \in 2^{<\omega})$, and $(\eta^\sigma_\tau : \sigma \subseteq \tau \in 2^{<\omega})$ such that:

iii) each $T^\sigma \subset \mathbb{N}^{<\omega} \times \mathbb{N}^{<\omega}$ is a tree such $U_\sigma = \{x : \exists y\ (x, y) \in [T^\sigma]\}$ and $T^\sigma \supseteq T^\tau$ for $\sigma \subseteq \tau$;

iv) $\mu_\sigma \in \mathbb{N}^{<\omega}$, $\mu_\sigma \subset \mu_\tau$ for $\sigma \subset \tau$, and $U_\sigma \subseteq N_{\mu_\sigma}$;

v) $\eta^\sigma_\tau \in \mathbb{N}^{<\omega}$, $\eta^\sigma_{\tau_0} \subset \eta^\sigma_{\tau_1}$ if $\sigma \subseteq \tau_0 \subset \tau_1$;

vi) for each τ

$$V_\tau = \{x \in U_\tau : x \supset \mu_\tau \wedge \bigwedge_{\sigma\subseteq\tau} \exists y \supset \eta^\tau_\sigma\ (x, y) \in [T^\sigma]\}$$

represents at least $\aleph_2$ E-clases. In particular $(\mu_\tau, \eta^\sigma_\tau) \in T^\sigma$.

Suppose we have done this. For $f \in \mathcal{C}$ let $x_f = \bigcup_n \mu_{f|n}$. We claim that $x_f \in \bigcap_n U_{f|n}$. Let $\sigma = f|n$. Then $(\mu_\tau, \eta^\sigma_\tau) \in T^\sigma$ for all $\tau \supseteq \sigma$. If $y = \bigcup_{\tau\supseteq\sigma} \eta^\sigma_\tau$, then $(x_f, y) \in [T^\sigma]$. Thus $x_f \in U_\sigma$. Thus $\{x_f : f \in \mathcal{C}\}$ is a perfect set of E-inequivalent elements.

We next sketch how to do the construction. Suppose we have defined U_τ, μ_τ and $(\eta^\sigma_\tau : \sigma \subseteq \tau)$ such that i)–vi) hold.

Since V_τ represents at least $\aleph_2$ E-classes, by Lemma 8.15 there is $\alpha < \omega$ and $a_0, a_1 \in V_\tau$ such that $a_0 \not\!E_\alpha a_1$ and $W_i = \{x \in V_\tau : xE_\alpha a_i\}$ represents at least $\aleph_2$ E-classes for $i = 0, 1$.

If $j \in \mathbb{N}$ and $\xi = (k_\sigma : \sigma \subseteq \tau)$ where each $k_\sigma \in \mathbb{N}$ let

$$W_i^{j,\xi} = \{x \in W_i : x \supseteq \mu_\tau\widehat{\ }j \wedge \bigwedge_{\sigma \subseteq \tau} \exists y \supseteq \eta_\tau^\sigma\widehat{\ }k_\sigma\ (x,y) \in [T^\sigma]\}.$$

Since

$$W_i = \bigcup_{j,\xi} W_i^{j,\xi}$$

some $W_i^{j,\xi}$ must represent at least $\aleph_2$ E-classes.

Let $U_{\tau\widehat{\ }i} = W_i^{j,\xi}$. Since $U_{\tau\widehat{\ }i}$ is $\mathbf{\Sigma}_1^1$ conditions i) and ii) are satisfied. Let $\mu_{\tau\widehat{\ }i} = \mu_\tau\widehat{\ }j$, let $\eta^\sigma_{\tau\widehat{\ }i} = \eta_\tau^\sigma\widehat{\ }k_\sigma$ for $\sigma \subseteq \tau$. Let $T^{\tau\widehat{\ }i} \subseteq T^\tau$ be a tree such that $U_{\tau\widehat{\ }i} = \{x : \exists y\ (x,y) \in T^{\tau\widehat{\ }i}\}$ and let $\eta_\tau^\tau = \emptyset$. Our choice of j, ξ insures that i)–vi) hold.

Burgess' Theorem has an important model theoretic corollary.

Corollary 8.16 (Morley's Theorem) *If $\mathcal{L}$ is a countable language and T is an $\mathcal{L}$-theory such that $I(T,\aleph_0) > \aleph_1$, then $I(T,\aleph_0) = 2^{\aleph_0}$. Indeed if ϕ is an $\mathcal{L}_{\omega_1,\omega}$-sentence and $I(\phi,\aleph_0) > \aleph_1$, then $I(\phi,\aleph_0) = 2^{\aleph_0}$.*

Morley's original proof uses the Perfect Set Theorem for $\mathbf{\Sigma}_1^1$-sets, but does not use Silver's Theorem. His proof is given in [11] §4.4.

Using Scott's analysis of countable models (see [11] §2.4) it is easy to see that isomorphism is an intersection of $\aleph_1$ Borel equivalence relations.

If $\mathcal{M}$ and $\mathcal{N}$ are countable $\mathcal{L}$-structures $\overline{a} \in M^n$ and $\overline{b} \in N^n$ we define $(\mathcal{M},\overline{a}) \sim_\alpha (\mathcal{N},\overline{b})$ as follows:

$(\mathcal{M},\overline{a}) \sim_0 (\mathcal{N},\overline{b})$ if and only if $\mathcal{M} \models \phi(\overline{a})$ if and only if $\mathcal{N} \models \phi(\overline{b})$ for all quantifier free formulas;

$(\mathcal{M},\overline{a}) \sim_{\alpha+1} (\mathcal{N},\overline{b})$ if and only if for all $c \in M$ there is $d \in N$ such that $(\mathcal{M},\overline{a},c) \sim_\alpha (\mathcal{N},\overline{b},d)$ and for all $d \in N$ there is $c \in M$ such that $(\mathcal{M},\overline{a},c) \sim_\alpha (\mathcal{N},\overline{b},d)$;

if α is a limit ordinal, then $(\mathcal{M},\overline{a}) \sim_\alpha (\mathcal{N},\overline{b})$ if and only if $(\mathcal{M},\overline{a}) \sim_\beta (\mathcal{N},\overline{b})$ for all $\beta < \alpha$.

Exercise 8.17 Prove that $\sim_\alpha$ is a Borel equivalence relation on $\mathrm{Mod}(\mathcal{L})$.

Proposition 8.18 *If $\mathcal{M}$ a countable $\mathcal{L}$-structure, there is $\alpha < \omega_1$ such that if $\mathcal{N}$ is countable and $\mathcal{M} \sim_\alpha \mathcal{N}$, then $\mathcal{M}$ and $\mathcal{N}$ are isomorphic.*

For a proof see [11] 2.4.15. It follows that $\bigcap_{\alpha<\omega_1} \sim_\alpha$ is the isomorphism equivalence relation. This proposition makes isomophism easier to analyze than general $\mathbf{\Sigma}_1^1$-equivalence relations. In particular for any $\mathcal{M}$ there is an α such that the $\sim_\alpha$-class of $\mathcal{M}$ is the isomorphism class. This makes the counting argument much easier.

Exercise 8.19 Give an example of a $\mathbf{\Sigma}_1^1$-equivalance relation E on a Polish space X and $x \in E$ such that if $E = \bigcap_{\alpha<\omega_1} E_\alpha$ where each E_α is a Borel equivalence relation, then for all $\alpha < \omega_1$ there is $y \in X$ such that $xE_\alpha y$ and $x \not\!E y$.

9 Tame Borel Equivalence Relations

In this section we will look at some general results about Borel equivalence relations. Let X be a Polish space and let E be a Borel equivalence relation on X. If $x \in X$ we let $[x]$ denote the equivalence class of X.

We start by looking at some of the simpler Borel equivalence relations.

Definition 9.1 $T \subseteq X$ is a *transversal* for E if

$$|T \cap [x]| = 1$$

for all $x \in X$.

We say that $s : X \to X$ is a *selector* for E if $s(x)Ex$ for all $x \in X$ and $s(x) = s(y)$ if xEy.

For example, let X be the set of all $f : \mathbb{N} \to \mathbb{R}$ such that f is Cauchy and let E be the equivalence relation

$$fEg \Leftrightarrow \forall n \exists m \forall k > m \ |f(k) - g(k)| < \frac{1}{n}.$$

Then the set T of constant sequences is a Borel transversal.

Lemma 9.2 *Let E be a Borel equivalence relation on a Polish space X. Then E has a Borel transversal if and only if E has a Borel-measurable selector.*

Proof

($\Rightarrow$) If T is a Borel transversal, let

$$s(x) = y \Leftrightarrow y \in T \text{ and } xEy.$$

Since the graph of s is Borel, s is Borel measurable by Lemma 2.3.

($\Leftarrow$) If s is a Borel measurable selector, then

$$T = \{x : s(x) = x\}$$

is a Borel selector.

Exercise 9.3 Suppose E is a Borel equivalence relation on X and Ω is a σ-algebra on X containing the Borel sets. Show that more generally E has a transversal in Ω if and only if E has an Ω-measurable selector.

Definition 9.4 Let E be an equivalence relation on X. We say that $(A_n : n \in \mathbb{N})$ is a *separating family* for E if

$$xEy \Leftrightarrow \forall n \ (x \in A_n \leftrightarrow y \in A_n).$$

We say that E is *tame* if there is a separating family $(A_n : n \in \mathbb{N})$ where each A_n is Borel. More generally, if Ω is a σ-algebra on X containing the Borel sets, we say that E is Ω-*tame* if there is a separating family $(A_n : n \in \mathbb{N})$ where each $A_n \in \Omega$.

Note that if E is tame, then E is Ω-tame for any σ-algebra containing the Borel sets.

We can give another characterization of tameness.

Proposition 9.5 *If E is a Borel equivalence relation on X, then E is tame if and only if there is a Borel measurable $f : X \to \mathcal{C}$ such that xEy if and only if $f(x) = f(y)$.*

Proof

($\Rightarrow$) If $(A_n : n \in \mathbb{N})$ is a Borel separating family, let $(f(x))(n) = 1$ if and only if $x \in A_n$. Then xEy if and only if $f(x) = f(y)$.

($\Leftarrow$) Let $A_n = \{x : (f(x))(n) = 1\}$. Then A_n is Borel and $(A_n : n \in \mathbb{N})$ is a separating family.

Proposition 9.5 leads us to the following key idea for comparing the complexity of Borel equivalence relations.

Definition 9.6 Suppose E is a Borel equivalence relation on X and E^* is a Borel equivalence relation on Y. We say that E is *Borel reducible* to E^* if there is Borel measurable $f : X \to Y$ such that

$$xEy \Leftrightarrow f(x)E^*f(y).$$

In this case we write $E \leq_B E^*$. As usual, we write $E <_B E^*$ if $E \leq_B E^*$ but $E^* \not\leq_B E$ and $E \equiv_B E^*$ if $E \leq_B E^*$ and $E^* \leq_B E$.

We say that E is *continuously reducible* to E^* if we can choose f continuous. In this case we write $E \leq_c E^*$.

If X is a Polish space we let $\Delta(X)$ be the equivalence relation of equality on X.

Exercise 9.7 Let $n = \{0, \ldots, n-1\}$. We view n and $\mathbb{N}$ as Polish spaces with the discrete topology.

a) Prove that

$$\Delta(1) <_B \Delta(2) <_B \ldots <_B \Delta(n) <_B \ldots <_B \Delta(\mathbb{N}) <_B \Delta(\mathcal{C}).$$

b) Suppose X is an uncountable Polish space. Show that $\Delta(X) \equiv_B \Delta(\mathcal{C})$.

c) If E is a Borel equivalence relation, then $E \leq_B \Delta(\mathbb{N})$ or $\Delta(\mathcal{C}) \leq_B E$. [Hint: This is an easy consequence of Silver's Theorem.]

d) Show that a Borel equivalence relation E is tame if and only if there is a Polish space X such that $E \leq_B \Delta(X)$.

d) Says that an equivalence relation is tame if and only if there is a Borel way to assign invariants in a Polish space.

We next show how tameness is related to the existence of selectors. If E is a Borel equivalence relation with a Borel selector, then the selector shows that

$$E \leq_B \Delta(X) \leq_B \Delta(\mathcal{C}).$$

Thus E is tame.

In general tame equivalence relations need not have Borel transversals. Suppose $C \subseteq \mathcal{N} \times \mathcal{N}$ is a closed set such that $\pi(C)$ is not Borel. Let E be the equivalence relation $(x, y)E(u, v)$ if and only if $x = u$ on C. Clearly π shows

E is tame. If T is a transversal for E is a Borel uniformization of C. If T is Borel, then, since $\pi|T$ is one-to-one, by Corollary 7.22, $\pi(C) = \pi(T)$ is Borel, a contradiction.

In some important cases these notions are equivalent.

Proposition 9.8 *Suppose E is a Borel equivalence relation on a Polish space X such that every E-class is K_σ. Then E is tame if and only if E has a Borel transversal.*

In particular this is true if every E-class is countable.

Proof We know that if E has a Borel transversal, then E is tame.

Suppose E is tame. There is a Borel measurable $f : X \to \mathcal{C}$ such that $xEy \leftrightarrow f(x) = f(y)$. Let $A = \{(x, y) : f(y) = x\}$. By 2.3, A is Borel and

$$A_x = \{y : f(y) = x\}$$

is K_σ for all $x \in \mathcal{C}$. By 7.36, A has a Borel uniformization B.

Let

$$T = \{y : \exists x \ (x, y) \in B\}.$$

Then T is a transversal of E and since T is the continuous injective image of B, T is Borel.

For general E we can use uniformization ideas to say something about transversals. Recall that C is the smallest σ-algebra containing the Borel sets and closed under the Souslin operator $\mathcal{A}$. We have shown that every C set is Lebesgue measurable and every analytic subset of $X \times X$ can be uniformized by a C-set (Theorem 4.25 and Exercise 5.33).

Proposition 9.9 *If E is a tame Borel equivalence relation on X, then E has a C-measurable transversal.*

Proof Let $f : X \to \mathcal{C}$ be Borel measurable such that xEy if and only if $f(x) = f(y)$. Let $A = \{(z, x) \in \mathcal{C} \times X : f(x) = z\}$. Let $B \in C$ uniformize A and let

$$T = \{x \in X : (f(x), x) \in B\}.$$

Then T is a C-measurable transversal of E.

What equivalence relations are not tame? There is a very natural example.

Definition 9.10 Let E_0 be the equivalence relation on $\mathcal{C}$ defined by

$$xE_0y \text{ if and only if } \exists n \forall m \geq n \ x(n) = y(n).$$

We call E_0 the *Vitali equivalence relation.*

The proof that E_0 is not tame detours through a bit of ergodic theory.

Definition 9.11 We say that μ is a *Borel probability measure* on X if there is a σ-algebra Ω on X containing the Borel sets, and a measure $\mu : \Omega \to [0, 1]$ with $\mu(X) = 1$.

Definition 9.12 We say that $A \subseteq X$ is *E-invariant* if whenever $x \in A$ and yEx, then $y \in A$. If μ is a Borel probability measure, we say that μ is E-ergodic, if $\mu(A) = 0$ or $\mu(A) = 1$ whenever A is μ-measurable and E-invariant.

Definition 9.13 We say that $A \subseteq X$ is *E-atomic* if there is $x \in X$ with $\mu([x]) > 0$.

If the equivalence relation is clear from the context we will refer to E as "atomic" rather than E-atomic.

Lemma 9.14 *If E is a tame Borel equivalence relation, then there is no E-ergodic, nonatomic Borel probability measure on X. Indeed, if μ is an E-ergodic, nonatomic Borel probability measure on X, then E is not μ-tame.*

Proof Suppose μ is an E-ergodic, nonatomic Borel probability measure on X and E is μ-tame. Suppose $(A_n : n \in \mathbb{N})$ is a μ-measurable separating family. If xEy and $x \in A_n$, then $y \in A_n$. Thus each A_n is E-invariant. Since μ is E-ergodic $\mu(A_n) = 0$ or $\mu(A_n) = 1$.

Let
$$B = \bigcap\{A_n : \mu(A_n) = 1\} \cap \bigcap\{X \setminus A_n : \mu(A_n) = 0\}.$$
Each of the sets in the intersection has measure 1, thus $\mu(B) = 1$. Let $x \in B$. Since A_n is a separating family, $[x] = B$. Thus μ is atomic, a contradiction.

We need one basic lemma from probability theory.

Lemma 9.15 (Zero-one law for tail events) *Let μ be the usual Lebesgue measure on $\mathcal{C}$. If $A \subseteq \mathcal{C}$ is E_0-invariant, then $\mu(A) = 0$ or $\mu(A) = 1$.*

Proof Since A is Lebesgue measurable, for any $\epsilon > 0$, there is an open set U such that $U \supseteq A$ and $\mu(U \setminus A) < \epsilon$.

If $U \subseteq \mathcal{C}$ is open, there is a tree T on $2^{<\omega}$ such that $\mathcal{C} \setminus U = [T]$. Let
$$S = \{\sigma \notin T : \forall \tau \subseteq \sigma \; \tau \in T\}.$$

Note that $U = \bigcup_{\sigma \in S} N_\sigma$ and $N_\sigma \cap N_\tau = \emptyset$ for σ, τ distinct elements of S. Thus
$$\mu(U) = \sum_{\sigma \in S} \mu(N_\sigma) = \sum_{\sigma \in S} \frac{1}{2^{|\sigma|}}.$$

But N_σ and A are independent events. Thus $\mu(N_\sigma \cap A) = \mu(N_\sigma)\mu(A)$ and
$$\mu(A) = \sum_{\sigma \in S} \mu(N_\sigma \cap A) = \sum_{\sigma \in S} \mu(N_\sigma)\mu(A) = \mu(U)\mu(A).$$
It follows that either $\mu(A) = 0$ or $\mu(U) = 1$. Thus either $\mu(A) = 0$ or A has outer measure 1. In the later case $\mu(A) = 1$ since A is measurable.

Corollary 9.16 *E_0 is not tame.*

Proof Let μ be Lebesgue measure on $\mathcal{C}$. By the zero-one law for tail events μ is E_0-ergodic. If $x \in \mathcal{C}$, then $[x]$ is countable and hence measure zero. Thus μ is nonatomic. Thus E_0 is not tame.

Indeed if Ω is the σ-algebra of Lebesgue measurable subsets of $\mathcal{C}$, our proof shows that E_0 is not Ω-tame. In particular E_0 is not C-tame.

The first major result on Borel equivalence relations is the next theorem of Harrington, Kechris and Louveau.[4] It says that E_0 is the simplest nontame Borel equivalence relation.

Theorem 9.17 (Glimm–Effros Dichotomy) *Suppose E is a Borel equivalence relation on a Polish space X. Then either*

i) E is tame or

ii) $E_0 \leq_B E$.

The proof of Theorem 9.17 heavily uses effective descriptive set theory. We postpone the proof. For now we will be content giving the following corollary.

Corollary 9.18 *Let E be a Borel equivalence relation on a Polish space X. The following are equivalent:*

i) E is tame;

ii) E has a C-measurable transversal;

iii) There is no Borel probability measure μ that is E-ergodic and nonatomic.

iv) $E_0 \not\leq_B E$.

Proof We have already shown i)$\Rightarrow$ ii), i)$\Rightarrow$ iii) and are assuming i) $\Leftrightarrow$ iv).

ii) $\Rightarrow$ i) Suppose E in not tame. Then $E_0 \leq_B E$. Let $f : \mathcal{C} \to X$ be a Borel reduction of E_0 to E. If T is a C-measurable transversal for E, then $f^{-1}(T)$ is a C-measurable transversal for E_0. But then E_0 is C-tame, a contradiction.

iii)$\Rightarrow$ i) If E is not tame, there is $f : \mathcal{C} \to X$ a Borel reduction of E_0 to E. Let μ be Lebesgue measure on $\mathcal{C}$. We define a measure ν on X by

$$\nu(A) = \mu(f^{-1}(A)).$$

Claim ν is a Borel probability measure on X.

We will only argue σ-additivity. If $A_0, A_1, \ldots \subseteq X$ are pairwise disjoint, then $f^{-1}(A_0)$,$f^{-1}(A_1), \ldots$ are disjoint and

$$\nu(\bigcup A_i) = \mu(f^{-1}(\bigcup A_i)) = \sum_{i=0}^{\infty} \mu(f^{-1}(A_i)) = \sum_{i=0}^{\infty} \nu(A_i)$$

as desired.

If $A \subseteq X$ is E-invariant, then $f^{-1}(A)$ is E_0-invariant. Thus

$$\nu(A) = \mu(f^{-1}(A) = 0 \text{ or } 1$$

[4]Glimm proved this when E is the orbit equivalence relation for a second countable locally compact group. Effros extended this to the case where E is an F_σ orbit equivalence relation for a Polish group. The general case is due to Harrington, Kechris and Louveau [3].

so ν.. For any $x \in X$, either $f^{-1}([x]_E) = \emptyset$ and $\nu([x]_E = 0$, or there is $y \in \mathcal{C}$ with $f(y) = x$. Then $f^{-1}([x])_E = [y]_{E_0}$ and $\nu([x]_E) = 0$. Thus ν is an E-ergodic nonatomic probability measure on X.

10 Countable Borel Equivalence Relations

Definition 10.1 Suppose G is a group. A map $\alpha : G \times X \to X$ is an *action.* If $\alpha(g, \alpha(h, x)) = \alpha(gh, x)$ for all $g, h \in G$ and $\alpha(e, x) = x$ for the identity element x.

If G and X are Borel subsets of Polish spaces and the action α is Borel measurable, we say that α is a *Borel action.*

When no confusion arises we write gx for $\alpha(g, x)$.

Definition 10.2 If $\alpha : G \times X \to X$ is a Borel action, the orbit equivalence relation E_G is given by

$$xEy \Leftrightarrow \exists g \in G \; gx = y.$$

For arbitrary Borel actions, the orbit equivalence relation is $\mathbf{\Sigma}^1_1$, but if G is countable

$$xEy \Leftrightarrow \bigvee_{g \in G} gx = y.$$

So E_G is a Borel equivalence relation.

Definition 10.3 A Borel equivalence relation E is *countable* if and only if every E-class is countable.

If G is a countable group, then the orbit equivalence relation is a countable Borel equivalence relation. Of course, there are also countable Borel equivalence relations like $\equiv_T$ and $\equiv_{\mathrm{hyp}}$, Turing equivalence and hyperarithmetic equivalence, that seem to have nothing to do with group actions. Remarkably, every countable Borel equivalence relation arises as an orbit equivalence relation.

Theorem 10.4 (Feldman–Moore) *If E is a countable Borel equivalence relation on a Borel set X, then there is a countable group G and a Borel action of E on X such that E is the orbit equivalence relation.*

Proof Consider $E \subseteq X \times X$. Since each section is countable, by 7.26 we can find Borel measurable functions $f_0, f_1, \ldots$ such that $f_i : A_i \to X$, the f_i have disjoint graphs and $E = \bigcup_i \text{Graph } f_i$.

For $i, j \in \mathbb{N}$ let

$$xR_{i,j}y \Leftrightarrow x \in A_i \wedge y \in A_j \wedge f_i(x) = y \wedge f_y(x) = x.$$

Note that $E = \bigcup_{i,j} R_{i,j}$.

For each i, j let $E_{i,j}$ be the equivalence relation generated by $R_{i,j}$. Then $E_{i,j}$ is Borel and $E = \bigcup_{i,j} E_{i,j}$. We claim that there is a Borel measurable $g_{i,j} : X \to X$ such that $E_{i,j}$-classes are the orbits of $g_{i,j}$.

For each x there is at most one y such that $xR_{i,j}y$ and at most one z such that $zR_{i,j}x$. Thus every $E_{i,j}$-class is of one of the following forms:

1) $\{x_i : i \in \mathbb{Z}\}$;
2) $\{x_i : i \in \mathbb{N}\}$;
3) $\{x_{-i} : i \in \mathbb{N}\}$; or
4) $\{x_i : i = 0, \ldots, n\}$ for some $n \in \mathbb{N}$, where $x_k R_{i,j} x_{k+1}$.

Let $B_i = \{x : [x] \text{ is of type i)}\}$. Then B_i is a Borel set.

We define $g_{i,j}$ as follows.

1) On classes of type 1) $g_{i,j}(x_k) = x_{k+1}$.

2) On classes of type 2)

$$g_{i,j}(x_k) = \begin{cases} x_1 & \text{if } k = 0 \\ x_{k-2} & \text{if } k > 0 \text{ is even} \\ x_{k+2} & \text{if } k \text{ is odd.} \end{cases}.$$

3) On classes of type 3)

$$g_{i,j}(x_k) = \begin{cases} x_{-1} & \text{if } k = 0 \\ x_{-k+2} & \text{if } k > 0 \text{ is even} \\ x_{-k-2} & \text{if } k \text{ is odd.} \end{cases}.$$

4) On classes of type 4)

$$g_{i,j}(x_k) = \begin{cases} x_{k+1} & \text{if } k < n \\ x_0 & \text{if } k = n \end{cases}.$$

Let G be the countable group of Borel permutations of X generated by $\{g_{i,j} : i, j \leq n\}$. We give G the discrete topology. The natural action of G on X is Borel and the orbit equivalence relation is E.

Universal Equivalence Relations

Definition 10.5 We say that a countable Borel equivalence relation E is *universal* if $E^* \leq_B E$ for all countable Borel equivalence relations E.

We will show how to use the Feldman-Moore Theorem to find natural universal equivalence relations Let X be a set and let G be a group if $f \in X^G$ and $g \in G$ define $gf \in X^G$ by

$$gf(h) = f(g^{-1}h).$$

Note that

$$g_1(g_2f)(h)) = (g_2f)(g_1^{-1}h) = f(g_2^{-1}g_1^{-1}h) = (g_1g_2)f(h).$$

Thus $(g, f) \mapsto gf$ is an action of G on X^G.

Our main goal is to show that if F_2 is the free group on two generators, then the orbit equivalence relation for the natural action of F_2 on 2^{F_2} is universal.

If G is a countable group and X is a standard Borel set, then $(g, f) \mapsto gf$ is a Borel action of G on X^G. We let $E(G, X)$ denote this action.

We first show that there is a universal G-action.

Lemma 10.6 *Suppose G is a countable group acting on a Borel set X. Let E_G be the orbit equivalence relation. Then $E_G \leq_B E(G, \mathcal{C})$.*

Proof Let $U_0, U_1, \ldots$ be Borel subsets of X such that if $x \neq y$ there is U_i such that only one of x and y are in U_i (we say U_i *separates points* of X).

We view elements of $\mathcal{C}^G$ as functions from $G \times \mathbb{N}$ to $\{0, 1\}$. Let $\phi : X \to \mathcal{C}^G$ be the function

$$\phi(x)(g, i) = 1 \Leftrightarrow g^{-1}x \in U_i.$$

Since $\phi(x)(e, i) = 1 \Leftrightarrow x \in U_i$ and the U_i separate points, we see that ϕ is one-to-one.

Note that

$$(h\phi(x))(g)(i) = 1 \Leftrightarrow \phi(x)(h^{-1}g)(i) = 1 \Leftrightarrow g^{-1}hx \in U_i \Leftrightarrow \phi(hx)(g)(i) = 1.$$

Thus $h\phi(x) = \phi(hx)$.

Suppose xE_Gy. Then there is $g \in G$ such that $y = hx$ and $\phi(y) = g\phi(x)$. Thus $\phi(x)E(G, \mathcal{C})\phi(y)$. Moreover, if $\phi(x)E(G, \mathcal{C})\phi(y)$, there is $g \in G$ such that $\phi(y) = g\phi(x) = \phi(gx)$. Since ϕ is one-to-one $y = gx$ and xE_Gy. Thus ϕ is a Borel reduction of E_G to $E(G, \mathcal{C})$.

Lemma 10.7 *Suppose G and H are countable groups and $\rho : G \to H$ is a surjective homomorphism then $E(H, X) \leq_B E(G, X)$ for any Borel X.*

Proof Let $\phi : X^H \to X^G$ be the function

$$\phi(f)(g) = f(\rho(g)).$$

Clearly, ϕ is one-to-one.

If $h \in H$ and $\rho(h_*) = h$, then

$$\phi(hf)(g) = (hf)(\rho(g)) = f(h^{-1}\rho(g)) = f(\rho(h_*^{-1}g))$$

and

$$h_*\phi(f)(g) = \phi(f)(h_*^{-1}g) = f(\rho(h_*^{-1}g)).$$

Thus $\phi(hf) = h_*\phi(f)$. Moreover, if there is $g \in G$ such that $g\phi(f_1) = \phi(f_2)$, then $f_2 = g\phi(f_1) = \phi(\rho(g)f_1)$. Since ϕ is one to one $\rho(g)f_1 = f_2$. Thus $f_1E_Gf_2$ if and only if $\phi(f_1)E(G, X)\phi(f_2)$.

For any cardinal κ let F_κ be the free group with κ generators.

Corollary 10.8 *If E is a countable Borel equivalence relation, then $E \leq_B E(F_{\aleph_0}, \mathcal{C})$.*

Proof By the Feldman-Moore Theorem there is a countable group G and a Borel action of E on X such that E is the orbit equivalence relation for the action. By Lemma 10.6 $E \leq_B E(G, \mathcal{C})$. There is a surjective homomorphism $\rho : F_{\aleph_0} \to G$. Thus by Lemma 10.7

$$E \leq_B E(G, \mathcal{C}) \leq_B E(F_{\aleph_0}, \mathcal{C}).$$

We will simplify this example a bit more after a couple of simple lemmas.

Lemma 10.9 *If G is a countable group and $H \subseteq G$, then $E(H, X) \leq_B E(G, X)$ for any Borel X.*

Proof Fix $a \in X$. Let $\phi : X^H \to G$ be the function

$$\phi(f)(g) = \begin{cases} f(g) & \text{if } g \in H \\ a & \text{otherwise.} \end{cases}$$

Clearly ϕ is one-to-one.

Let $h \in H$. If $g \in H$, then

$$\phi(hf)(g) = (hf)(g) = f(h^{-1}g) = (h\phi(f))(g).$$

If $g \notin H$, then $h^{-1}g \notin H$ so

$$\phi(hf)(g) = a = (h\phi(f))(g).$$

Since ϕ is one-to-one, we may argue as above that $f_1 E(H, X) f_2$ if and only if $\phi(f_1) E(G, X) \phi(f_2)$.

Suppose a, b are free generators of F_2. Then $\{a^n b a^n : n = 1, 2, \ldots\}$ freely generate a subgroup of F_2 isomorphic to $F_{\aleph_0}$. Thus $E(F_2, \mathcal{C})$ is also a universal countable Borel equivalence relation.

Lemma 10.10 *If G is a countable group, then $E(G, \mathcal{C}) \leq_B E(G \times \mathbb{Z}, 2)$.*

Proof We identify $\mathcal{C}^G$ with $2^{G \times \mathbb{N}}$. Let $\phi : \mathcal{C}^G \to 2^{G \times \mathbb{Z}}$ be the function

$$\phi(f)(g, i) = \begin{cases} f(g, i) & \text{if } i \leq 0 \\ 1 & \text{if } i = -1 \\ 0 & \text{if } i < -1. \end{cases}$$

Clearly ϕ is one-to-one.

Suppose $f \in \mathcal{C}^G$ and $h \in G$. Then $\phi(hf) = (h, 0)\phi(f)$. Suppose $\phi(f_1) = (h, m)\phi(f)$. Then

$$\phi(f_1)(g, i) = \phi(f)(h^{-1}g, i - m)$$

for all $g \in G$, $i \in \mathbb{Z}$. We claim that $m = 0$. Let $i = -1$, Then $\phi(f_1)(g, -1) = 1$. Thus $-1 - m \geq -1$ and $m \leq 0$. On the other hand, let $i = m - 1$. Then $\phi(f_1)(g, m - 1) = \phi(f)(h^{-1}g, -1) = 1$. Thus $m - 1 \geq -1$ and $m \geq 0$. Thus $m = 0$. Thus

$$\phi(f_1) = (h, 0)\phi(f) = \phi(hf).$$

Since ϕ is one-to-one, $f_1 = hf$. Thus

$$f_1 E(G, \mathcal{C}) f_2 \Leftrightarrow \phi(f_1) E(G \times \mathbb{Z}, 2).$$

Theorem 10.11 *If E is a countable Borel equivalence relation, then $E \leq_B E(F_2, 2)$.*

Proof

$$\begin{aligned}
E &\leq_B E(F_{\aleph_0}, \mathcal{C}) \\
&\leq_B E(F_{\aleph_0} \times \mathbb{Z}, 2) \quad \text{by 10.10} \\
&\leq_B E(F_{\aleph_0}, 2) \quad \text{by 10.7} \\
&\leq_B E(F_2, 2) \quad \text{by 10.9 since we can embed } F_{\aleph_0} \text{ into } F_2
\end{aligned}$$

11 Hyperfinite Equivalence Relations

Definition 11.1 We say that a Borel equivalence relation E is *finite* if every equivalence class is finite. We say that E is *hyperfinite* if there are finite Borel equivalence relations $E_0 \subseteq E_1 \subseteq \ldots$ such that $E = \bigcup E_n$.

The equivalence relation E_0 is hyperfinite. Let F_n be the equivalence relation on $\mathcal{C}$

$$xF_ny \Leftrightarrow \forall m > n\; x(m) = y(m).$$

Then $E_0 = \bigcup F_n$ and each F_n is finite.

The main goal of this section will be to give the following characterizations of hyperfinite equivalence relations.

Theorem 11.2 *Let E be a countable Borel equivalence relation. The following are equivalent:*
i) E is hyperfinite;
ii) E is the orbit equivalence relation for a Borel action of $\mathbb{Z}$;
iii) $E \leq_B E_0$.

We first show that, for countable Borel equivalence relations, finite $\Rightarrow$ tame $\Rightarrow$ hyperfinite.

Proposition 11.3 *If E is a finite Borel equivalence relation, then E is tame.*

Proof There is a Borel action of a countable group G on X such that E is the orbit equivalence relation. Without loss of generality we may assume that $X = \mathbb{R}$ so we can linearly order X. Then

$$T = \{x \in X : \forall g \in G\; x \leq gx\}$$

is a Borel transversal for E.

Proposition 11.4 *If E is tame countable Borel equivalence relation, then E is hyperfinite.*

Proof There is a countable group G such that E is the orbit equivalence relation on X. Suppose $G = \{g_0, g_1, \ldots\}$ where $g_0 = e$. Since E is tame, there is a Borel measurable selector $s : X \to X$. Let

$$xE_ny \Leftrightarrow xEy \text{ and } \left(x = y \vee \left(\bigwedge_{i=0}^{n} x = g_is(x) \wedge \bigwedge_{i=0}^{n} y = g_is(x)\right)\right).$$

Then $xE_ns(x)$ if and only if $x \in \{g_is(x) : i = 0, \ldots, n\}$ and if $x \not\!E_ns(x)$ then $|[x]_{E_n}| = 1$. Thus E_n is a finite equivalence relation and $\bigcup E_n = E$.

Since E_0 is hyperfinite, the converse is false. There is a partial converse.

Theorem 11.5 *Let E be a countable Borel equivalence relation, then E is hyperfinite if and only if there are tame Borel equivalence relations $E_0 \subseteq E_1 \subseteq E_2 \subseteq \dots$ with $E = \bigcup_n E_n$.*

For a proof see [2] Theorem 5.1.

We mention a few important closure properties for hyperfinite equivalence relations.

Definition 11.6 If E is an equivalence relation on X we say that $A \subseteq X$ is *full* for E if for all $x \in X$ there is $y \in A$ such that xEy.

Proposition 11.7 *i) If $E \subseteq F$ and F is hyperfinite, then E is hyperfinite.*

ii) If E is hyperfinite and $A \subseteq X$ is Borel, then $E|A$ is hyperfinite.

iii) If E is a countable Borel equivalence relation, $A \subseteq X$ is Borel and full for E, and $E|A$ is hyperfinite, then A is hyperfinite.

iv) If E is a countable Borel equivalence relation, $E \leq_B E^*$ and E^* is hyperfinite, then E is hyperfinite.

Proof i) and ii) are obvious.

iii) Suppose $E_0 \subseteq E_1 \subseteq E_2 \subseteq \dots$ are finite Borel equivalence relations on A such that $E|A = \bigcup E_i$. There is a countable group $G = \{g_0, g_1, \dots,\}$ such that E is the orbit equivalence relation for a Borel action of G on X. For $x \in X$, let n_x be least such that $g_{n_x} x \in A$.

Let xF_ny if and only if

$$xEy \wedge (x = y \vee ({}_x\leq n \wedge n_y \leq n \wedge g_{n_x} x F_n g_{n_y} y)).$$

Then F_n is a finite equivalence relation and $\bigcup F_n = E$.

iv) Let $f : X \to Y$ be a Borel reduction E to a hyperfinite E^*. Since E is countable, the map f has countable fibers. Thus by 7.21, $B = f(X)$ is Borel and there is a Borel measurable $s : B \to X$ such that $f(s(y)) = y$ for all $y \in f(X)$. Let $A = s(B) = \{x \in X : s(f(x)) = x\}$. Then A is Borel and full in E. By ii) $E^*|B$ is hyperfinite. But $E|A$ is Borel isomorphic to $E^*|B$. By iii) E is hyperfinite.

$\mathbb{Z}$-actions

Suppose E is a Borel equivalence relation on X and $<_{[x]}$ is a linear order of $[x]$. We say that $[x] \mapsto <_{[x]}$ is Borel if there is a Borel $R \subseteq X \times X \times X$ such that

i) $R(x, y, z) \Rightarrow (xEy \wedge xEz)$;

ii) $R(x, y, z) \Rightarrow y <_{[x]} z$;

iii) if xEx_1 then $R(x, y, z) \Leftrightarrow R(x_1, y, z)$.

Theorem 11.8 *Let E be a Borel equivalence relation on X. The following are equivalent:*

i) E is hyperfinite;

ii) There is a Borel $[x] \mapsto <_{[x]}$ such that each infinite E-class has order type $\mathbb{Z}$, ω or ω^*.

iii) There is a Borel $[x] \mapsto <_{[x]}$ such that each infinite E-class has order type $\mathbb{Z}$.

iv) There is a Borel action of $\mathbb{Z}$ on X such that E is the orbit equivalence relation.

v) There is a Borel automorphism $T : X \to X$ such that E-equivalence classes are T-orbits.

Proof

It is clear that iv) $\Leftrightarrow$ v)

i) $\Rightarrow$ ii) Let $E_0 \subseteq E_1 \subseteq E_2 \subseteq \ldots$ be finite Borel equivalence relations such that $E = \bigcup E_n$. We may assume that E_0 is equality. We may also assume that there is an ordering $<$ of X.

We inductively define $<_{[x]_{E_n}}$ as follows.

1) $<_{[x]_{E_0}}$ is trivial, since $[x]_{E_0} = \{x\}$.

2) Suppose y, zE_nx and $yE_{n-1}z$, then $y <_{[x]_{E_n}} z$ if and only if $y <_{[y]_{E_{n-1}}} z$.

3) Suppose y, zE_nx and $y \not{E}_{n-1}z$. Let $\widehat{y}$ be the $<_{[y]_{E_{n-1}}}$-least element of $[y]_{n-1}$ and $\widehat{z}$ be the $<_{[z]_{E_{n-1}}}$-least element of $[z]_{E_{n-1}}$. If $\widehat{y} < \widehat{z}$, then $y <_{[x]_{E_n}} z$. Otherwise $z <_{[x]_{E_n}} y$.

In other words: we order $[x]_{E_n}$ be breaking it into finitely many E_{n-1} classes $C_1, \ldots, C_m$. We then order the classes C_i by letting y_i be the $<_{[y_i]_{E_{n-1}}}$-least element and saying that $C_i < C_j$ if $y_i < y_j$.

Let $<_{[x]_E} = \bigcup <_{[x]_{E_n}}$. If xE_ny and $x <_{[x]} z <_{[x]} y$, then xE_nx. It follows that $<_{[x]}$ is a discrete union of finite orders. Thus $<_{[x]}$ is either a finite order or has order type ω, ω^* or $\mathbb{Z}$.

We need only argue that the assignments $[x] \mapsto <_{[x]_{E_n}}$ is Borel. The only difficulty is picking $\widehat{x}$ the $<_{[x]_{E_n}}$-least element of $[x]_{E_n}$. There is a countable groups G and a Borel actions of G on X such that E_n is the orbit equivalence relation of G_n. Then

$$y = \widehat{x} \Leftrightarrow (yE_nx \wedge \forall g \in G_n \ y \leq_{[x]_{E_n}} gx).$$

This is easily seen to be Borel.

ii) $\Rightarrow$ iii) We may assume that E is the orbit equivalence relation for the action of a countable group G. Since

$$\{x : \exists g \in G \forall h \in G \ hx \leq_{[x]} gx\}$$

and

$$\{x : \exists g \in G \forall h \in G \ gx \leq_{[x]} hx\}$$

are Borel we can determine the order type of each class. If a class has order type ω of ω^* we can reorder it so that it has order type $\mathbb{Z}$. For example if $[x]$ is $x_0 <_{[x]} x_1 <_{[x]} < \ldots$ we define a new order $<^*$ so that

$$\ldots x_5 <^* x_3 <^* x_1 <^* x_0 <^* x_2 <^* x_4 < \ldots$$

The ω^* case is similar. This can clearly be done in a Borel way.

iii) $\Rightarrow$ iv) We define a Borel automorphism $g : X \to X$ such that E is the orbits of g. If x is the $<_{[x]}$-maximal element of $[x]$, then $[x]$ is the $<_{[x]}$-least element of $[x]$. Otherwise let $g(x)$ be the $<_{[x]}$-successor of x. Arguing as above g is Borel. We let $\mathbb{Z}$ act on X by $nx = g^{(n)}x$. Clearly E is the orbit equivalence relation.

iv) $\Rightarrow$ iii) Let $g : X \to X$ be a Borel automorphism such that E-classes are g-orbits. Then $X_0 = \{x : \exists n \neq 0 : g^{(n)}x = x\}$ is Borel. On X_0 we can define $<_{[x]}$ using $<$ is a fixed linear order of X. Thus, without loss of generality, we may assume that every E-class is infinite. But then we can define $x <_{[x]} y$ if and only if there is an $n > 0$ such that $g^{(n)}x = y$. Clearly this is a $\mathbb{Z}$-ordering of $[x]$.

iii) $\Rightarrow$ i) We may assume that E is an equivalence relation on $\mathcal{C}$. For each equivalence class C we define a tree $T_C \subseteq 2^{<\omega}$ by

$$T_C = \{\sigma \in 2^{<\omega} : \exists x \in C \ x \supset \sigma\}.$$

There is a Borel automorphism g such that E-classes are g-orbits. Since

$$T_{[x]} = \{\sigma : \exists n \in \mathbb{Z} : g^{(n)}x \supset \sigma\},$$

the function $x \mapsto T_{[x]}$ is Borel measurable. Clearly T_C is infinite. Let $z_C \in [T_C]$ be the leftmost path in T_C.

Claim The functions $x \mapsto z_{[x]}$ is Borel measurable.

We define $\sigma_0^x \subset \sigma_1^x \subset \ldots$ such that $\{\tau \in T_{[x]} : \tau \supseteq \sigma_i^x\}$ is infinite. Let $\sigma_0^x = \emptyset$ and $\sigma_{i+1}^x = \sigma_i^x\frown j$ where j is least such that $\{\tau \in T_{[x]} : \tau \supseteq \sigma_i^x\frown j\}$ is infinite. Then $z_{[x]} = \bigcup \sigma_i^x$. It is easy to see that $(T_{[x]}, z_{[x]})$ is Π_1^0. Thus $x \mapsto z_{[x]}$ is Borel measurable.

There are several cases to consider. It will be clear that deciding which case we are in is Borel.

<u>case 1</u>: $z_C \in C$.

For $x \in C$ we define xE_ny if and only if $x = y$ or there are i, j with $|i|, |j| \leq n$ such that $x = g^{(i)}z_{[x]}$ and $y = g^{(j)}z_{[x]}$.

For $m \in \mathbb{N}$ let $C_m = \{x \in C : x|m = z_C|m\}$.

<u>case 2</u>: There is an m such that C_m has a $<_C$-least element.

Let m be least such that C_m has a least element w_C. For $x \in C$, we define xE_ny if and only if $x = y$ or there are i, j with $|i|, |j| \leq n$ such that $x = g^{(i)}w_{[x]}$ and $y = g^{(j)}w_{[x]}$.

<u>case 3</u>: There is an m such that C_m has a $<_C$-maximal element.

Similar.

<u>case 4</u>: Otherwise.

We have $C_0 \supseteq C_1 \supseteq C_2 \supseteq \ldots$. Since we are not in case 1, $\bigcap C_i = \emptyset$. Since we are not in case 2 or 3, C_i has no smallest or largest element.

We define E_n on C by: xE_ny if and only if ($x \in C_n$ and $x = y$) or and there is $i > 0$ such that $g^{(i)}x = y$ and $g_{(j)}x \notin C_n$ for $j = 0, \ldots, i$.

Clearly each E_n class is finite and if xEy, then xE_ny for all sufficiently large n.

The i)$\Leftrightarrow$ iii) is due to Slaman and Steel. The direction iv) $\Rightarrow$ i) is due to Weiss.

It follows immediately that there is a universal hyperfinite Borel equivalence.

Corollary 11.9 *If E is a hyperfinite Borel equivalence relation, then $E \leq_B E(\mathbb{Z}, \mathcal{C})$.*

Recall that an action of G on X is a free action if $gx \neq hx$ for any $x \in X$ and $g \neq h$. Our proof shows the following.

Corollary 11.10 *If E is a hyperfinite equivalence relation on a standard Borel space X and every E class if infinite, then E is the orbit equivalence relation for a free Borel action of $\mathbb{Z}$ on X.*

Reducibility to E_0

Theorem 11.11 (Doughrety-Jackson-Kechris) *If E is a hyperfinite Borel equivalence relation, then $E \leq_B E_0$.*

Corollary 11.12 *If E is a nontame hyperfinite Borel equivalence relation then $E \equiv_B E_0$.*

Proof By Theorem 9.17 $E_0 \leq_B E$ and by Theorem 11.11 $E \leq_B E_0$.

By Theorem 11.8 and Lemma 10.6 every hyperfinite Borel equivalence relation is Borel-reducible to $E(\mathbb{Z}, \mathcal{C})$. Thus we may assume that $E = E(\mathbb{Z}, \mathcal{C})$.

We say that $X \subseteq \mathcal{C}^{\mathbb{Z}}$ is tame if X is E-invaraint and $E|X$ is tame.

Lemma 11.13 *Suppose $X \subseteq \mathcal{C}^{\mathbb{Z}}$ is tame, and $f : \mathcal{C}^{\mathbb{Z}} \setminus X \to \mathcal{C}$ is a Borel reduction of $E|Y$ to E_0. Then $E \leq_B E_0$.*

Proof Let $g : X \to \mathcal{C}$ be a Borel measurable function such that

$$xEy \Leftrightarrow g(x) = g(y)$$

for $x, y \in X$. Since there is a perfect set of E_0-inequivalent elements, there is a continuous $p : \mathcal{C} \to \mathcal{C}$ such that $p(x) \not{E}_0 p(y)$ for $x \neq y$. Let $\langle , \rangle : \mathcal{C}^2 \to \mathcal{C}$ be the ususal bijection

$$< x, y\rangle = (x(0), y(0), x(1), y(1), \ldots).$$

Finally let $\overline{0}, \overline{1} \in \mathcal{C}$ denote the infinte sequences that are constantly 0 and constantly 1, respectively.

Define $\widehat{f} : \mathcal{C}^Z \to \mathcal{C}$ by

$$\widehat{f}(x) = \begin{cases} \langle p(g(x)), \overline{0}\rangle & \text{if } x \in X \\ \langle f(x), \overline{1}\rangle & \text{if } x \notin X \end{cases}.$$

If $x \in X$ and $y \notin X$, then $f(x) \not{E}_0 f(y)$, since the even part of $f(x)$ is $\overline{0}$ and the even part of $f(y)$ is $\overline{1}$.

If $x, y \in X$, then

$$\widehat{f}(x) E_0 \widehat{f}(y) \Leftrightarrow p(g(x)) E_0 p(g(y)) \Leftrightarrow g(x) = g(y) \Leftrightarrow x E y.$$

If $x, y \notin X$, then

$$\widehat{f}(x) E_0 \widehat{f}(y) \Leftrightarrow f(x) E_0 f(y) \Leftrightarrow x E y.$$

Thus $\widehat{f}$ is the desired reduction.

Lemma 11.14 *If $X_0, X_1, \ldots, \subseteq \mathcal{C}^{\mathbb{Z}}$ are tame, then $\bigcup X_n$ is tame.*

Proof Since each X_i is invariant we may assume that the X_i are disjoint. If $f_i : X_i \to \mathcal{C}$ is a Borel reduction and $f : \bigcup X_i \to \mathcal{C}$ is the function $f(x) = 0^i 1 \widehat{\ } f_i(x)$ for $x \in X_i$, then f is a Borel reducition of $E|X$ to $\Delta(\mathcal{C})$.

The two lemmas allow us to work "modulo tame sets", i.e. if X is tame we may ignore it and assume we are just working with $E|(\mathcal{C}^{\mathbb{Z}} \setminus X)$.

Proof of Theorem 11.11 We will view each $x \in \mathcal{C}^{\mathbb{Z}}$ as a $\mathbb{Z} \times \mathbb{N}$ array of zeros and ones. The columns are $\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots$ where $x_i \in \mathcal{C}$. If $\sigma \in (2^n)^n$, we view σ as $(\sigma_0, \ldots, \sigma_{n-1})$ where each $\sigma_i \in 2^n$. We say that σ *occurs* in x at k if $\sigma_i = x_{k+i}|n$ for $i = 0, \ldots, n$.

Fix $\sigma \in (2^n)^n$. Let Y be the set of all $x \in \mathcal{C}^{\mathbb{Z}}$ such that there is a largest k such that σ occurs in x at k. We will argue that Y is tame. Suppose $x \in X$ and k is maximal such that σ occurs in x at k. If $n \in \mathbb{Z}$, then $(nx)_i(j) = x_{i-n}(j)$. Thus $k - n$ is the largest i such that σ occurs in nx at i. Thus Y is $\mathbb{Z}$-invariant. Let $s : X \to X$ be the function $s(x) = kx$ where k is maximal such that σ occurs in x at k. Then $s(x)$ is the unique element of $[x]$ where 0 is the largest i such that σ occurs at i. Thus s is E-invariant and Y is tame.

Similarly, the set of x such that there is a least k such that σ occurs in x at k is tame. By throwing out these tame Borel sets, we may restrict attention to a Borel set X that for all σ and $x \in X$, the set of k such that σ occurs in x at k is unbounded in both directions.

If $\sigma \in (2^n)^n$ and $m < n$ we let $\sigma|m = (\sigma_0|m, \ldots, \sigma_{m-1}|m)$. Fix $<_n$ a linear order of $(2^n)^n$ such that if $\sigma, \tau \in (2^n)^n$ and $\sigma|m <_m \tau|m$ for some $m < n$, then $\sigma <_n \tau$.

For $x \in X$ let $f_n(x)$ be the $<_n$-least element of $(2^n)^n$ occuring in x. Our assumptions on $<_n$, insure that $f_n(x)|m = f_m(x)$ for $m < n$. Define $f : X \to \mathcal{C}^{\mathbb{N}}$ by

$$f(x) = (y_0, y_1, \ldots)$$

where $f_n(x) = (y_0|n, y_1|n, \ldots, y_{n-1}|n)$ for all n. Note that each f_n and f are E-invariant.

We say that $g \in \mathcal{C}^{\mathbb{N}}$ *occurs* in x at k if $x_{k+i} = g(i)$ for all $i \in \mathbb{N}$. Let Y be the set of $x \in X$ such that $f(x)$ occurs in x and there is a least k such that

$f(x)$ occurs in x at k. Then Y is Borel, E-invariant and the function $s(x) = kx$ where k is least $f(x)$ occurs at k is a Borel selector. Thus Y is tame. Let W be the set of all $x \in X$ such that $f(x)$ occurs in x at k for arbitrarily small k. If $x \in W$, then the action of $\mathbb{Z}$ on $[x]$ is periodic. Thus $[x]$ is finite and W is tame. Throwing out Y and W we may assume that $f(x)$ does not occur in x for all $x \in X$.

For $x \in X$ and $n \in \mathbb{N}$ define

$$\begin{aligned} k_0^x &= 0 \\ k_{2n+1}^x &= \text{the least } k \text{ such that } k > k_{2n}^x \text{ and } f_{2n+1}(x) \text{ occurs in } x \text{ at } k. \\ k_{2n+2}^x &= \text{the largest } k \text{ such that } k < k_{2n}^x \text{ and } f_{2n+2}(x) \text{ occurs in } x \text{ at } k. \end{aligned}$$

Then

$$\ldots \leq k_4^x \leq k_2^x \leq k_0^x < k_1^x \leq k_3^x \leq \ldots.$$

Since $f(x)$ does not occur in x, $k_{2n}^x \to \infty$ and $k_{2n+2}^x \to -\infty$.

We make the usual identification between $\mathcal{C}$ and $\mathcal{P}(\mathbb{N})$ by identifying sets with their characteristic functions. Under this identification

$$AE_0B \Leftrightarrow A \triangle B \text{ is finite }.$$

Fix a bijection

$$p : \mathbb{N} \times (2^{<\omega})^{<\omega} \to \mathbb{N}.$$

For $x \in X$ and $n \in \mathbb{N}$ let

$$t_n^x = |k_{n+1}^x - k_n^x| + 1$$

and let $r_n^x \in (2^{<\omega})^{t_n^x}$ be $(\sigma_0, \ldots, \sigma_{t_n^x - 1})$ where

$$\sigma_i = x_{\min\{k_n^x, k_{n+1}\}+1}|n.$$

This looks more confusing then it is. Suppose n is even. Then $k_n^x < k_{n+1}^x$ and r_n^x is just the block of the matrix x where we look take rows $0, \ldots, n-1$ and columns k_n^x to k_{n+1}^x.

Let $G(x) = \{p(n, r_n^x) :\in \mathbb{N}\}$. From $G(x)$, and knowing $k_0^x = 0$, we can reconstruct the sequence $(k_i^x : i \in \mathbb{N})$ and x. Thus G is one-to-one.

Suppose $G(x) \triangle G(y)$ is finite. Then there is an m such that $r_n^x = r_n^y$ for all $n > m$. It follows that y is obtained by shifting x. Thus xEy.

Suppose xEy. There is $m \in \mathbb{Z}$ such that $x_{m+i} = y_i$ for all $i \in \mathbb{N}$. Without loss of generality assume $m > 0$. Let n_0 be least such that $k_{2n_0+1}^x > m$. Since $f(x) = f(y)$,

$$k_{2n_0+1}^x = m + k_{2n_0+1}^y.$$

Thus $k_n^x = m + k_n^y$ for all $n > n_0$. It follows that $G(x) \triangle G(y)$ is finite.

Thus G reduces E to E_0.

Growth Properties

Our next goal is to show that there are countable groups $G \neq \mathbb{Z}$ such that every G-action is hyperfinite.

Definition 11.15 Suppose G is a finitely generated group. We say that G has *polynomial growth* if there is a finite $X \subseteq G$ closed under inverse such that $G = \bigcup X^n$ and there are $C, d \in \mathbb{Z}$ such that

$$|X^n| \in O(n^d)$$

for all $n > 0$.

For example $\mathbb{Z}^d$ has polynomial growth. Suppose $X = \{0, \pm e_1, \ldots, \pm e_d\}$ where

$$e_i(j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Then $(m_1, \ldots, m_d) \in X^n$ if and only if $\sum |m_i| \leq n$. Clearly, $|X_n| \leq (2n+1)^d \in O(n^d)$.

Since every finitely generated Abelian group is a quotient of $\mathbb{Z}^d$ for some d, every finitely generated Abelian group has polynomial growth.

The free group F_2 does not have polynomial growth. Let a, b generate F_2 and let $X = \{a, b, a^{-1}, b^{-1}\}$. Then X^n is the number of words of length at most n and

$$|X^n| = \sum_{i=0}^{n-1} 43^i = 4(3^n - 1) \in O(3^n).$$

Theorem 11.16 (Gromov) *Suppose G is a finitely generated group. Then G is of polynomial growth if and only if G is nilpotent-by-finite.*

We will prove that all Borel actions of finitely generated groups of polynomial growth induce hyperfinite orbit equivalence relations. In fact we will work in a more general context which will also allow us to understand actions of some nonfinitely generated groups like $\mathbb{Q}^d$.

Definition 11.17 Let G be a countable group. We say that G has the *mild growth property* of order c, if there is a sequence of finite sets $K_0 \subseteq K_1 \subseteq K_2 \ldots$ such that:

i) $\bigcup K_i = G$;
ii) $1 \in K_0$;
iii)$K_i = K_i^{-1}$ for all i;
iv) $K_i^2 \subseteq K_{i+1}$ for all i;
v) $|K_{i+4}| < c|K_i|$ infinitely many i.

Lemma 11.18 *If G is a finitely generated group of polynomial growth $O(n^d)$, then G has the mild growth property of order $16^d + 1$.*

Proof Let X be a set of generators closed under inverse such that $|X^n| \leq Cn^d$ for all $n > 0$. Let $K_n = X^{2^n}$. Clearly i)—iii) hold. Since $X^{2^n} X^{2^n} = X^{2^{n+1}}$ iv) holds. We need only argue v). Suppose not. Then there is n_0 such that

$$|K_{n+4}| \geq (16^d + 1)|K_n|$$

for all $n \geq n_0$. Then

$$|K_{4k+n_0}| \geq (16^d + 1)^k |K_{n_0}|$$

for all k. But

$$|K_{4k+n_0}| = |X^{2^{4k+n_0}}| \leq C2^{n_0 d} 2^{4kd} = C2^{n_0 d}(16^d)^k$$

and

$$(16^d + 1)^k |K_{n_0}| \leq C2^{n_0 d}(16^d)^k$$

for all k. But this is clearly impossible.

Lemma 11.19 *Suppose $G_0 \subset G_1 \subset G_2 \subset \ldots$ are finitely generated groups with the mild growth property of order c. Then $\bigcup G_i$ has the mild growth property of order c.*

Proof Let $K_{i,0} \subseteq K_{i,1} \subseteq \ldots$ witness that G_i has the mild growth property of order c. Let $\sigma : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ be a bijection such that if $\sigma(i) = (j, k)$, then $j \leq i$.

We will build $K_0 \subseteq K_1 \subseteq \ldots \subset G$. For notational convenience let $K_{-1} = \{1\}$.

Suppose we have $K_i \subseteq G_i$ for $i < 5k$. We will show how to define K_{5k+i} for $i = 0, \ldots, 4$. Let

$$K = K_{5k-1}^2 \cup K_{\sigma(k)} \subseteq G_{5k}.$$

We can find an n such that

$$K \subseteq K_{5k,n} \text{ and } |K_{5k,n+4}| \leq c|K_{5k,n}|.$$

Let $K_{5k+i} = K_{5k,n+i}$ for $i = 0, \ldots, 4$. It is easy to see that i)–iv) hold and

$$|K_{5k}| \leq c|K_{5k+4}| \text{ for all } k.$$

Since $\mathbb{Q}^d = \bigcup_{n=1}^{\infty} \frac{1}{n!}\mathbb{Z}^d$, $\mathbb{Q}^d$ has the mild growth property.

Theorem 11.20 (Jackson-Kechris-Louveau) *Let G be a countable group with the mild growth property. If E is the orbit equivalence relation for a Borel action of G on a Borel space X, then E is hyperfinite.*

In particular the orbit equivalence relation for any Borel action of a finitely generated Abelian group is hyperfinite and the orbit equivalence relation for any Borel action of $\mathbb{Q}^d$ is hyperfinite. It is still an open question if any action of a countable Abelian group is hyperfinite.

The theorem will follow from several lemmas.

Definition 11.21 Let F be a symmetric, reflexive Borel binary relation on a Borel space X. We say that F is *locally finite* if $\{y : yFx\}$ is finite for all x. We say that $Y \subseteq X$ is *F-discrete* if $\neg(xFy)$ for all distinct $x, y \in Y$ and we say that Y is *maximal F-discrete* if it is discrete and for all $x \in X$ there is $y \in Y$ with xFy.

Lemma 11.22 *Let F be a locally finite, symmetric, reflexive Borel binary relation on X. Then there is a maximal F-discrete $Y \subseteq X$.*

Proof Let $(X_n : n \in \mathbb{N})$ be a family of Borel subsets of X that separates points and is closed under finite interesections. For $x \in X$ let $\phi(x)$ be the least n such that $B_n \cap \{y : yFx\} = \{x\}$. For each n, $\phi^{-1}(n)$ is F-discrete. Let $Y_0 = \phi^{-1}(0)$ and

$$Y_{n+1} = Y_n \cup \phi^{-1}(n+1) \setminus \bigcup j \leq n \bigcup_{y \in Y_n} \{x : xFy\}.$$

Each Y_i is Borel and $\bigcup Y_i$ is maximal F-discrete.

Definition 11.23 Let $F_0 \subseteq F_1 \subseteq F_2$ be a sequence of locally finite, symmetric,reflexive Borel binary relations on X. We say that the sequences satisfies the *Weiss condition* if $F_n^2 \subseteq F_{n+1}$ for all n and there is a integer c such that for all $x \in X$ there are infinitely many n such that any F_n-discrete set contained in $\{y : yF_{n+2}x\}$ has cardinality at most c.

Note that $\bigcup F_i$ is an equivalence relation.

Lemma 11.24 *If G is a group with the mild growth property and E is the orbit equivalence relation for a Borel action of G, then there are locally finite, symmetric, reflexive Borel binary relations $F_0 \subseteq F_1 \subseteq \ldots$ satisfying the Weiss condition such that $E = \bigcup F_i$.*

Proof Let $K_0 \subseteq K_1 \subseteq \ldots$ witness that G has the mild growth property of order c. Let xF_ny if and only if there is $g \in K_n$ with $gx = y$. Since $1 \in K_0$ and $g \in K_n$ if and only if $g^{-1} \in K_n$, F_n are locally finite, reflexive and symmetric. Clearly $\bigcup F_n = E$.

We need only show it satisfies the Wiess condition. Since $K_n^2 \subseteq K_{n+1}$, $F_n^2 \subseteq F_{n+1}$. Let $x \in X$. Given $m \in \mathbb{N}$ there is a $n > m$ such that $|K_{n+4}| \leq c|K_n|$. Suppose $x_1, \ldots, x_N$ is an F_{n+1}-discrete set and $x_iF_{n+3}x$. There are $g_1, \ldots, g_N \in K_{n+3}$ such that $g_ix = x_i$ for $i = 1, \ldots, N$.

Claim $K_ng_i \cap K_ng_j = \emptyset$ for $i < j \leq N$.

Suppose $a, b \in K_n$ and $ag_i = bg_j$. Then

$$g_jg_i^{-1} = b^{-1}a \in K_{n+1}.$$

Thus

$$x_ix_j^{-1} = g_ig_j^{-1} \in K_{n+1}$$

and $x_iF_{n+1}x_j$, a contradiction.

If $h \in K_n$, then $hg_i \in K_{n+4}$. Thus

$$N|K_n| \leq |K_{n+4}| \leq c|K_n|$$

and $N \leq c$. Hence there are infinitely many n such that any F_{n+1}-discrete subset of $\{y : yF_{n+3}x\}$ has cardinality at most c and $(F_n : n \in \mathbb{N})$ has the Weiss condition.

Lemma 11.25 *Suppose $E \subseteq E^*$ are countable Borel equivalence relations. If E is hyperfinite and every E^*-class contains finitely many E-classes, then E^* is hyperfinite.*

Proof Since E^* is the orbit equivlance relation for a Borel action of some countable group G. Then $|[x]_{E^*}/E| = k$ if and only if there are $g_1, \ldots, g_k \in G$ such that $g_i x \not{E} g_j x$ for $i \neq j$ and for all $g \in G$ $gxEg_ix$ for some $i = 1, \ldots, k$. Thus $\{x : |[x]_{E^*}/E| = k\}$ is Borel and, without loss of generality we may assume that there is a fixed k such that each E^* class contains exactly k E-classes.

Let $G = \{g_0, g_1, \ldots\}$. We inductively define functions $f_1(x), \ldots, f_k(x)$ by $f_1(x) = x$. Let $N_{i+1}(x)$ be the least n such that $g_n x \not{E} f_j(x)$ for all $j \leq i$ and $f_{i+1}(x) = g_{N_{i+1}(x)} x$. Then

$$[x]_{E^*} = \bigcup_{i=1}^{k} [f_i(k)]_E.$$

Suppose $E = \bigcup E_n$ where $E_0 \subseteq E_1 \subseteq \ldots$ are finite Borel equivalence relations. Let xE_n^*y if and only if there is σ a permutaion on $\{1, \ldots, k\}$ such that $f_i(x)E_n f_{\sigma(i)}(y)$ for all i.

Clearly E_n^* is an equivalence relation and $E_n^* \subseteq E_{n+1}^*$. If xE_n^*y, then $xE_n f_i(y)$ for some i. Thus each E_n^* class is finite. If xE^*y, then there is a permutation σ such that $f_i(x)Ef_{\sigma(i)}(y)$ for all i. There is an m such that $f_i(x)E_n f_{\sigma(i)}(y)$ for all i and all $m > n$. Thus $E^* = \bigcup E_n^*$ is hyperfinite.

Proof of Theorem 11.20 By Lemma 11.24 we can find $F_0 \subseteq F_1 \subseteq \ldots$ a sequence of locally finite, symmetric, reflexive Borel binary relations with the Weiss condition such that $\bigcup F_n = E$. Let Y_n be a Borel maximal F_n-disjoint set. Let $s_n : X \to Y_n$ be a Borel measurable function such that $s_n(x)F_nx$. Let $\pi_n : X \to X$ be $s_n \circ s_{n-1} \circ \ldots \circ s_0$ and let xE_ny if and only if $\pi_n(x) = \pi_n(y)$.

Clearly E_n is an equivalence relation and $E_n \subseteq E_{n+1}$. Since each s_n is finite-to-one, π_n is finite-to-one and E_n is a finite equivalence relation. An easy induction shows that if xE_ny, then xEy. Thus $E^* = \bigcup E_n$ is a hyperfinite equivalence relation and $E \subseteq E^*$. By Lemma 11.25 it suffices to show that every E-class contains at most finitely many E^*-classes.

Suppose $(F_n : n \in \mathbb{N})$ satisifies the Weiss condition with constant c. Suppose $x_1, \ldots, x_N$ are E-equivalent but E^*-inequivalent. We can find arbitrarily large n such that $x_1, \ldots, x_N F_n x_1$ and any F_n-discrete subset of $\{y : yF_{n+2}x_1\}$ has cardinality at most c. Then $\pi_n(x_1), \ldots, \pi_n(x_N)$ are distinct elements of Y_n and hence are F_n-discrete. Since $F_i^2 \subseteq F_{i+1}$, we see, by induction, that $\pi_n(x_i)F_{n+1}x_i$. Since $x_iF_nx_1$, $\pi_n(x_i)F_{n+2}x_1$. Thus $N < c$. Thus every E-class contains at most c, E^*-classes and E is hyperfinite.

Ammenability

Throughout this section Γ will be a countable group.

Theorem 11.26 *Suppose Γ acts freely on a standard Borel space X and μ is a Γ-invariant probability measure on X. If the orbit equivalence relation E is hyperfinite, then Γ is ammenable.*

The proof we give was pointed out to me by Greg Hjorth. It uses one of the many useful characterizations of ammenability.

Suppose K is a compact metric space. Let $P(K)$ be the space of all Borel probability measures on K. We topologize $P(K)$ with the weakest topology such the maps

$$\mu \mapsto \int f d\mu$$

are continuous for all bounded continuous $f : K \to \mathbb{R}$. The space $P(K)$ is also a compact metric space and if K is a Polish space so is $P(K)$ (see [6] 17.E).

A continuous action of Γ on K induces an action of Γ on $P(K)$ by

$$g\mu(A) = \mu(g^{-1}A).$$

We say that μ is Γ-invariant if $g\mu = \mu$ for all $g \in \Gamma$.

Theorem 11.27 *A countable group Γ is ammenable if and only if for every compact metric space K and every continuous action of Γ on K, there is a Γ-invariant measure in $\mu(K)$.*

Definition 11.28 Let G and H be countable groups acting on a standard Borel space X. We say that a Borel measurable $\alpha : G \times X \to H$ is a *Borel cocycle* if

$$\alpha(gh, x) = \alpha(g, hx)\alpha(h, x)$$

for all $g, h \in G$ and $x \in X$.

If $\alpha : G \times H \to X$ is a Borel cocyle and H acts on Y, we say that $f : X \to Y$ is *α-invariant* if and only if

$$\alpha(g, x)f(x) = f(gx)$$

for all $g \in G$, $x \in X$.

Proof of Theorem 11.26

Suppose Γ acts freely on X, μ is a Γ-invariant probability measure and the orbit equivalence relation E is hyperfinite. Since E is hyperfinite and every class is infinite it is also the orbit equivalence relation for a Borel action of $\mathbb{Z}$ on X.

We define a Borel cocycle $\alpha : \mathbb{Z} \times X \to \Gamma$ such that $\alpha(n, x) = g$ if and only if $nx = gx$. Since the action of Γ is free this is a well-defined cocycle. Note that there is also a Borel cocycle $\beta : \Gamma \times X \to \mathbb{Z}$ such that $\alpha(n, x) = g$ if and only if $\beta(g, x) = n$.

Suppose Γ acts continuously on a compact metric space K. We need the following theorem of Zimmer. This is a special case of Theorem B3.1 of [5].

Theorem 11.29 (Zimmer) *There is an α-invariant, μ-measurable $x \mapsto \nu_x$ from X to $P(K)$.*

Assuming Zimmer's result we will complete the proof. We claim that there is a Γ-invariant Borel probability measure on K.

For $A \subseteq K$ Borel let

$$\nu(A) = \int_X \nu_x(A)\ d\mu.$$

Since μ and each ν_x are probability measures, ν is a probability measure, we need only show that it is Γ-invariant.

$$\begin{aligned}\nu(gA) &= \int_X \nu_x(gA)\ d\mu \\ &= \int_X g^{-1}\nu_x(A)\ d\mu \\ &= \int_X \nu_{\beta(g^{-1},x)x}(A)\ d\mu \\ &= \int_X \nu_{g^{-1}x}(A)\ d\mu.\end{aligned}$$

But

$$\int_X F(x)\ d\mu = \int_X gF(x)\ d\mu = \int_X F(g^{-1}x)\ d\mu$$

for any μ-measurable $F : X \to \mathbb{R}$ and $g \in \Gamma$. Thus

$$\int_X \nu_x(gA)\ d\mu = \int_X \nu_{g^{-1}x}(A)\ d\mu = \int_X \nu_x(A)\ d\mu$$

and $\nu(gA) = \nu(A)$.

Thus ν is a Γ-invariant Borel probability measure on K. It follows that Γ is ammenable.

We sketch the proof of Zimmer's result. Let $l_1^\infty(X, P(K))$.

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