Five Exercises

0) Define $N(X,T) := |\{x \in X \mid H(x) \leq T\}|$. Calculate $N(X,T)$ for the following defiinable sets:

a) $X = \{(x,y) \mid y = x^2\}$

b) $X = \{(x,y) \mid y = 2^x\}$

1) Show that the following three sets are the same:

- X^{alg}, the union of all connected infinite semialgebraic subsets of X.
- The union of all connected *one-dimensional* semialgebraic subsets of X.
- The union of all infinite connected subsets of algebraic curves contained in X.

2) Let E be an elliptic curve defined over $\mathbb{Q}^{\mathrm{alg}}$. Prove that there is a non-torsion point in $E(\mathbb{Q}^{\mathrm{alg}})$. [Hint: There are curves $C \subset E \times E$ of genus at least 2 defined over $\mathbb{Q}^{\mathrm{alg}}$.]

3) Let $0 < \epsilon < 1$. Prove that $\phi(n) > n^\epsilon$ for sufficiently large n. Can you do even better?

4) a) For $f \in x\mathbb{C}[[x]]$ show that $e^f = \sum_{n=0}^{\infty} \frac{f^n}{n!}$ is a well defined element of $\mathbb{C}[[x]]$.

b) Suppose $f_1, \ldots, f_n \in x\mathbb{C}[[x]]$ are linearly independent over $\mathbb{C}$. Prove that the transcendence degree of $\mathbb{C}(f_1, \ldots, f_n, e^{f_1}, \ldots, e^{f_n})$ over $\mathbb{C}$ is at least $n+1$.