Lecture Notes on Ax's Theorem
UIC Seminar–Fall 1999

I will give a proof of Ax's differential field version of Schanuel's conjecture, following ideas of Rosenlicht. The methods are similar to those used in the proof of non-minimality of differential closures and in arguements about non-elementary integrals.

§1. Preliminaries

Let K/k be fields of characteristic 0. If M is a K-vector space we say $d : K \to M$ is a *derivation* if $d(a+b) = da + db$ and $d(ab) = adb + bda$.

Lemma 1.1 There is a K-vector space $\Omega_{K/k}$ and a derivation $d : K \to \Omega_{K/k}$ such that $d|k = 0$ and if M is a K-vector space and $\delta : K \to M$ is a derivation with $\delta|k = 0$, then there is a unique K-linear $\xi : \Omega_{K/k} \to M$ with $\delta = \xi \circ d$.

Proof

Take $\Omega_{K/k}$ to be the K-vector space generated by $\{dx : x \in K\}$ mod the relations:
$d(x+y) - dx - dy$
$d(xy) - xdy - ydx$
da
for $a \in k$, $x, y \in K$.

Lemma 1.2 $\dim_K \Omega_{K/k} = \mathrm{td}(K/k)$. In particular $dx = 0$ if and only if x is algebraic over k.

Proof If $x_1, \ldots, x_n \in K$ are algebraically dependent over k, choose $p(X_1, \ldots, X_n) \in k[\overline{X}]$ of minimal degree with $p(\overline{x}) = 0$. Since

$$0 = dp(\overline{x}) = \sum_{i=1}^{n} \frac{\partial p}{\partial X_i}(\overline{x}) dx_i,$$

and each $\frac{\partial p}{\partial X_i}(\overline{x}) \neq 0$, $dx_1, \ldots, dx_n$ are linearly dependent over K.

Suppose $x_1, \ldots, x_n$ are algebraically independent over k and $\sum f_i dx_i = 0$. Let $\delta_i : K \to K$ be a derivation such that $\delta_i|k = 0$ and

$$\delta_i(x_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

and let $\xi_i : \Omega_{K/k} \to K$ be K-linear with $\delta_i = \xi_i \circ d$. Then

$$0 = \xi_j(\sum f_i dx_i) = \sum f_i \xi_j(dx_i) = f_j.$$

Hence $dx_1, \ldots, dx_n$ are linearly independent over K.

If $\delta : K \to K$ is a derivation such that $\delta|k : k \to k$ we can define $D : \Omega_{K/k} \to \Omega_{K/k}$ by

$$D(\sum f_i dx_i) = \sum \delta(f_i)dx_i + f_i d(\delta x_i).$$

This is well defined and a simple computation shows that

$$D(f\omega) = \delta(f)\omega + fD\omega.$$

If there are several fields around we write $d_{K/k}$ for d.

Lemma 1.3 Suppose $K \supset l \supset k$. There is a K-linear map $\phi : \Omega_{K/k} \to \Omega_{K/l}$ such that $d_{K/l}f = \phi(d_{K/k}f)$.

Proof

Define ϕ on the generators by sending $d_{K/k}x$ to $d_{K/l}x$ and show that the ϕ respects the relations.

§2 Ax's Theorem

Henceforth we assume that $\delta : K \to K$ is a derivation with field of constants k. Recall that k is algebraically closed in K.

We need one lemma for Ax's theorem.

Lemma 2.1 Suppose $c_1, \ldots, c_n \in k$ are linearly independent over $\mathbf{Q}$, $v, u_1, \ldots, u_n \in K$ and $\sum c_i \frac{du_i}{u_i} = dv$. Then $du_1 = \ldots = du_n = 0$.

Proof

Suppose not. Suppose u_1 is transcendental over k. Let $u_1, t_1, \ldots, t_m$ be a transcendence base for $k(u_1, \ldots, u_n, v)$ over k. Let $l = k(t_1, \ldots, t_m)$. Using 1.2 and 1.3 we see that $d_{K/l}(u_1) \neq 0$ and

$$\sum c_i \frac{d_{K/l}u_i}{u_i} = d_{K/l}(v).$$

Replacing k by l we may, without loss of generality, assume that $u_2, \ldots, u_n, v$ are algebraic over $k(u_1)$ and hence have Puiseux series expansions

$$u_j = \sum_{m=q_j}^{\infty} a_{j,m} u_1^m$$

$$v = \sum_{m=r}^{\infty} b_m u_1^m$$

where each $a_{j,q_j} \neq 0$.

Then

$$du_j = (\sum_{m=q_j}^{\infty} m a_{j,m} u_i^{m-1}) du_1$$

and

$$dv = (\sum_{m=r}^{\infty} m b_m u_1^m) du_1.$$

So

$$\frac{du_j}{u_j} = (q_j u_1^{-1} + \text{ higher terms}) du_1.$$

Comparing the u_1^{-1} coefficients we see that

$$\sum c_1 + \sum_{j=1}^{n} q_j c_j = 0.$$

Theorem 2.2 Let $y_1, \ldots, y_n, z_1, \ldots, z_n \in K$ such that

$$\delta y_i = \frac{\delta z_i}{z_i}$$

(think of $z_i = \exp y_i$). If $\mathrm{td}(k(y_1, \ldots, y_n, z_1, \ldots, z_n)/k) \leq n$, then

i) there are $m_1, \ldots, m_n \in \mathbf{Z}$ not all zero such that $\prod z_i^{m_i} \in k$ and

ii) there are $m_1, \ldots, m_n \in \mathbf{Z}$ not all zero such that $\sum m_i y_i \in k$.

Proof

If all the $\delta y_i = 0$, then we are done. So we assume $\delta(y_i) \neq 0$. We replace δ by the derivation $x \mapsto \frac{\delta x}{\delta y_i}$ and assume without loss of generality that $\delta(y_1) = 1$. Let $\omega_i = \dfrac{dz_i}{z_i}$. By 1.2 $\dim_K \Omega_{K/k} \leq n$ so there are $f_1, \ldots, f_n, g \in K$ not all zero such that $\sum f_i \omega_i + g dy_1 = 0$. We can choose $f_1, \ldots, f_n, g$ such that a minimal number of them are nonzero and at least one of them is 1.

Note that

$$\begin{aligned} D(dy_i - \frac{dz_i}{z_i}) &= d(\delta y_i) - \frac{1}{z_i} d(\delta z_i) + \frac{\delta z_i}{z_i^2} dz_i \\ &= d(\frac{\delta z_i}{z_i}) - \frac{1}{z_i} d(\delta z_i) + \frac{\delta z_i}{z_i^2} dz_i \\ &= \frac{z_i d(\delta z_i) - (\delta z_i) dz_i}{z_i^2} - \frac{1}{z_i} d(\delta z_i) + \frac{\delta z_i}{z_i^2} dz_i = 0. \end{aligned}$$

Thus

$$\begin{aligned} 0 &= D(\sum f_i \omega_i + g dy_1) \\ &= \sum (\delta(f_i)\omega_i + f_i D\omega_i) + \delta g dy_1 + dD(dy_1) \\ &= \delta(g) y_i + \sum \delta(f_i) \omega_i \end{aligned}$$

Since one of $f_1, \ldots, f_n, g$ is 1, the minimality assumption implies that

$$\delta f_1 = \ldots = \delta f_n = \delta g = 0.$$

Thus we can find $c_0, \dots, c_n \in k$ such that $\sum c_i \omega_i + c_0 dy_1 = 0$. Let

$$v = (c_0 + c_1 y_1) + c_2 y_2 + \dots c_n y_n$$

then $\sum c_i \frac{dz_i}{z_i} = dv$.

Suppose $c_1, \dots, c_m$ is a basis for $c_1, \dots, c_n$ over $\mathbf{Q}$. Let $c_j = \sum_{i=1}^{m} l_{i,j} c_i$ where $l_{i,j} \in \mathbf{Q}$. We may assume (changing v if needed) that all the $l_{i,j} \in \mathbf{Z}$. Then

$$\begin{aligned} \sum_{j=1}^{n} c_j \frac{dz_i}{z_i} &= \sum_{i=1}^{m} c_i (\sum_{j=1}^{n} l_{i,j} \frac{dz_j}{z_j}) \\ &= \sum_{i=1}^{m} c_i \frac{dw_i}{w_i} \end{aligned}$$

where $w_i = \prod_{j=1}^{n} z_j^{l_{i,j}}$.

By lemma 2.1 each $dw_i = 0$. Since k is algebraically closed in K, by 1.2 each $w_i \in k$. Thus i) holds. But

$$0 = \frac{\delta w_i}{w_i} = \sum_{j=1}^{m} l_{i,j} \frac{\delta z_j}{z_j} = \sum_{j=1}^{m} l_{i,j} \delta(y_i) = \delta(\sum_{j=1}^{m} l_{i,j} y_j).$$

Thus $\sum_{j=1}^{m} l_{i,j} y_j \in k$.

Schanuel's Conjecture for Power series Let $y_1, \dots, y_n \in \mathbf{C}[[X_1, \dots, X_n]]$ such that $y_i(0) = 0$. If $y_1, \dots, y_n$ are linearly independent over $\mathbf{Q}$, then

$$\mathrm{td}(y_1, \dots, y_n, e^{y_1}, \dots, e^{y_n}) > n.$$

Proof

If not, then by Theorem 2.2 there are rational numbers m_i, not all zero, such that $\sum m_i y_i \in \mathbf{C}$. Since all of the $y_i(0) = 0$, we must have $\sum m_i y_i = 0$.