# Exercise for o-minimalty Lectures I and II

## Real Closed Fields

**Exercise 1** Let $x$ and $y$ be algebraically independent over $\mathbb{R}$. Show that $\mathbb{R}(x, y)$ is formally real and that we can find orders $<_1$ and $<_2$ of $\mathbb{R}(x, y)$ such that $x <_1 y$ and $y <_2 x$.

**Exercise 2** Let $F$ be a real closed field. We say that a function $g : F^n \to F$ is *algebraic* if there is a nonzero polynomial $p(X_1, \ldots, X_n, Y)$ over $F$ such that for all $a \in F^n$, $p(a, g(a)) = 0$.

a) Use quantifier elimination to show that every semialgebraic function is algebraic.

b) Show that if $f : \mathbb{R} \to \mathbb{R}$ is semialgebraic, then there are disjoint intervals $I_1, \ldots, I_m$ and a finite set $X$ such that $\mathbb{R} = I_1 \cup \ldots \cup I_m \cup X$ and $f$ is analytic on each $I_j$. (Hint: Use the Implicit Function Theorem for $\mathbb{R}$.)

**Exercise 3** (Real Nullstellensatz) Let $F$ be a real closed field, and let $P$ be a prime ideal in $F[X_1. \ldots, F[X_n]]$. We say that $P$ is *real* if whenever $\sum p_i^2 \in P$, then all the $p_i \in P$. Prove that $I(V_F(P)) = P$ if and only if $P$ is real.

**Exercise 4** Prove that for all $n$ and $d$ there are $M$ and $D$ such that if $f(X_1, \ldots, X_n) = \frac{g}{h}$ where $g$ and $h$ are real polynomials of degree at most $d$ and $f$ is positive semidefinite, then there are polynomials $g_1, \ldots, g_M$, $h_1, \ldots, h_M$ of degree at most $D$ such that

$$f = \sum_{i=1}^{M} \frac{g_i^2}{h_i^2}.$$

**Exercise 5** If $K$ is a field, let $K[[t]]$ denote the field of formal power series over $K$ in variable $t$, and let $K((t))$ denote its fraction field, the field of formal Laurent series over $K$. Let

$$K\langle\langle t\rangle\rangle = \bigcup_{n=1}^{\infty} K\left(\left(t^{\frac{1}{n}}\right)\right)$$

be the field of formal *Puiseux series* over $K$. Series in $K\langle\langle t\rangle\rangle$ are of the form $\sum_{i=m}^{\infty} a_i t^{\frac{i}{n}}$ for some $m, n \in \mathbb{Z}$ with $n > 0$. An important theorem is that if $K$ is algebraically closed, then $K\langle\langle t\rangle\rangle$ is also algebraically closed. It follows that if $R$ is real closed then $R\langle\langle t\rangle\rangle$ is real closed.

a) Show that $R \prec R\langle\langle t\rangle\rangle$, and $t$ is a positive infinitesimal element of $R\langle\langle t\rangle\rangle$.

b) Suppose that $r \in R$ and $f : (0, r) \to R$ is definable. Show that there is $\mu \in R\langle\langle t\rangle\rangle$ such that $R\langle\langle t\rangle\rangle \models f(t) = \mu$. Suppose that $\mu = at^q+$ higher-degree terms. Show that $f$ is asymptotic to $ax^q$ at 0. In other words, show that

$$R \models \forall\epsilon > 0 \exists\delta > 0 \ \left(0 < x < \delta \to \left|\frac{f(x)}{ax^q} - 1\right| < \epsilon\right).$$

## Algebraically Closed Fields

**Exercise 6** a) Use the quantifier elimination test to prove that the theory of algebraically closed fields has quantifier elimination.

b) For $p = 0$ or $p$ prime let $\mathrm{ACF}_p$ be the theory of algebraically closed fields of characteristic $p$. Conclude that $\mathrm{ACF}_p$ is complete.

c) Let $\phi$ be a sentence in the language of fields. Show that the following are equivalent.

i) $\mathbb{C} \models \phi$;

ii) $\mathrm{ACF}_0 \models \phi$;

iii) $\mathrm{ACF}_p \models \phi$ for all sufficiently large $p$;

iv) there are infinitely many primes $p$ such that there is an algebraically closed field $K_p$ of characteristic $p$ such that $K \models \phi$.

d) [Ax's Theorem] For $p$ let $\mathbb{F}_p^{\mathrm{alg}}$ be the algebraic closure of the $p$ element field $\mathbb{F}_p$. Show that in every injective polynomial map $f : K^n \to K^n$ is onto. [Hint: First note that this is true in the finite field $\mathbb{F}_{p^m}$ and that $\mathbb{F}_p^{\mathrm{alg}} = \bigcup \mathbb{F}_{p^m}$]. Conclude that the same is true in $\mathbb{C}$.

**Exercise 7** Let $K \subset L$ be algebraically closed fields. Let $V, W \subseteq L^n$ be Zariski closed sets defined over $K$. Suppose that there is $f : V \to W$ a bijective polynomial map defined over $L$. Show that there is $g : V \cap K^n \to W \cap K^n$ a bijective polynomial map defined over $K$.

**Exercise 8 [Nullstellensatz]** Let $K$ be an algebraically closed field. Let $P$ be a prime ideal and in $K[X_1, \ldots, X_n]$ and $g \in K[X_1, \ldots, X_n] \setminus P$. Let $L$ be the algebraic closure of the fraction field of $K[X_1, \ldots, X_n]/P$.

Show that $K \prec L$. Show that there is $x \in K^n$ such that $f(x) = 0$ for $f \in P$ but $g(x) \neq 0$. Conclude that

$$P = I(V(P)) = \{g \in K[X_1, \ldots, X_n] : \text{ if } f(x) = 0 \text{ for all } f \in P, \text{ then } g(x) = 0\}.$$

## Model Completeness

**Exercise 9** a) Show that the following are equivalent.

i) There is a universal formula $\psi(\overline{v})$ such that $T \models \forall \overline{v}(\phi(\overline{v}) \leftrightarrow \psi(\overline{v}))$.

ii) If $\mathcal{M}$ and $\mathcal{N}$ are models of $T$ with $\mathcal{M} \subset \mathcal{N}$, $\overline{a} \in M$, and $\mathcal{N} \models \phi(\overline{a})$, then $\mathcal{M} \models \phi(\overline{a})$.

b) Show that a theory $T$ has a universal axiomatization if and only if $\mathcal{N} \models T$ whenever $\mathcal{M} \subset \mathcal{N}$ and $\mathcal{M} \models T$.

**Exercise 10** We say that $T$ has a $\forall\exists$-axiomatization if it can be axiomatized by sentences of the form $\forall v_1 \ldots \forall v_n \exists w_1 \ldots \exists w_m\ \phi(\overline{v}, \overline{w})$ where $\phi$ is a quantifier-free formula.

a) Suppose that $T$ has a $\forall\exists$-axiomatization, $(I, <)$ is a linear order, and $(\mathcal{M}_i : i \in I)$ is a chain of models of $T$. Show that $\bigcup \mathcal{M}_i$ is a model of $T$.

We will show that the converse also holds. Suppose that whenever $(\mathcal{M}_i : i \in I)$ is a chain of models of $T$, then $\bigcup M_i \models T$. Let $\Gamma = \{\phi : \phi$ is a $\forall\exists$-sentence and $T \models \phi\}$. Let $\mathcal{M} \models \Gamma$. We will show that $\mathcal{M} \models T$.

b) Show that there is $\mathcal{N} \models T$ such that if $\psi$ is an $\exists\forall$-sentence and $\mathcal{M} \models \psi$, then $\mathcal{N} \models \psi$.

c) Show that there is $\mathcal{N}' \supseteq \mathcal{M}$ with $\mathcal{N}' \equiv \mathcal{N}$ such that if $a_1, \ldots, a_n \in \mathcal{M}$, $\phi(v_1, \ldots, v_n)$ is universal and $\mathcal{M} \models \phi(a_1, \ldots, a_n)$, then $\mathcal{N}' \models \phi(a_1, \ldots, a_n)$.

d) Show that there is $\mathcal{M}' \supseteq \mathcal{N}'$ such that $\mathcal{M} \prec \mathcal{M}'$.

e) Iterate the constructions from c) and d) to build a chain of structures

$$\mathcal{M} = \mathcal{M}_0 \subseteq \mathcal{N}_1 \subseteq \mathcal{M}_1 \subseteq \mathcal{N}_2 \ldots$$

such that $\mathcal{M}_i \prec \mathcal{M}_{i+1}$ for $i = 0, 1, \ldots$ and each $\mathcal{N}_i \equiv \mathcal{N}_{i+1}$. Let $\mathcal{M}^* = \bigcup \mathcal{M}_i = \bigcup \mathcal{N}_i$. Show that $\mathcal{M}^* \models T$ and $\mathcal{M} \prec \mathcal{M}^*$.

f) Conclude that $T$ is $\forall\exists$-axiomatizable.

**Exercise 11** We say that $\mathcal{M} \models T$ is *existentially closed* if whenever $\mathcal{N} \models T$, $\mathcal{N} \supseteq \mathcal{M}$, and $\mathcal{N} \models \exists \overline{v}\ \phi(\overline{v}, \overline{a})$, where $\overline{a} \in M$ and $\phi$ is quantifier-free, then $\mathcal{M} \models \exists \overline{v}\ \phi(\overline{v}, \overline{a})$.

a) Show that if $T$ is $\forall\exists$-axiomatizable, then $T$ has an existentially closed model. Indeed, if $\mathcal{M} \models T$, there is $\mathcal{N} \supseteq \mathcal{M}$ an existentially closed model of $T$ with $|N| = |M| + |\mathcal{L}| + \aleph_0$.

b) Suppose that $T$ has an infinite nonexistentially closed model. Prove that $T$ has a nonexistentially closed model of cardinal $\kappa$ for any infinite cardinal $\kappa \geq |\mathcal{L}|$. [Hint: Suppose that $\mathcal{M} \subset \mathcal{N}$ are models of $T$ and $\mathcal{N}$ satisfies an existential formula not satisfied in $\mathcal{M}$. Consider models of the theory of $\mathcal{N}$ where we add a unary predicate for $M$.]

c) Suppose that $T$ is $\kappa$-categorical for some infinite $\kappa \geq |\mathcal{L}|$ and axiomatized by $\forall\exists$-sentences. Prove that all models of $T$ are existentially closed. Conclude that every algebraically closed field is existentially closed.

**Exercise 12**

a) Show that if $\mathcal{M} \subseteq \mathcal{N}$ and $\mathcal{M}$ is existentially closed, then there is $\mathcal{M}_1 \models T$ such that $\mathcal{M} \subseteq \mathcal{N} \subseteq \mathcal{M}_1$ with $\mathcal{M} \prec \mathcal{M}_1$. [Note: See Exercise **??**.]

b) Show that $T$ is model-complete if and only if every model of $T$ is existentially closed. [Hint: ($\Leftarrow$) Suppose that $\mathcal{M}_0 \subseteq \mathcal{N}_0$ are models of $T$. Use a) to build $\mathcal{M}_0 \subseteq \mathcal{N}_0 \subseteq \mathcal{M}_1 \subseteq \mathcal{N}_1 \subseteq \mathcal{M}_2 \subseteq \ldots$, a chain of models of $T$ such that $M_i \prec \mathcal{M}_{i+1}$ and $\mathcal{N}_i \prec \mathcal{N}_{i+1}$.]

c) Suppose that $T$ is a $\forall\exists$-axiomatizable theory with infinite models that is $\kappa$-categorical for some infinite cardinal $\kappa$. Show that $T$ is model-complete. [Hint: See Exercise **??**.]

d) Show that $T$ is model-complete if and only if for any formula $\phi(\overline{v})$ there is a quantifier-free formula $\psi(\overline{v}, \overline{w})$ such that $T \models \phi(\overline{v}) \leftrightarrow \exists \overline{w}\ \psi(\overline{v}, \overline{w})$. [Hint: Use Exercise **??**.]

e) Show that any model-complete theory has a $\forall\exists$ axiomatization. [Hint: Use Exercise **??**.]

## Basic o-minimality

**Exercise 13** Suppose $(M, +, <, \ldots)$ is an o-minimal ordered group.

a) Prove that any definable subgroup must be convex and conclude there are no non-trivial definable subgroups.

b) Prove that $M$ is divisible and abelian.

**Exercise 14** Suppose $(M, +, \cdot, \ldots)$ is an o-minimal ordered field, prove that $M$ is real closed.

**Exercise 15** Suppose $(M, <, \ldots)$ is o-minimal, $a < b \in M$ and $f : (a, b) \to M$ is strictly increasing. Prove that $f$ is continuous on an interval $I \subseteq (a, b)$.

**Exercise 16** [**Definable Skolem Functions**] Suppose $(M, <, +, \ldots)$ is an o-minimal group and $X \subseteq M^{m+n}$ is definable. For $a \in M^m$ let

$$X_a = \{b \in M^n : (a, b) \in X\}.$$

Let $Y = \{a \in M^n : X_a \neq \emptyset\}$. Prove that there is a definable $f : Y \to M^n$ such that $(y, f(y)) \in X$ for all $y \in Y$. [Hint: First consider the case $n = 1$.]

**Exercise 17** [**Curve Selection**] Suppose $(M, +, <, \cdot, \ldots)$ is an expansion of an o-minimal ordered field. Suppose $X \subset M^n$ is definable and $a$ is in the closure of $X$. Prove that there is a definable $f : (0, b) \to M^n$ such that $f(x) \in X$ for all $x$ and $\lim_{x \to 0} f(x) = a$. [Hint: Consider $\{(\epsilon, x) : x \in X \text{ and } d(x, a) < \epsilon\}$.]

## Definable Closure and Exchange

Fix $(M, <, \ldots)$ o-minimal . If $A \subseteq M$, we let $\mathrm{dcl}(A)$ be the definable closure of $A$. Recall that $b \in \mathrm{dcl}(A)$ if there is a formula $\phi(v)$ with parameters from $A$ such that

$$\{x \in M : \phi(x)\} = \{b\}.$$

**Exercise 18** [**Exchange**] Suppose $c \in \mathrm{dcl}(A \cup \{b\})$. Then $c \in \mathrm{dcl}(A)$ or $b \in \mathrm{dcl}(A \cup \{c\})$.

We say that $A \subset M$ is *independent* if and only if $a \notin \mathrm{dcl}(A \setminus \{a\})$ for all $a \in A$. We say that $B_0 \subseteq B$ is a *basis* for $B$ if and only if $B_0$ is independent and $B \subseteq \mathrm{dcl}(B_0)$.

**Exercise 19** Prove that any two bases for $B$ have the same cardinality.

**Exercise 20** Prove that if $K$ is a real closed field $a_1, \ldots, a_n \in K$ are independent if and only if they are algebraically independent.

## Consequences of Cell Decomposition

**Exercise 21** Suppose $\mathcal{M}$ is o-minimal and $\mathcal{N}$ is elementarily equivalent to $\mathcal{M}$. Prove that $N$ is o-minimal.

**Exercise 22** Suppose $(M, <, +, \cdot, \ldots)$ is an expansion of an ordered field. Then every $k$-cell is definably homeomorphic to $(0, 1)^k$.

**Exercise 23** A definable set $X \subseteq M^n$ is *definably connected* if if we can not find definable disjoint open sets $U, V$ such that $U \cap X \neq \emptyset$, $V \cap X \neq \emptyset$ and $X \subset U \cup V$. Prove that every cell is definably connected.

**Exercise 24** We say $B \subseteq A$ is a *definably connected component* if it is a maximal definably connected subset of $A$. Prove that every definable set is a finite disjoint union of definably connected sets.

**Exercise 25** Suppose $A \subset M^{n+m}$ is definable. There is $N$ such that $A_a$ has at most $N$ connected components for all $a \in M^n$

## Dimension

Let $\mathcal{M} = (M, <, \ldots)$ be o-minimal.

**Exercise 26** Let $A \subseteq M^n$ be defined by $\phi(v_1, \ldots, v_n, \mathbf{a})$ where $\mathbf{a} \in M^m$. Prove that the following are equivalent for $k \leq n$.

i) $A$ contains a $k$-cell;

ii) there is a coordinate projection map $\pi : M^n \to M^k$ such that $\pi(A)$ has interior;

iii) there is $\mathcal{N}$ an elementary extension of $\mathcal{M}$ and $b_1, \ldots, b_n \in N^n$ such that $\mathcal{N} \models \phi(b_1, \ldots, b_m, \mathbf{a})$ and there is $B_0 \subseteq B$ independent over $\mathbf{a}$ of cardinality $k$.

We let dim $A$ be the maximal $k$ such that i)–iii) hold.

**Exercise 27** Suppose $A$ is definable and $f : A \to M^n$ is definable and injective. Then dim $A =$ dim $f(A)$.

**Exercise 28** Suppose $A \subseteq M^{n+m}$ is definable, then for each $k \leq m$

$$\{a \in M^n : A_a \text{ has dimension } k\}$$

is definable.