Lectures on the model theory of real and complex exponentiation

A.J. Wilkie
School of Mathematics
The Alan Turing Building
University of Manchester
Manchester M13 9PL
UK

July 17, 2012

1 Introduction

We shall be interested in the structures $\mathbb{R}_{exp}$ and $\mathbb{C}_{exp}$, the expansions of the rings of real and complex numbers respectively, by the corresponding exponential functions and, in particular, the problem of describing mathematically their definable sets.

In the real case I shall give the main ideas of a proof of the theorem stating that the theory T_{exp} of the structure $\mathbb{R}_{exp}$ is *model complete.*

Definition 1. *A theory T in a language L is called* model complete *if it satisfies one of the following equivalent conditions:*
(i) for every formula $\phi(x)$ of L, there exists an existential formuls $\psi(x)$ of L such that $T \models \forall x(\phi(x) \leftrightarrow \psi(x))$;
(ii) for all models $\mathbb{M}_0$, $\mathbb{M}$ of T with $\mathbb{M}_0 \subseteq \mathbb{M}$ we have that $\mathbb{M}_0 \preccurlyeq_1 \mathbb{M}$ (i.e. existential formulas are absolute between $\mathbb{M}_0$ and $\mathbb{M}$);
(iii) for all models $\mathbb{M}_0$, $\mathbb{M}$ of T with $\mathbb{M}_0 \subseteq \mathbb{M}$ we have that $\mathbb{M}_0 \preccurlyeq \mathbb{M}$ (i.e. all formulas are absolute between $\mathbb{M}_0$ and $\mathbb{M}$);
(iv) for all models $\mathbb{M}$ of T, the $L_{\mathbb{M}}$-theory $T \cup Diagram(\mathbb{M})$ is complete.

This is, of course, a theorem. The actual definition that gives rise to the name is (iv).

I shall prove that (ii) holds for $T = T_{exp}$ and in this case the task can be further reduced:

Exercise 1: Suppose that all pairs of models $\mathbb{M}_0$, $\mathbb{M}$ of T_{exp} with $\mathbb{M}_0 \subseteq \mathbb{M}$ have the property that any quasipolynomial (see below) with coefficients in $\mathbb{M}_0$ and having a solution in $\mathbb{M}$, also has a solution in $\mathbb{M}_0$. Prove that T_{exp} is model complete.

Definition 2. *Let $\mathbb{M}_0 = \langle M_0, ...\rangle$ be a substructure of a model $\mathbb{M} = \langle M, ...\rangle$ of T_{exp}. Then a function (from M^n to M) that can be written in the form*

$$\langle x_1, \dots, x_n\rangle \mapsto P(x_1, \dots, x_n, \exp(x_1), \dots, \exp(x_n)),$$

where P is a polynomial (in $2n$ variables) with coefficients in M_0, is called a quasipolynomial *with coefficients in $\mathbb{M}_0$.*

In fact I shall only prove, in the notation of Exercise 1, that a quasipolynomial with solutions in M^n has one, $\langle b_1, \dots, b_n\rangle$ say, that is $\mathbb{M}_0$-bounded, i.e. for some $a \in M$ with $a > 0$, we have that $-a \leq b_i \leq a$ for $i = 1, \dots, n$.

To go on to find a solution in M_0^n requires a separate argument, which I shall not go into in these notes. The method is fairly routine and appeared several years before the eventual proof of the model completeness of T_{exp} (see [W3]). The case $n = 1$ of both arguments is reasonably straightforward and serves as a good introduction to the general case:

Exercise 2 Let $\mathbb{M}_0 = \langle M_0, ...\rangle$ and $\mathbb{M} = \langle M, ...\rangle$ be models of T_{exp} with $\mathbb{M}_0 \subseteq \mathbb{M}$. Use the two step method discussed above to show that every zero in M of a one variable (nonzero) quasipolynomial with coefficients in M_0, actually lies in M_0. [That is, first show that such zeros are $\mathbb{M}_0$-bounded, and then show that they lie in M_0.]

Of course it is (i) in Definition 2 that is the desired conclusion, but by establishing it via the model theoretic statement (ii) we get no information on how to *effectively* find $\psi(x)$ from $\phi(x)$. We do know that this can be done in principle if Schanuel's Conjecture is true, but even with this assumption a transparent (say, primitive recursive) algorithm is still lacking.

In the complex case, where we know that model completeness fails (see [M]), we discuss the conjecture of Zilber stating that every definable subset of

$\mathbb{C}$ (in the structure $\mathbb{C}_{exp}$) is either countable or cocountable. This property (of an expansion of the complex field say) is called *quasi-minimality*. (A structure is called just *minimal* if every definable subset of its domain is either finite or cofinite. When discussing Zilber's conjecture I shall be assuming that the reader has studied some complex analysis as well as model theory. Many of the arguments will combine the two. Here is an example.

Exercise 3: Let $\mathbb{M}$ be an expansion of the ring of complex numbers in which an entire function $f : \mathbb{C} \to \mathbb{C}$ is definable. Prove that if $\mathbb{M}$ is minimal then f is a polynomial.)

I shall study these rather specific topics in the more general context of expansions of the underlying fields (real or complex) by analytic functions.

2 Analytic functions

In this section $\mathbb{K}$ denotes either $\mathbb{R}$ or $\mathbb{C}$.

Let U be an open neighbourhood of a point $\omega \in \mathbb{K}^n$ (for the usual topology) and let $f : U \to \mathbb{K}$ be an infinitely differentiable function. Then we may form the Taylor series of f at ω:

$$Tf := \sum_{\alpha \in \mathbb{N}^n} \frac{f^{(\alpha)}(\omega)}{\alpha!} \cdot (x - \omega)^\alpha.$$

It is important to realise that in the real case this is just a formal series in the variables $x = \langle x_1, \ldots, x_n \rangle$. It may not converge for any values of the variables (other than ω itself) and even if it does, the sum may bear no relation to $f(x)$. If there exists an open neighbourhood V of ω with $V \subseteq U$ such that the series converges to $f(x)$ for each $x \in V$, then we say that the function f is *analytic* at ω. If it is analytic at all points ω in its domain U then we just say that f is *analytic*. In the complex case, the continuous differentiability of f on U is in fact sufficient to guarantee that f is analytic.

I now state some theorems that will be important when we come to discuss quantifier elimination and model completeness for expansions of $\mathbb{K}$ by analytic functions.

Throughout these notes, $\Delta_{\mathbb{K}}^{(n)}(\omega; r)$ denotes the *box neighbourhood* (also called the *polydisk*, in the case that $\mathbb{K} = \mathbb{C}$) centred at $\omega \in \mathbb{K}^n$ and polyradius $r \in \mathbb{R}^n_{>0}$ defined by

$$\Delta_{\mathbb{K}}^{(n)}(\omega; r) := \{x \in \mathbb{K}^n : |x_i - \omega_i| < r_i \text{ for } i = 1, \ldots, n\}.$$

The subscript $\mathbb{K}$ will be omitted if it is clear from the context, and the superscript also in the case that $n = 1$.

For polyradii r and s we write $s < r$, or $s \leq r$, if corresponding coordinates are so ordered. In the following statements we assume that $n > 1$ and make the convention that primed variables are $(n-1)$-tuples. Further, if y is already designated as an n-tuple, then y' is its initial $(n-1)$-tuple.

Theorem 1 (The Implicit Function Theorem for one dependent variable)**.** *Suppose that $F : \Delta_{\mathbb{K}}^{(n)}(0; r) \to \mathbb{K}$ is analytic. Suppose further that*

$$F(0) = 0 \neq \frac{\partial F}{\partial x_n}(0).$$

Then there exists $s \in \mathbb{R}^n_{>0}$ with $s \leq r$ and an analytic function $\phi : \Delta_{\mathbb{K}}^{(n-1)}(0'; s') \to \mathbb{K}$ such that $\phi(0') = 0$ and for all $x' \in \Delta_{\mathbb{K}}^{(n-1)}(0'; s')$ we have $F(x', \phi(x')) = 0$. Further, for each such x', $\phi(x')$ is the only $y \in \mathbb{K}$ satisfying $|y| < s_n$ and $F(x', y) = 0$.

The uniqueness statement here is important for definability issues. It guarantees that (at least, in the real case) if the data is definable then so is the implicit function ϕ. In the following more general result the non-singularity condition is relaxed. However, it suffers from the disadvantage that the functions asserted to exist are not necessarily definable from the data.

Theorem 2 (The Weierstrass Preparation Theorem)**.** *Suppose that $F : \Delta_{\mathbb{K}}^{(n)}(0; r) \to \mathbb{K}$ is analytic. Suppose further that $p \geq 1$ and*

$$F(0) = \frac{\partial F}{\partial x_n}(0) = \cdots = \left(\frac{\partial}{\partial x_n}\right)^{p-1} F(0) = 0 \neq \left(\frac{\partial}{\partial x_n}\right)^{p} F(0).$$

Then there exists $s \in \mathbb{R}^n_{>0}$ with $s \leq r$, analytic functions $\phi_1, \ldots, \phi_p : \Delta_{\mathbb{K}}^{(n-1)}(0'; s') \to \mathbb{K}$ and $u : \Delta_{\mathbb{K}}^{(n)}(0; s) \to \mathbb{K}$ such that $\phi_1(0') = \ldots = \phi_p(0') = 0$, u does not vanish, and for all $x \in \Delta_{\mathbb{K}}^{(n)}(0; s)$

$$F(x) \ = \ u(x) \cdot (x_n^p + \phi_1(x') \cdot x_n^{p-1} + \cdots + \phi_p(x')).$$

Theorem 2 forms the foundation for the local theory of analytic functions and analytic sets (= zero sets of analytic functions). For the geometric theory the non-singularity hypothesis constitutes no loss of generality:

Exercise 4: Let $F : \Delta_{\mathbb{K}}^{(n)}(0; r) \to \mathbb{K}$ be an analytic function. Suppose that $F(0) = 0$ but that F is not identically zero. Prove that after a linear change of coordinates (which may be taken to have an integer matrix) the hypothesis of Theorem 2 holds (for some $p \geq 1$ and possibly a smaller r).

Unfortunately, as we shall see in the next section, this is not good enough for model theoretic considerations: one cannot just, e.g., permute variables when one is considering a fixed projection of, say, the zero set of an analytic function into lower dimensions. One has to have some form of preparation theorem that deals the case that $\left(\frac{\partial}{\partial x_n}\right)^p F(0) = 0$ for all p, or, equivalently, when the function $F(0', \cdot)$ is identically zero.

Theorem 3 (The Denef-van den Dries Preparation Theorem)**.** *Let* $F : \Delta_{\mathbb{K}}^{(n)}(0; r) \to \mathbb{K}$ *be an analytic function. Then there exist* $d \geq 1$*,* $s \leq r$ *and analytic functions* $\phi_j : \Delta_{\mathbb{K}}^{(n-1)}(0; s') \to \mathbb{K}$*, and* $u_j : \Delta_{\mathbb{K}}^{(n)}(0; s) \to \mathbb{K}$ *(for* $j \leq d$*) with the* u_j*'s nonvanishing, such that for all* $x \in \Delta_{\mathbb{K}}^{(n)}(0; s)$

$$F(x) = \sum_{j=0}^{d} \phi_j(x') \cdot x_n^j \cdot u_j(x).$$

Finally in this section I state the many variable version of the Implicit Function Theorem. The role of the derivative is played by the *Jacobian*:

Definition 3. *Let* $F : U \times V \to \mathbb{K}^m$ *be an analytic map (i.e. its coordinate functions are analytic functions) where* $U \subseteq \mathbb{K}^n$ *and* $V \subseteq \mathbb{K}^m$ *are open sets. Write* x *for the* U*-variables and* y *for the* V*-variables. Then the* Jacobian J_F *of* F *with respect to* y *is given by the determinant of the matrix*

$$\left(\frac{\partial F_i}{\partial y_j}\right)_{1 \leq i,j \leq m}.$$

It is an analytic function from $U \times V$ *to* $\mathbb{K}$.

Theorem 4 (The Implicit Function Theorem for several dependent variables)**.** *Let* $F : U \times V \to \mathbb{K}^m$ *be as in the definition. Let* $\langle a, b\rangle \in U \times V$ *be such that* $F(a, b) = 0$ *and* $J_F(a, b) \neq 0$*. Then there exists an open neighbourhood* $U_a \times V_b \subseteq U \times V$ *of* $\langle a, b\rangle$ *and an analytic map* $\Phi : U_a \to V_b$ *such that for all* $x \in U_a$ *we have* $F(x, \Phi(x)) = 0$*. Further, for each such* x*,* $\Phi(x)$ *is the unique* $y \in V_b$ *satisfying* $F(x, y) = 0$*.*

It hardly seems worth giving a reference for the Implicit Function Theorem-just consult your favourite analysis text. The same is almost true for the Weierstrass Preparation Theorem though for this (and for the formal case) I would recommend [R]. As for Theorem 3, I give a sketch of the proof in [W2] but for more details you will have to consult the original paper [DvdD].

3 The structure $\mathbb{R}_{an}$ and its reducts

In this and the next three sections we are only concerned with the case that $\mathbb{K} = \mathbb{R}$.

The local theory of real analytic functions and real analytic sets is captured by the definability theory of the structure $\mathbb{R}_{an}$. This is the expansion of the ordered ring of real numbers $\bar{\mathbb{R}} := \langle \mathbb{R}; +, \cdot, -, 0, 1, < \rangle$ by all $r \in \mathbb{R}$ as distinguished elements and all *restricted* analytic functions:

Definition 4. *Let $n \geq 1$, $\omega \in \mathbb{R}^n$, $r \in \mathbb{R}^n_{>0}$ and $f : \Delta^{(n)}(\omega; r) \to \mathbb{R}$ be an analytic function. Let s be a polyradius with $s < r$. Then the function $\widetilde{f} : \mathbb{R}^n \to \mathbb{R}$ defined by*

$$\widetilde{f}(x) = \begin{cases} f(x) & \text{if } x \in \Delta^{(n)}(\omega; s), \\ 0 & \text{otherwise.} \end{cases}$$

is called a restricted analytic function*. Of course it is not analytic on $\mathbb{R}^n$, only on $\Delta^{(n)}(\omega; s)$.*

Definition 5. *(i)* $\mathbb{R}_{an} := \langle \bar{\mathbb{R}}, \{r\}_{r \in \mathbb{R}}, \{\textit{all restricted analytic functions}\} \rangle$.
(ii) $T_{an} := Th(\mathbb{R}_{an})$.
(iii) $\mathbb{R}^D_{an} := \langle \mathbb{R}_{an}, D \rangle$*, where* $D : \mathbb{R}^2 \to \mathbb{R}$ *is defined by*

$$D(x, y) = \begin{cases} x/y & \text{if } y \neq 0 \text{ and } |x| \leq |y|, \\ 0 & \text{otherwise.} \end{cases}$$

(iv) $T^D_{an} := Th(\mathbb{R}^D_{an})$.

We have the following results.

Theorem 5 (Gabrielov)**.** *T_{an} is model complete.*

Theorem 6 (Denef-van den Dries)**.** *T_{an}^D admits elimination of quantifiers.*

Since both the graph of the function D and its complement in $\mathbb{R}^3$ are existentially definable in the language $\mathcal{L}(\mathbb{R}_{an})$, we see immediately that the two structures $\mathbb{R}_{an}^D$ and $\mathbb{R}_{an}$ have the same definable (= existentially definable) sets, and that Theorem 6 implies Theorem 5. The theory T_{an} does not have quantifier elimination (in the language $\mathcal{L}(\mathbb{R}_{an})$), as is immediate from the following

Exercise 5 Let $A \subseteq \mathbb{R}^n$ be quantifier-free definable in the language $\mathcal{L}(\mathbb{R}_{an})$. Prove that there exists $R \in \mathbb{R}_{>0}^n$ and a semi-algebraic set $B \subseteq \mathbb{R}^n$ such that $A \cap (\mathbb{R}^n \setminus \Delta^{(n)}(0;R)) = B \cap (\mathbb{R}^n \setminus \Delta^{(n)}(0;R))$. Produce an example to show that this is false with $\mathbb{R}_{an}^D$ in place of $\mathbb{R}_{an}$.

It is rather a deep fact that any subset of $\mathbb{R}^2$ which is quantifier free definable in the language $\mathcal{L}(\mathbb{R}_{an}^D)$ is in fact quantifier free definable in the language $\mathcal{L}(\mathbb{R}_{an})$. So your example here must necessarily have $n \geq 3$. But in any case, this argument that the two languages have different quantifier-free expressive power is not particularly interesting as it relies on the slightly arbitrary decision not to restrict addition and multiplication. One should really find a truly local example:

Exercise* 6 (Osgood) Find a set $A \subseteq \mathbb{R}^3$ which is quantifier-free definable in the language $\mathcal{L}(\mathbb{R}_{an}^D)$ and has the property that for no $\epsilon > 0$ is $A \cap \Delta^{(3)}(0;\epsilon)$ quantifier-free definable in the language $\mathcal{L}(\mathbb{R}_{an})$.

The following exercises are set with a view towards the proof of Theorem 6.

Exercise 7 Let $A \subseteq \mathbb{R}^n$ be a quantifier-free definable set in the language $\mathcal{L}(\mathbb{R}_{an})$. Prove that for each $\omega \in \mathbb{R}^n$ there exists $s_\omega \in \mathbb{R}_{>0}^n$ such that the set $A \cap \Delta^{(n)}(\omega; s_\omega)$ is a finite union of sets of the form

$$\{x \in \Delta^{(n)}(\omega; s_\omega) : f(x) = 0, g_1(x) > 0, \ldots, g_{N_2}(x) > 0\} \qquad (*)$$

for some analytic functions $f, g_1, \ldots, g_{N_2} : \Delta^{(n)}(\omega; s_\omega) \to \mathbb{R}$.

Exercise 8 Let A be as in Exercise 6 and assume that $n \geq 2$. Suppose that for all $\omega \in \mathbb{R}^n$, each f and g_j arising in the conclusion of Exercise 7 has the (pleasant, but not usually realised) property that either it does not vanish at ω, or else is *regular in* x_n at ω. I.e. there is some $p \geq 1$ (depending on the function) such that it satisfies the hypothesis of Theorem 2. Prove that the projection, $\{x' \in \mathbb{R}^{n-1} : \exists x_n \in \mathbb{R}, \langle x', x_n \rangle \in A\}$, of A onto the first $n-1$ coordinates, is quantifier-free definable in the language $\mathcal{L}(\mathbb{R}_{an})$. [Hint: First consider the case when A has the form (*). As well as Exercise 7 and Theorem 2 (translated to arbitrary points in $\mathbb{R}^n$), you will need Tarski's Theorem on the quantifier elimination for $\bar{\mathbb{R}}$. Then use Exercise 5 and the compactness of bounded closed subsets of $\mathbb{R}^n$.]

If you managed to do these, then you should not have too much trouble with the Exercise 9. This time you will need Theorem 3 as well as the following fact: if $f : \Delta(0; r) \to \mathbb{R}$ is an analytic function and $f(0) = 0$ then there is an analytic function $g : \Delta(0; r) \to \mathbb{R}$ such that for all $x \in \mathbb{R}$, $f(x) = x \cdot g(x)$.

Exercise 9 Let $A \subseteq \mathbb{R}^2$ be a quantifier-free definable set in the language $\mathcal{L}(\mathbb{R}_{an})$. Prove that the projection of A onto the first coordinate is also quantifier-free definable in the language $\mathcal{L}(\mathbb{R}_{an})$.

Having studied these special cases let us now consider the general situation, where Theorem 3 is needed and where there is no possibility of dividing the coefficients ϕ_j by a common factor and thereby reducing to the situation of Exercise 8 (as you should have done in solving Exercise 9).

As in Exercise 8, it is sufficient to consider sets of the form $(*)$, and for simplicity I consider the projection of the set

$$A := \{x \in \Delta^{(n)}(0; r) : F(x) = 0\}$$

onto the first $n-1$ coordinates. We may obviously assume that the analytic function $F : \Delta^{(n)}(0; r) \to \mathbb{R}$ is not identically zero and the difficult case is when the function $F(0', \cdot)$ is identically zero.

We choose a representation of F as displayed in Theorem 3. The idea is to divide by the largest (in modulus) of the ϕ_j's and then appeal to Weirstrass. To this end, let

$$S := \{c \in [-1, 1]^{d+1} : \text{at least one coordinate of } c \text{ is } 1\}.$$

Let $v = \langle v_0, \ldots, v_d \rangle$ be new variables, let $c \in S$ and consider the analytic function $F_c : (-1,1)^{d+1} \times \Delta^{(n)}(0;s) \to \mathbb{R}$ given by

$$F_c(v,x) := \sum_{j=0}^{d} (c_j + v_j) \cdot x_n^j \cdot u_j(x).$$

Now if p is minimal such that $c_p \neq 0$, then, by direct calculation, $(\frac{\partial}{\partial x_n})^p(0,0) = p! \cdot c_p \cdot u_p(0) \neq 0$. So we are in the Weierstrass situation (of Exercise 8) with respect to the variables v, x and hence we obtain a quantifier-free formula of $\mathcal{L}(\mathbb{R}_{an})$, $\Theta_c(v,x')$ say, which defines the projection of the zero set of F_c onto the $\langle v, x' \rangle$- coordinates, at least in some sufficiently small box neighbourhood $\Delta^{((d+1)+(n-1))}(\langle 0, 0' \rangle; \langle t_c, s_c' \rangle)$ of $\langle 0, 0' \rangle$.

Now S is compact. So there is a finite set $\sigma \subseteq S$ such that S is covered by the collection $\{\Delta^{(d+1)}(c; t_c) : c \in \sigma\}$. Choose $\tau' \in \mathbb{R}^{n-1}_{>0}$ such that $\tau' < s_c'$ for all $c \in \sigma$.

It follows that a point $x' \in \Delta^{(n-1)}(0; \tau')$ lies in our projection of A (at least, for sufficiently small x_n-one has to do another compactness argument to cover all $x_n \in [-r_n, r_n]$) if and only if:

either $\phi_j(x') = 0$ for $j = 0, \ldots, d$,

or else for some j_0 with $0 \leq j_0 \leq d$, $\phi_{j_0}(x') \neq 0$,

and $|\phi_{j_0}(x')| \geq |\phi_j(x')|$ for $j = 0, \ldots .d$,

and for some $c \in \sigma$ we have $|\phi_j(x') - c_j \cdot \phi_{j_0}(x')| < (t_c)_j \cdot |\phi_{j_0}(x')|$ for $j = 0, \ldots, d$,

and $\Theta_c(h_0(x'), \ldots, h_d(x'), x')$ holds, where $h_j(x')$ denotes the term $D(\phi_j(x'), \phi_{j_0}(x')) - c_j$ of $\mathcal{L}(\mathbb{R}^D_{an})$ for each $j = 0, \ldots, d$.

Now, you will have noticed that we started with a quantifier-free formula of $\mathcal{L}(\mathbb{R}_{an})$ and we ended up with a quantifier-free formula of $\mathcal{L}(\mathbb{R}^D_{an})$, so it would appear that we cannot proceed further to eliminate more existentially quantified variables. However, this is easily dealt with. One first proves a generalization of Theorem 3 in which x_n is replaced by a tuple $x_n, \ldots, x_{n+m-1}$ of variables. This is done by induction on m, the representation of $F(x', x_n, \ldots, x_{n+m-1})$ looking the same as in Theorem 3 except that the subscript j becomes a multi-index $\alpha \in \mathbb{N}^m$ with $|\alpha| \leq d$. And now the point is that in successively eliminating the quantifiers $\exists x_{n+m-1}, \exists x_{n+m-2}, \ldots, \exists x_n$ (as above) we apply the D function to terms involving the variables x' only. There is one extra complication which I should point out, but about which I will not go into detail, namely that, even after dividing by the largest $\phi_\alpha(x')$, we might not have guaranteed the regularity of F with respect to any of the variables $x_n, \ldots, x_{n+m-1}$. This can be achieved, however, by a suitable (polynomial) invertible transformation of these variables (leaving x' fixed).

For our purposes in these notes, one of the most important consequences of Theorem 6 is contained in the following

Exercise 10 Prove that every subset of $\mathbb{R}$ that is definable in the language $\mathcal{L}(\mathbb{R}_{an})$ is a finite union of intervals and points.

It might now hardly seem worth commenting on the (very difficult) proof of Theorem 5. However, apart from pointing out the fact that [G] appeared twenty years before [DvdD], it turned out that Gabrielov's arguments were much more suited to dealing with reducts of $\mathbb{R}_{an}$ than those of Denef-van den Dries and in [G2] he proves the following result.

Theorem 7 (Gabrielov)**.** *Let $\widetilde{\mathbb{R}}$ be a reduct of $\mathbb{R}_{an}$ having the property that for each restricted analytic function $\widetilde{f}$ appearing in its signature, and for each j, $\widetilde{\frac{\partial f}{\partial x_j}}$ also appears. (Or, more generally, that $\widetilde{\frac{\partial f}{\partial x_j}}$ is the interpretation of some term of the language $\mathcal{L}(\widetilde{\mathbb{R}})$.) Then $Th(\widetilde{\mathbb{R}})$ is model complete in its language $\mathcal{L}(\widetilde{\mathbb{R}})$.*

The reason that the Denef-van den Dries method fails here is essentially due to the remark I made just before the statement of the Weierstrass Preparation Theorem. To see the extent to which their method can be applied to reducts consult [vdD2].

The importance of Theorem 7 for us is the

Corollary 1. *Let $\mathbb{R}_{rexp} := \langle \bar{\mathbb{R}}; \exp \restriction (0,1) \rangle$ where $\exp : \mathbb{R} \to \mathbb{R}, x \mapsto e^x$ is the usual exponential function. Then $Th(\mathbb{R}_{rexp})$ is model complete. So is the theory of the structure $\mathbb{R}_{rexp,rtrig} := \langle \bar{\mathbb{R}}; \exp \restriction (0,1), \sin \restriction (0,2\pi), \cos \restriction (0,2\pi) \rangle$.*

4 Some topics in o-minimality

Let $\mathbb{M} = \langle M, <, \ldots \rangle$ be a structure, where $<$ is a dense, total ordering without endpoints of its domain M. Then $\mathbb{M}$ is called *o-minimal* if every parametrically defianable subset of M is a finite union of open intervals and points.

Clearly any reduct of an o-minimal structure is o-minimal, so we immediately deduce the following fact from Exercise 10.

Corollary 2. *Any reduct of* $\mathbb{R}_{an}$*, in particular* $\mathbb{R}_{rexp}$ *and* $\mathbb{R}_{rexp,rtrig}$*, is o-minimal.*

In this section I shall be listing those properties of o-minimal structures that I shall be using in the sequel. Unless otherwise stated, all proofs may be found in the excellent book [vdDb]. From now on, notions of definability are always with reference to the structure under consideration and are *without* parameters. We thereby achieve *uniformity in parameters* in our results.

Theorem 8 (The Monotonicity Theorem)**.** *Let* $\mathbb{M} = \langle M, <, \ldots \rangle$ *be an o-minimal structure and suppose that* $f : M \to M$ *is a definable function. Then there are points* $a_1 < a_2 < \ldots < a_p$ *in* M *such that (setting* $a_0 = -\infty$ *and* $a_{p+1} = +\infty$*) for each* $j = 0, \ldots, p$*,* f *is either constant or strictly monotone and continuous (for the order topology) on the interval* (a_j, a_{j+1})*.*

Exercise 11 Prove that the a_j's in Theorem 8 may be taken to be definable (even without knowing the proof of Theorem 8).

Theorem 9 (Existence of definable Skolem functions)**.** *Let* $\mathbb{M} = \langle M, <, +, 0, \ldots \rangle$ *be an o-minimal structure where* $\langle M, <, +, 0 \rangle$ *is an ordered abelian group. Then* $Th(\mathbb{M})$ *admits definable Skolem functions. This means that for any definable set* $A \subseteq M^{n+1}$*, there exists a definable function* $f : M^n \to M$ *such that for all* $a \in M^n$*, if there is some* $b \in M$ *such that* $\langle a, b \rangle \in A$*, then* $\langle a, f(a) \rangle \in A$*.*

Henceforth, all o-minimal structures will be assumed to be equipped with an ordered abelian group structure as in Theorem 9.

Let us fix one such, $\mathbb{M} = \langle M, <, +, 0, \ldots \rangle$ say.

It follows from Theorem 9 that for any subset S of M, $Dcl(S)$ is the domain of an elementary substructure of $\mathbb{M}$ which we denote by $\widetilde{Dcl(S)}$. Here $Dcl(S)$ denotes the *definable closure* of S:

$$Dcl(S) := \{f(s_1, \ldots, s_n) : n \geq 0, s_1, \ldots, s_n \in S, f \text{ a definable function}\}.$$

By convention, a 0-place definable function is a definable element of M, so $\widetilde{Dcl(\emptyset)}$ is an isomorphic copy of the unique prime model of $Th(\mathbb{M})$.

I should mention at this point the foundational theory of o-minimality developed by Pillay and Steinhorn in [PS] and [KPS]. They prove the Cell Decomposition Theorem which has the consequence that any structure elementarily equivalent to an o-minimal structure is itself o-minimal. (This certainly does not follow from the definition which involves sets defined *with* parameters.) I do not need these results explicitly in these lectures (though a slightly more detailed account would certainly need to discuss them) so I return to properties of the $Dcl(\cdot)$ operator.

Exercise 12 Prove that $Dcl(\cdot)$ is a pre-geometry on M. [Hint: all is clear apart from Exchange. For this use Theorem 8 and Exercise 11.]

So by Exercise 12 we may assign a cardinal number (usually finite in our applications) to any subset $S \subseteq M$:

$$rank(S) := \max\{|I| : I \subseteq S, I \text{ is } Dcl(\cdot)\text{-independent}\}.$$

Thus, for example, $rank(M)$, also written $rank(\mathbb{M})$, is the smallest number of elements of M required to generate M under the definable functions. Exercise 12 guarantees that this is well-defined in the sense that any two minimal sets of generators have the same cardinality.

Exercise 13 Let $\mathbb{M} = \langle M, <, +, \cdot, 0, 1\rangle$ be a real closed, ordered field (i.e. $\mathbb{M} \equiv \bar{\mathbb{R}}$). Prove that for any $S \subseteq M$, $rank(S)$ is the transcendence degree (over $\mathbb{Q}$) of the subfield of $\mathbb{M}$ generated by S.

5 Some valuation theory for o-minimal structures

Let $\widetilde{\mathbb{R}}$ be an o-minimal expansion of $\bar{\mathbb{R}}$ and let $\mathbb{M} \equiv \widetilde{\mathbb{R}}$. The set of *finite* and *infinitesimal* elements of M (= the domain of $\mathbb{M}$) are defined, respectively, by:

$$Fin(\mathbb{M}) := \{a \in M : |a| < N \text{ for some } N \in \mathbb{Q}\},$$

$$\mu(\mathbb{M}) := \{a \in M : |a| < q \text{ for all } q \in \mathbb{Q}_{>0}\},$$

where we have identified $\mathbb{Q}$ with the prime subfield of $\mathbb{M}$.

Exercise 14 Prove that $Fin(\mathbb{M})$ is (the domain of) a subring of $\mathbb{M}$ and that $\mu(\mathbb{M})$ is the unique maximal ideal of $Fin(\mathbb{M})$.

Thus $Fin(\mathbb{M})/\mu(\mathbb{M})$ is a field, called the *residue field* of $\mathbb{M}$, and is denoted $Res(\mathbb{M})$.

Exercise 15 Let $\mathcal{S}$ be the collection of all elementary substructures $\mathbb{A} = \langle A, \ldots \rangle$ of $\mathbb{M}$ such that $A \subseteq Fin(\mathbb{M})$. Prove that $\mathcal{S}$ satisfies the hypotheses of Zorn's Lemma (in particular, that $\mathcal{S} \neq \emptyset$). Deduce that there exists $\mathbb{A}_0 = \langle A_0, \ldots \rangle \in \mathcal{S}$ such that for each $a \in Fin(\mathbb{M})$, there exists a unique $b \in A_0$ (the "standard part" of a) such that $|a - b| \in \mu(\mathbb{M})$.

Thus, the field $Res(\mathbb{M})$ can be expanded to an $\mathcal{L}(\mathbb{M})$-structure in such a way that it has an isomorphic copy, $\mathcal{R}(\mathbb{M})$ say, in $\mathbb{M}$ (contained in $Fin(\mathbb{M})$) and, further, we have that $\mathcal{R}(\mathbb{M}) \preccurlyeq \mathbb{M}$. We call $rank(\mathcal{R}(\mathbb{M}))$ the *residue rank* of $\mathbb{M}$ and denote it by $resrank(\mathbb{M})$.

Obviously $resrank(\mathbb{M}) \leq rank(\mathbb{M})$ and we wish to investigate the deficiency $rank(\mathbb{M}) - resrank(\mathbb{M})$, which, in some sense, measures the number of "degrees of infinity" in $\mathbb{M} \setminus Fin(\mathbb{M})$. The following is completely standard algebra, and not particularly special to the situation here.

Exercise 16 Prove that there exists a unique (up to isomorphism) ordered $\mathbb{Q}$-vector space $\langle \Gamma, <, +, 0 \rangle$ (called the *value group* of $\mathbb{M}$) and a unique surjective function $\nu : M \setminus \{0\} \to \Gamma$ (called the *valuation map* of $\mathbb{M}$) having the following properties.

For all $a, b \in \mathbb{M} \setminus \{0\}$,

(i) $\nu(a \cdot b) = \nu(a) + \nu(b)$;

(ii) $\nu(a + b) \geq \min\{\nu(a), \nu(b)\}$;

(iii) $\nu(a) = 0$ if and only if $a \in Fin(\mathbb{M}) \setminus \mu(\mathbb{M})$.

[Hint: $Fin(\mathbb{M}) \setminus \mu(\mathbb{M})$ is a multiplicative subgroup of $\mathbb{M} \setminus \{0\}$. Then Γ is the quotient group written additively.]

We continue to use the notation of Exercise 16 throughout this section.

Exercise 17 Prove that if $a, b \in \mathbb{M} \setminus \{0\}$ and $\nu(a) > \nu(b)$, then $\nu(a+b) = \nu(b)$.

You should now be able to establish the following classical inequality.

Exercise 18 Let $a_1, \ldots, a_n \in \mathcal{R}(\mathbb{M})$ be algebraically independent (over $\mathbb{Q}$) and let $b_1, \ldots, b_m \in \mathbb{M} \setminus \{0\}$ be such that $\nu(b_1), \ldots, \nu(b_m)$ are linearly independent (over $\mathbb{Q}$) elements of Γ. Prove that $a_1, \ldots, a_n, b_1, \ldots, b_m$ are algebraically independent (over $\mathbb{Q}$). Deduce that in the case $\widetilde{\mathbb{R}} = \bar{\mathbb{R}}$ (see exercise 13), we have the inequality

$$rank(\mathbb{M}) \geq resrank(\mathbb{M}) + dim_{\mathbb{Q}}(\Gamma).$$

The main step in the proof of the model completeness of T_{exp} is to generalize Exercise 18 to the case that $\widetilde{\mathbb{R}}$ is $\mathbb{R}_{rexp}$. In fact we have the following

Theorem 10 (The valuation inequality)**.** *Let $\widetilde{\mathbb{R}}$ be any reduct of $\mathbb{R}_{an}$ that expands $\bar{\mathbb{R}}$. Then for any $\mathbb{M} \equiv \widetilde{\mathbb{R}}$ with $rank(\mathbb{M})$ finite, we have the inequality*

$$rank(\mathbb{M}) \geq resrank(\mathbb{M}) + dim_{\mathbb{Q}}(\Gamma).$$

I can say very little here about the proof of Theorem 10 except that one uses the method of Exercise 18 after approximating analytic functions by polynomials via their Taylor expansions. Various proofs of Theorem 10 now exist in the literature for much wider classes of o-minimal structures, and I refer you to [S] for a very readable account.

Actually, I will need a relativized version of Theorem 10 which follows fairly easily from it. I now assume that $Th(\mathbb{M})$ is model complete.

Firstly, and in general, for $\mathbb{A} = \langle A, <, \ldots \rangle$ and $\mathbb{B} = \langle B, <, \ldots \rangle$ o-minimal structures in the same language with $\mathbb{A} \preccurlyeq \mathbb{B}$, let us write $rank_{\mathbb{A}}(\mathbb{B})$, the *rank of* $\mathbb{B}$ *over* $\mathbb{A}$, for the rank (as defined above) of the expansion $\langle \mathbb{B}, a \rangle_{\{a \in A\}}$ of $\mathbb{B}$, where each element of A is distinguished.

To return to situation at hand, let $\mathbb{M}_0 = \langle M_0, <, \ldots \rangle$ be an elementary substructure of $\mathbb{M}$. Then one may easily modify the Zorn's Lemma argument of Exercise 15 to show that the copies of the residue fields may be chosen in such a way that $\mathcal{R}(\mathbb{M}_0) \preccurlyeq \mathcal{R}(\mathbb{M})$. (Recall that both are models of $Th(\mathbb{M})$, and it is at this point that one needs model completeness.) So we may define $resrank_{\mathbb{M}_0}(\mathbb{M}) := rank_{\mathcal{R}(\mathbb{M}_0)}(\mathcal{R}(\mathbb{M}))$.

One can also easily show that Γ_0 (the value group of $\mathbb{M}_0$) is a sub-$\mathbb{Q}$-vector space of Γ.

Then in this situation we have

Theorem 11 (The relativized valuation inequality)**.** *If* $rank_{\mathbb{M}_0}(\mathbb{M})$ *is finite then*

$$rank_{\mathbb{M}_0}(\mathbb{M}) \geq resrank_{\mathbb{M}_0}(\mathbb{M}) + dim_{\mathbb{Q}}(\Gamma/\Gamma_0).$$

6 The model completeness of T_{exp}

In this section I give a proof of the model completeness of T_{exp}, at least insofar as I indicated after the statement of Definition 2 in section 1. But let us proceed for the moment as if we are trying to establish the hypotheses of Exercise 1 as stated.

So consider models $\mathbb{M}_0$, $\mathbb{M}$ of T_{exp} with $\mathbb{M}_0 \subseteq \mathbb{M}$. Let their domains be M_0 and M respectively and suppose we are given a quasipolynomial $f : M^n \to M$ say, with coefficients in M_0 and having a zero, b say, in M^n. The first step in finding such a zero in M_0^n is a *reduction to the non-singular case*:

Lemma 1. *It is sufficient to show that whenever* $F : M^n \to M^n$ *is a quasipolynomial map with coefficients in* M_0*, and* $b \in M^n$ *is a* non-singular *solution to the equation* $F(x) = 0$*, i.e.*

$$F(b) = 0 \text{ and } J_F(b) \neq 0,$$

then $b \in M_0$*.*

(Here, J_F *is the Jacobian with respect to all the variables* $x = x_1, \ldots, x_n$*.)*

There is now a simpler argument than that given in [W] of a much more general result than this, and I refer you to [JW] for the proof of this lemma.

It is perhaps overdue for me also to mention Khovanski's paper [K]. There it is shown that quasipolynomial maps as in the lemma have only finitely many non-singular zeros. Khovanski only works over the reals, but his (completely effective) upper bound for the number of such zeros is independent of any parameters occuring as coefficients, and hence one obtains the finiteness in all models of T_{exp}.

Now let $n \geq 1$ be given and assume, for a contradiction, that we have a counterexample, i.e. for some quasipolynomial map $F : M^n \to M^n$ (with coefficients in M_0) and some $b \in M^n$, we have $F(b) = 0$ and $J_F(b) \neq 0$ but $b \notin M_0^n$. Suppose further that these have been chosen to maximise the number r of coordinates of b that lie in $Fin(\mathbb{M})$. We may assume that these are the first r coordinates, $b_1, \ldots, b_r$, of b. Choose $N \in \mathbb{N}$ such that $-N < b_j < N$ for $j = 1, \ldots, r$.

Case 1 $r = n$.

Now we may consider $\mathbb{M}_0$ and $\mathbb{M}$ as models of T_{rexp} ($:= Th(\mathbb{R}_{exp})$) by simply restricting their exponential functions to $(0, 1)$. Let us call these structures $\mathbb{M}_0^*$ and $\mathbb{M}^*$ respectively. Since the exponentiation of $\mathbb{M}$ restricted to the interval $(-N, N)$ is definable in $\mathbb{M}^*$, the map F restricted to $(-N, N)^n$ is definable in $\mathbb{M}^*$ using parameters from $\mathbb{M}_0^*$, and we have, by Khovanski's theorem, that

$$\mathbb{M}^* \models \exists^{=k} x \in (-N, N)^n \, (F(x) = 0 \wedge J_F(x) \neq 0)$$

for some $k \geq 1$.

But $\mathbb{M}_0^* \preccurlyeq \mathbb{M}^*$ (by Corollary 1) and hence

$$\mathbb{M}_0^* \models \exists^{=k} x \in (-N, N)^n \, (F(x) = 0 \wedge J_F(x) \neq 0),$$

whence $b \in M_0^n$, a contradiction.

Case 2 $r < n$.

Then $b_1, \ldots, b_r \in Fin(\mathbb{M})$ and $b_{r+1}, \ldots, b_n \in \mathbb{M} \setminus Fin(\mathbb{M})$.

In fact, it is easy to see that if C is an $(n-r) \times (n-r)$ invertible matrix with rational entries, and $\gamma \in M_0^{n-r}$, then the n-tuple $\langle b_1, \ldots, b_r, \gamma + \langle b_{r+1}, \ldots, b_n \rangle C \rangle$ also constitutes a counterexample (for a suitably transformed map F).

Hence, in particular, for all $\gamma \in M_0$ and $q_{r+1}, \ldots, q_n \in \mathbb{Q}$ (not all zero) we have

$$\gamma + q_{r+1} \cdot b_{r+1} + \cdots + q_n \cdot b_n \notin Fin(\mathbb{M}) \qquad (*)$$

(by the maximality of r).

We now work in the theory T_{rexp} and consider its models $\mathbb{M}_0^*$ and $\mathbb{M}^*$ as discussed in Case 1. Let M_1 be the definable closure in $\mathbb{M}^*$ of the set $M_0 \cup \{b_1, \ldots, b_n, \exp(b_{r+1}), \ldots, \exp(b_n)\}$ and let $\mathbb{M}_1^*$ be the elementary substructure of $\mathbb{M}$ with domain M_1.

Now each coordinate function F_j of the map F has the form

$$F_j : M^n \to M, \langle x_1, \ldots, x_n \rangle \mapsto P_j(x_1, \ldots, x_n, \exp(x_1), \ldots, \exp(x_n))$$

where P_j is a polynomial (in $2n$ variables) with coefficients in M_0. Let us consider the function G_j given by

$$G_j : M^{n+(n-r)} \to M, \langle x_1, \ldots, x_n, y \rangle \mapsto P_j(x_1, \ldots, x_n, \exp(x_1), \ldots, \exp(x_r), y),$$

where $y = y_{r+1}, \ldots, y_n$

Notice that these functions are definable in $\mathbb{M}^*$ (with parameters from M_0) if the variables $x_1, \ldots x_r$ are constrained to the interval $(-N, N)$.

It is immediate that $\langle b_1, \ldots, b_n, \exp(b_{r+1}), \ldots, \exp(b_n) \rangle$ is a zero of the map $G := \langle G_1, \ldots, G_n \rangle : M^{n+(n-r)} \to M^n$ and one may easily check, by direct calculation, that it is a non-singular zero with respect to some sub-n-tuple of the variables $x_1, \ldots, x_n, y_{r+1}, \ldots, y_n$. It now follows from Khovanski's theorem that the coordinates of the corresponding sub-n-tuple of $\langle b_1, \ldots, b_n, \exp(b_{r+1}), \ldots, \exp(b_n) \rangle$ are all contained in the definable closure (over the parameters M_0) of the remaining $n - r$ coordinates. (We are using here the fact that in totally ordered structures, "algebraic closure = definable closure".). We conclude that $rank_{\mathbb{M}_0^*}(\mathbb{M}_1^*) \leq n - r$.

However,

$$\nu(\exp(b_{r+1})), \ldots, \nu(\exp(b_n))$$

are $\mathbb{Q}$-linearly independent elements of the vector space Γ_1^*/Γ_0^* (where Γ_1^* and Γ_0^* denote the value groups of $\mathbb{M}_1^*$ and $\mathbb{M}_0^*$ respectively). Since if not, then

$$\beta \cdot \exp(q_{r+1}b_{r+1} + \ldots + q_n b_n) \in Fin(\mathbb{M}_1^*) \setminus \mu(\mathbb{M}_1^*)$$

for some $\beta \in M_0$, and some $q_{r+1}, \ldots, q_n \in \mathbb{Q}$ (not all zero).

We may assume that $\beta > 0$ and since $\mathbb{M}_0 \models T_{exp}$, there is some $\gamma \in M_0$ such that $\exp(\gamma) = \beta$. But then

$$\exp(\gamma + q_{r+1}b_{r+1} + \ldots + q_n b_n) \in Fin(\mathbb{M}_1^*) \setminus \mu(\mathbb{M}_1^*)$$

whence

$$\gamma + q_{r+1}b_{r+1} + \ldots + q_n b_n \in Fin(\mathbb{M})$$

which contradicts $(*)$.

So $dim_{\mathbb{Q}}(\Gamma_1^*/\Gamma_0^*) \geq n - r$.

Since we have already established that $rank_{\mathbb{M}_0^*}(\mathbb{M}_1^*) \leq n - r$, it follows from Theorem 11 that $Resrank_{\mathbb{M}_0^*}(\mathbb{M}_1^*) = 0$, so

$$\mathcal{R}(\mathbb{M}_0^*) = \mathcal{R}(\mathbb{M}_1^*),$$

and that

$$\nu(\exp(b_{r+1})), \ldots, \nu(\exp(b_n))$$

generate Γ_1^* (as a $\mathbb{Q}$-vector space) over Γ_0^*.

It now takes a very easy valuation theoretic argument (together with the fact that M_0 is closed under taking logarithms of positive elements) to establish the following claim, and I leave the details of the proof to you.

Claim For all nonzero $a \in \mathbb{M}_1^*$ there exists $\gamma \in M_0$ and a homogeneous linear form with rational coefficients $\lambda : M^{n-r} \to M$ and an infinitesimal $\epsilon \in \mu(\mathbb{M}_1^*)$ such that

$$a = (1 + \epsilon) \cdot \exp(\gamma + \lambda(b_{r+1}, \ldots, b_n)).$$

In order to obtain our contradiction we use the claim in the obvious way to define inductively a sequence $a_1, a_2, \ldots$ of elements of M_1, where a_1 is an arbitrary positive element of $M_1 \setminus Fin(\mathbb{M}_1^*)$, and, for each $j \geq 1$,
(i) $a_j = (1 + \epsilon_j) \cdot \exp(a_{j+1})$ for some $\epsilon_j \in \mu(\mathbb{M}_1)$, and
(ii) $a_{j+1} = \gamma_{j+1} + \lambda_{j+1}(b_{r+1}, \ldots, b_n)$ for some $\gamma_{j+1} \in M_0$ and some homogeneous linear form λ_{j+1} with rational coefficients.

It is clear that $\langle a_j : j \geq 1 \rangle$ is a decreasing sequence of positive elements of M_1 and for all $l, m, j \geq 1$,

$$l < a_{j+1}^m < a_j \quad (**).$$

Now since the $\mathbb{Q}$-vector space of $(n - r)$-variable rational linear forms is finite dimensional it follows from (ii) that for some $p \geq 1$ and rational numbers $q_1, \ldots, q_p$ (not all zero), we have

$$q_1 a_1 + \cdots + q_p a_p \in M_0.$$

Thus by $(**)$ there certainly exists some $\gamma \in M_0$ such that $0 < a_{j_0} < \gamma$, where j_0 is the least j such that $q_j \neq 0$. If $j_0 > 1$ we may set $j = j_0 - 1$ in (i) and (using the fact that M_0 is closed under exponentiation) deduce that there also exists some $\gamma' \in M_0$ such that $0 < a_{j_0 - 1} < \gamma'$. Continuing in this way we eventually arrive at some $\gamma'' \in M_0$ such that $0 < a_1 < \gamma''$. Since a_1 was an arbitrary positive infinite element of M_1 this shows that every element of M_1 is $\mathbb{M}_0$-bounded. This is true, in particular, for $b_1, \ldots, b_n$ and this is all that I claimed we would show here.

7 The complex exponential field and analytic continuation for definable functions

We now consider the case that $\mathbb{K} = \mathbb{R}$.

As mentioned in the introduction, this section is motivated by the conjecture of Zilber stating that every $\mathbb{C}_{exp}$-definable subset of $\mathbb{C}$ is either countable or co-countable. As far as I know, even sets of the form

$$\{z \in \mathbb{C} : \exists w \in \mathbb{C}\ F(z,w) = 0\} \qquad (*)$$

where $F(z,w)$ is a (two variable) term of the language $\mathcal{L}(\mathbb{C}_{exp})$ have not been shown to satisfy Zilber's conjecture.

Our approach to this particular case is as follows. Let us suppose that

$$F(0,0) = 0 \neq \frac{\partial F}{\partial w}(0,0).$$

Then by the Implicit Function Theorem there exists $\epsilon > 0$ and a complex analytic function $\phi : \Delta(0;\epsilon) \to \mathbb{C}$ such that for all $z \in \Delta(0;\epsilon)$, we have $F(z,\phi(z)) = 0$. We must show that the set $(*)$ is co-countable and it seems reasonable to conjecture that the function element ϕ has an analytic continuation (which necessarily preserves the equation $F(z,\phi(z)) = 0$) to all but countably many points in the complex plane. Indeed, one can fairly easily show that if one proves a suitably generalized version of this *analytic continuation conjecture* (in which w is allowed to be an n-tuple of variables and F an n-tuple of terms in the $1+n$ variables z, w, and where the countably many exceptional points have a certain specific form) then Zilber's conjecture (even for subsets of $\mathbb{C}$ defined by formulas of the language $\mathcal{L}_{\omega_1,\omega}(\mathbb{C}_{exp})$) would follow.

Let us now consider issues of definability. The approach to Zilber's conjecture suggested above transcends $\mathcal{L}(\mathbb{C}_{exp})$-definablility (at least, if Zilber's conjecture is true!): one cannot define *restricted* functions $\phi : \Delta(0;\epsilon) \to \mathbb{C}$ without the resource of the real line and the usual metric. So we follow the Peterzil-Starchenko idea of doing complex analysis definably in a suitable o-minimal structure via the usual identifications $\mathbb{C} \sim \mathbb{R} \oplus i\mathbb{R} \sim \mathbb{R} \times \mathbb{R}$. (Actually, we will only be considering a fixed o-minimal expansion $\widetilde{\mathbb{R}}$ of the ordered field of real numbers $\overline{\mathbb{R}}$, so many of the subtleties of [PeS] will not be required here.)

In order to study the complex exponential function in this way we require an o-minimal structure in which it is definable (considered as a function from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R} \times \mathbb{R}$).

Exercise 19 Prove that there is no such structure.

However, Miller and van den Dries generalized the proof of the model completeness of T_{exp} to show that the expansion $\mathbb{R}_{an,exp}$ of the structure $\mathbb{R}_{an}$ by the (unrestricted) real exponential function is model complete and o-minimal (see [vdDM]).

Exercise 20 Prove that for any $N \in \mathbb{N}$ the restriction of the complex exponential function to the strip $\{x + iy \in \mathbb{C} : -N < y < N\}$ is definable in the structure $\mathbb{R}_{an,exp}$.

Exercise 21 Prove that any (analytic) branch of the complex logarithm function restricted to the slit plane $\{x + iy \in \mathbb{C} : y \neq 0 \text{ or } x > 0\}$ is definable in the structure $\mathbb{R}_{an,exp}$.

So some useful complex analytic functions are definable in the structure $\mathbb{R}_{an,exp}$. For many others see [W4].

To return to the approach to Zilber's conjecture suggested above, I propose the program of investigating analytic continuation for functions definable in the structure $\mathbb{R}_{an,exp}$ and I conclude these notes with a first step in this direction.

We first require the analytic cell decomposition theorem for $\mathbb{R}_{an,exp}$ (see [vdDM]). Since we only need it for subsets of the complex plane I state it only in the two (real) dimensional case. Definability here, and for the remainder of these notes, is with reference to the structure $\mathbb{R}_{an,exp}$.

Theorem 12 (van den Dries-Miller)**.** *Let $\mathcal{A}$ be a finite collection of definable subsets of $\mathbb{R}^2$. Then there are points $a_1 < a_2 < \cdots < a_p$ in $\mathbb{R}$ and, for each $j = 0, \ldots, p$, a finite collection $\mathcal{F}_j$ of definable (real) analytic functions with domain (a_j, a_{j+1}) (where we have set $a_0 = -\infty$ and $a_{p+1} = \infty$ and we also include the two functions with constant value ∞, $-\infty$ in $\mathcal{F}_j$) such that*

(i) for f, g distinct functions in $\mathcal{F}_j$, either for all $x \in (a_j, a_{j+1})$, $f(x) < g(x)$ (written $f \prec g$) or else for all $x \in (a_j, a_{j+1})$, $f(x) > g(x)$;

(ii) for each $A \in \mathcal{A}$ and each $f \in \mathcal{F}_j$, either graph$(f) \subseteq A$ or else graph$(f) \cap A = \emptyset$;
(iii) for each $f, g \in \mathcal{F}_j$ with $f \prec g$ and for which there is no $h \in \mathcal{F}_j$ with $f \prec h \prec g$, we have that either $(f, g) \subseteq A$ or else $(f, g) \cap A = \emptyset$, where $(f, g) := \{\langle x, y \rangle \in \mathbb{R}^2 : a_j < x < a_{j+1}, f(x) < y < g(x)\}$.

The sets graph(f) and (f, g) mentioned above are called *1-cells* and *2-cells* respectively and the collection of all of them is called an *analytic cell decomposition compatible with* $\mathcal{A}$. (Strictly speaking we should also include, for each $j = 0, \ldots, p$, a compatible partition of the ordinates $\{a_j\} \times \mathbb{R}$ into open intervals and points.)

The results contained in the following two exercises will be required for the proof of our analytic continuation theorem. They both require a little o-minimal theory, and the second some elementary complex analysis (Cauchy's Theorem) as well.

We use the notation $\bar{X}$ to denote the closure of a set $X \subseteq \mathbb{C}$ in the usual topology on $\mathbb{C}$.

Exercise 22 In the notation of Theorem 12, suppose that $A \in \mathcal{A}$ and that A is a (nonempty) *regular open* set (i.e. A is the interior of its closure in $\mathbb{R}^2$). Prove that there exists a finite set $S \subseteq \bar{A} \setminus A$ with the property that for all $a \in \bar{A} \setminus (A \cup S)$, there is a *unique* 2-cell, V_a say, in the cell decomposition such that $V_a \subseteq A$ and $a \in \bar{V_a} \setminus V_a$.

Exercise 23 Let $F : \bar{C} \to \mathbb{C}$ be a definable continuous function, where $C \subseteq \mathbb{C}$ is an analytic 2-cell, and assume that F has infinitely many zeros. Suppose further that $F \restriction C$ is (complex) analytic. Prove that F is identically zero.

Theorem 13. *Let $U \subseteq \mathbb{C}$ be a definable, regular open set and let $\phi : U \to \mathbb{C}$ be a definable (complex) analytic function. Then there exists a finite set $T \subseteq \bar{U} \setminus U$ such that for all $w \in \bar{U} \setminus (T \cup U)$, ϕ has an analytic continuation to some open set containing w*

Proof. Consider an analytic cell decomposition compatible with the collection $\mathcal{A} = \{U, \{z \in U : |\phi(z)| \leq 1\}\}$. If C is a 2-cell of this decomposition and $C \subseteq U$ then either $|\phi(z)| \leq 1$ throughout C or else $|\phi(z)| > 1$ throughout C. Now to prove the theorem it follows from Exercise 22 that we may assume that U is in fact such a 2-cell (exercise). Say

$$C = \{x + iy : a < x < b, f(x) < y < g(x)\},$$

where $f, g : (a, b) \to \mathbb{R}$ are definable real analytic functions.

I show that ϕ has an analytic continuation across all but finitely many points of $graph(g)$ (assumed $\neq \infty$), other cases being dealt with similarly.

Consider first the case that ϕ is bounded on C. Then by o-minimality it follows that ϕ has a continuous extension, $\bar{\phi}$ say, to all but finitely many points of $graph(g)$, and by a further use of analytic cell decomposition we may assume that $\bar{\phi} \circ g$ is real analytic at all but finitely many points of (a, b). So, avoiding such points, let $x_0 \in (a, b)$ and choose $\epsilon > 0$ so that $a < x_0 - \epsilon < x_0 + \epsilon < b$ and, further, so that the (real) Taylor series of both g and $\bar{\phi} \circ g$ at x_0 define complex analytic functions $G : \Delta(x_0; \epsilon) \to \mathbb{C}$ and $\Phi : \Delta(x_0; \epsilon) \to \mathbb{C}$ respectively. We have to show that ϕ has an analytic continuation to some open set containing $g(x_0)$.

Define the complex analytic function $H : \Delta(x_0; \epsilon) \to \mathbb{C}$ by $H(z) := z + iG(z)$.

Since the Taylor coefficients of G are real it follows that $H'(x_0) \neq 0$ and hence (by reducing ϵ if necessary) that H is a holomorphic homeomorphism from $\Delta(x_0; \epsilon)$ onto an open set, V say. Further, H maps the interval $(x_0 - \epsilon, x_0 + \epsilon)$ onto $graph(g \restriction (x_0 - \epsilon, x_0 + \epsilon))$. We may also suppose, by reducing ϵ further, that H is definable (its real and imaginary parts being restricted analytic functions (when extended by 0) of two real variables).

Now consider the function

$$\Psi := \phi - \Phi \circ H^{-1} : (C \cup graph(g \restriction (x_0 - \epsilon, x_0 + \epsilon))) \cap V \to \mathbb{C}.$$

By our construction Ψ is definable, continuous, holomorphic on $C \cap V$, and identically zero on the curve $graph(g \restriction (x_0 - \epsilon, x_0 + \epsilon))$ which obviously contains infinitely many points of the boundary of $C \cap V$. It now easily follows from Exercise 23 that Ψ is identically zero throughout $(C \cup graph(g \restriction (x_0 - \epsilon, x_0 + \epsilon))) \cap V$ and hence that $\Phi \circ H^{-1}$ provides an analytic continuation of ϕ to V, as required.

The case that $|\phi(z)| > 1$ for all $z \in C$ is dealt with by applying the above argument to the function $1/\phi$ and then inverting the analytic continuation. (The proof actually shows that ϕ is necessarily locally bounded at all but finitely many points of $graph(g)$.)

This completes the proof of Theorem 13. □

REFERENCES

[DvdD] J. Denef and L. van den Dries. p-Adic and Real Subanalytic Sets, Annals of Mathematics 128 (1988), 79-138.

[vdD] L van den Dries, On the elementary theory of restricted elementary functions, J. Symbolic Logic 53 (1988), 796-808.

[vdDb] Lou van den Dries. Tame topology and o-minimal structures, volume 248 of London Mathematical Society Lecture Note Series, Cambridge University Press, Cambridge, 1998

[vdDM] Lou van den Dries and Chris Miller, On the real exponential field with restricted analytic functions, Israel J of Math 85 (1994), 19-56.

[G] A. Gabrielov, Projections of Semi-Analytic Sets, Functional Analysis and its Applications, 2 (1968), 282-291.

[G2] A Gabrielov, Complements of subanalytic sets and existential formulas for analytic functions, Inv. Math. 125 (1996), 1-12.

[JW] G O Jones and A J Wilkie, Locally polynomially bounded structures, Bulletin of the London Mathematical Society, 40 (2) (2008) pp. 239-248.

[K] A. Khovanski, On a Class of Systems of Transcendental Equations, Soviet Math. Doklady 22 (1980), 762-765.

[KPS] J F Knight, A Pillay, and C Steinhorn, Definable sets in ordered structures, II, Trans. Amer. Math. Soc. 295 (1986), 593-605.

[M] David Marker, A remark on Zilber's pseudoexponentiation, J Symb. Logic, 71(3) (2006), 791-798

[R] Jesus M. Ruiz, The basic theory of power series, Edizioni ETS (2009) Pisa.

[PS] A. Pillay and C. Steinhorn, Definable Sets in Ordered Structures I, Trans. Amer. Math. Soc. 295 (1986), 565-592

[PeS] Ya'acov Peterzil and Sergei Starchenko, Expansions of algebraically closed fields in o-minimal structures, Sel. math., New ser. 7 (2001) 409-445.

[S] P Speissegger, Lectures on o-minimality, in B Hart, M Valeriote (eds), Lectures on algebraic model theory, Fields Institute Monographs, AMS, 2000.

[W] A J Wilkie, Model completeness results for expansions of the ordered field of real numbers by restricted Pfaffian functions and the exponential function, J Amer. Math. Soc. 9, No. 4, (1996), 1051-1094.

[W2] A J Wilkie, Model theory of analytic and smooth functions, in 'Models and Computability', eds S B Cooper and J K Truss, LMS lecture note series 259, 407-419.

[W3] A J Wilkie, On the theory of the real exponential field, Illinois J of Math, Vol 33, No. 3, (1989), 384-408.

[W4] A J Wilkie, Some results and problems on complex germs with definable Mittag-Leffler stars, to appear in the Notre Dame J of Formal Logic.