Continuous One-to-One Functions

Fall 2004

Theorem 1 *If* $f : \mathbb{R} \to \mathbb{R}$ *is continuous and one-to-one, then* f *is either increasing or decreasing.*

Lemma 2 *If* $f : [a, b] \to \mathbb{R}$ *is continuous and one-to-one,* $f(a) < f(b)$ *and* $a < c < b$*, then* $f(a) < f(c) < f(b)$*.*

Proof Suppose not. Since f is one-to-one, we must have $f(c) < f(a)$ or $f(c) > f(b)$.

Suppose $f(c) < f(a)$. Choose $\alpha \in \mathbb{R}$ with $f(c) < \alpha < f(a) < f(b)$. By the Intermediate Value Theorem, there are x, y such that $c < x < a < y < b$ and $f(x) = f(y) = \alpha$, contradicting the fact that f is one-to-one.

Thus we must have $f(c) > f(b)$. But then we choose α such that $f(b) < \alpha < f(c)$. Again we find x, y such that $c < x < a < y < b$ and $f(x) = f(y) = \alpha$, contradicting the fact that f is one-to-one. Thus we must have $f(a) < f(c) < f(b)$.

Corollary 3 *If* $f : [a, b] \to \mathbb{R}$ *is continuous, one-to-one, and* $f(a) < f(b)$*, then* f *is increasing on* $[a, b]$*.*

Proof Otherwise, there are $a \leq c < d \leq b$ with $f(c) < f(d)$. By Lemma 2 $f(a) \leq f(c) < f(b)$. If $b = d$ we are done. If $c < d < b$, then we apply Lemma 2 to the restriction of f to $[c, b]$. Hence, $f(c) < f(d)$, as desired.

An analogous argumen shows that if $f : [a, b] \to \mathbb{R}$ is continous, one-to-one, and $f(a) > f(b)$, then f is decreasing on $[a, b]$.

Proof of Theorem 1 Suppose not. If f is not increasing there are $a < b$ with $f(a) > f(b)$. If f is not decreasing there are $c < d$ with $f(c) < f(d)$. Pick M such that $a, b, c, d \in [-M, M]$. Apply the corollary to the restriction of f to $[-M, M]$. This function is either increasing or decreasing, a contradiction.