MTHT 430 Analysis for Teachers
Midterm Exam
Wednesday October 13

1) Give complete, precise definitions of the following concepts:

a) $f : X \to Y$ is one-to-one
If $x, y \in X$ and $f(x) = f(y)$, then $x = y$.

b) $f : X \to Y$ is onto.
If $y \in Y$, then there is $x \in X$ such that $f(x) = y$.

c) $\alpha = \sup A$ where $A \subseteq \mathbb{R}$.
α is an upper bound for A and if b is an upper bound for A, then $\alpha \leq b$.

2) a) State the Completeness Axiom
If $A \subseteq \mathbb{R}$ is nonempty and bounded above, then there is $\alpha \in \mathbb{R}$ a least upper bound for A.

b) State the Triangle Inequality.
If $a, b \in \mathbb{R}$, then

$$|a + b| \leq |a| + |b|.$$

3) Give examples of the following phenomena.

a) $f : \mathbb{R} \to \mathbb{R}$ that is one-to-one but not onto.
$f(x) = e^x$ is one-to-one but not onto since we can not find x with $f(x) = 0$ or $f(x) = -1$.

b) $f : \mathbb{N} \to \mathbb{N}$ that is onto but not one-to-one.
Let

$$f(x) = \begin{cases} 1 & \text{if } x = 1 \\ x - 1 & \text{otherwise} \end{cases}.$$

Then f is onto, but $f(1) = f(2) = 1$.

c) $f : \mathbb{R} \to \mathbb{R}$ and $A, B \subseteq \mathbb{R}$ such that $f(A \cap B) \not\supseteq f(A) \cap f(B)$.
Let $f(x) = x^2$. Suppose $A = (0, +\infty)$ and $B = (0, +\infty)$. Then $A \cap B = \emptyset$. So $f(A \cap B) = \emptyset$. But $f(A) = f(B) = (0, +\infty)$. So

$$f(A) \cap f(B) = (0, +\infty) \neq f(A \cap B).$$

d) A sequence of intervals $J_n = (a_n, b_n)$ where $a_n < b_n$ for all n and $J_1 \supseteq J_2 \supseteq J_2 \supseteq \ldots$, but

$$\bigcap_{n=1}^{\infty} J_n = \emptyset.$$

Let $J_n = (0, 1/n)$. Then $\bigcap J_n = \emptyset$.

e) A sequence of intervals $J_n = (a_n, b_n)$ where

$$a_1 < a_2 < a_3 < \ldots < a_n < \ldots < b_n < \ldots < b_2 < b_1$$

but

$$\bigcap_{n=1}^{\infty} J_n \neq \emptyset.$$

Let $J_n = (-\frac{1}{n}, \frac{1}{n})$. Then $\bigcap J_n = \{0\}$.

4) Suppsose $a, b, x, y > 0$ and $\dfrac{a}{b} < \dfrac{x}{y}$. Prove that

$$\frac{a}{b} < \frac{a+x}{b+y}.$$

We want

$$\begin{aligned} \frac{a}{b} &< \frac{a+x}{b+y} \\ a(b+y) &< b(a+x) \\ ab + ay &< ba + bx \\ ay &< bx \end{aligned}$$

But since $a/b < x/y$, $ay < bx$. Thus the desired inequality is true.

5) Prove that

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$$

for all $n \geq 1$.

We prove this by induction on n. If $n = 1$, this is true since $1/(1(2)) = 1/2$.

Suppose the equality is true for $n = k$. Then

$$\begin{aligned}
\sum_{i=1}^{k+1} \frac{1}{i(i+1)} &= \sum_{i=1}^{k} \frac{1}{i(i+1)} + \frac{1}{(k+1)(k+2)} \\
&= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}, \text{ by our induction hypothesis} \\
&= \frac{k(k+2)+1}{(k+1)(k+2)} \\
&= \frac{k^2+2k+1}{(k+1)(k+2)} \\
&= \frac{(k+1)^2}{(k+1)(k+2)} \\
&= \frac{k+1}{k+2}
\end{aligned}$$

Thus, by induction, the claim is true for all n.

6) Prove that if $B \subseteq A \subseteq \mathbb{R}$ are nonempty and bounded above, then $\sup B \leq \sup A$.

If $b \in B$, then $b \in A$ so $b \leq \sup A$. Thus $\sup A$ is an upper bound for B. Since $\sup B$ is the least upper bound for B, $\sup B \leq \sup A$.