MTHT 430 Analysis for Teachers
Problem Set 11

Do problems 2 ii) iv), 3, 5 from Chapter 7 of Spivak's *Calculus*.

1) Suppose $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$, are continuous and $f(x) = g(x)$ for all $x \in \mathbb{Q}$. Prove that $f(x) = g(x)$ for all $x \in \mathbb{R}$.

2) Suppose $f : \mathbb{R} \to \mathbb{R}$ and $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.

a) Prove that $f(0) = 0$.

b) Prove by induction that $f(nx) = nf(x)$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$.

Let $\alpha = f(1)$.

c) Prove that $f(x) = \alpha x$ for all $x \in \mathbb{N}$.

d) Prove that $f(-x) = -f(x)$ for all $x \in \mathbb{R}$. Conclude that $f(x) = \alpha x$ for all $x \in \mathbb{Z}$.

e) Prove that $f(\frac{x}{n}) = \frac{f(x)}{n}$ for all $x \in \mathbb{R}$. Conclude that $f(x) = \alpha x$ for all $x \in \mathbb{Q}$.

f) Suppose in addition that f is continuous. Prove that $f(x) = \alpha x$ for all $x \in \mathbb{R}$. [Remark: You have proved that the only continuous homomorphisms of the additive group $(\mathbb{R}, +)$ are of the form $f(x) = \alpha x$.]