MTHT 430 Analysis for Teachers
Problem Set 7

1) Suppose $A, B \subseteq \mathbb{R}$ are nonempty and bounded above. Prove the following.
   a) If $\sup A > c$, then there is $a \in A$ with $a > c$.
   b) If $\sup A < \sup B$, there is $b \in B$ an upper bound for $A$.
   c) For all $\epsilon > 0$ there is $a \in A$ such that $\sup A - \epsilon < a \leq \sup A$

2) Suppose $A, B \subseteq \mathbb{R}$ are nonempty and bounded above. Let
   \[ A + B = \{a + b : a \in A, b \in B\}. \]
   a) Prove that $\sup A + \sup B$ is an upper bound for $A + B$.
   b) Prove that $\sup(A + B) \leq \sup A + \sup B$.
   c) Prove that $\sup(A + B) \geq \sup A + \sup B$ and conclude that
      \[ \sup(A + B) = \sup A + \sup B. \]
   [Hint: Suppose for contradiction that $\sup A + \sup B > \sup(A + B)$. As a first step use 1a) to show there is $a \in A$ with $a + \sup B > \sup(A + B)$.

3) Verify the following limits using the definition of convergence.
   a) $\lim_{n \to \infty} \frac{1}{6n^2 + 1} = 0$
   b) $\lim_{n \to \infty} \frac{3n + 1}{2n + 5} = \frac{3}{2}$
   c) $\lim_{n \to \infty} \frac{2}{\sqrt{n + 3}} = 0$.

4) Consider the sequence
   \[ 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, \ldots \]
   where each string of zeros has one more zero than the previous. Does this sequence converge or diverge? If it converges to a limit $L$, prove that it converges to $L$. If it diverges, prove that it diverges.