MTHT 430 Analysis for Teachers

Problem Set 8

1) Verify the following limits **using the definition** of convergence.

a) $\lim_{n\to\infty} \frac{n+3}{n^3+4} = 0$

[Hint: It might be useful to note that $n+3 \leq 2n$ for $n \geq 3$ and $n^3+4 > n^3$.]

b) $\lim_{n\to\infty} \left(\sqrt{n+1} - \sqrt{n}\right) = 0.$

[Hint: It might be useful to note that

$$\left(\sqrt{n+1} - \sqrt{n}\right)\left(\sqrt{n+1} + \sqrt{n}\right) = n+1-n = 1.]$$

2) a) Suppose $(a)_{n=1}^{\infty}$ converges to a and $a_n \leq b$ for all $n \in \mathbb{N}$. Prove that $a \leq b$. [Remark: A similar argument shows that if $a_n \geq c$ for all $n \in \mathbb{N}$, then $a \geq c$. In particular if $a_n \in [c, b]$ for all $n \in \mathbb{N}$, then $a \in [c, b]$.]

b) Suppose $(a)_{n=1}^{\infty}$ converges to a and $a_n < b$ for all $n \in \mathbb{N}$. Must we have $a < b$? Prove or give a counterexample.

3) Let $a_1 = 1$ and for each $n \in \mathbb{N}$ define

$$a_{n+1} = \frac{3a_n + 4}{4}.$$

a) Use induction to prove that the sequence satisfies $a_n < 4$ for all $n \in \mathbb{N}$.

b) Prove that the sequence $(a_1, a_2, \ldots)$ is increasing.[Hint: This can either be proved directly or by an inductive argument where we let $P(n)$ be the assertion $a_n < a_{n+1}$.]

c) Prove that the sequence $(a_n)_{n=1}^{\infty}$ converges.

4) Give an example of each of the following or prove that it is impossible.

a) A sequence where no a_n is 0 or 1 but there are subsequences converging to both values.

b) A monotone sequence that diverges but has a convergent subsequence.

c) An unbounded sequence with a convergent subsequence.

d) A sequence that has a subsequence that is bounded but containing no convergent subsequence.