# MTHT 430 Analysis for Teachers

Final Exam Study Guide

1) Know all important definitions and how to apply them:

absolute value, binomial coefficient $\binom{n}{i}$, functions, composition of functions, one-to-one and onto functions, inverse functions, increasing and decreasing functions, images and inverse images, upper(lower bounds), sup and inf, the Completeness Axiom, sequences, convergence of sequences, limits of functions, continuous functions,

2) Know the main results and techiques, how to prove them and how to apply them:

triangle inequality, $\begin{pmatrix} n+1 \\ i \end{pmatrix} = \begin{pmatrix} n \\ i-1 \end{pmatrix} + \begin{pmatrix} n \\ i \end{pmatrix}$, binomial theorem, one-to-one onto functions have inverses, Archimedian Property, Density of $\mathbb{Q}$ in $\mathbb{R}$, Existence of Square Roots, Nested Interval Property, Monotone Convergence Theorem, Bolzano–Weierstrass Theorem, properties of limits, Intermediate Value Theorem, Extreme Value Theorem.

3) Know how to do proofs by induction, proofs by by contradiction, proofs that two sets are equal, proofs of "if-then" and "if and only if" statements.

## Review Problems/Sample Exam Questions

1) Define the following concepts

a) $f : X \to Y$ is one-to-one

b) $(a_n)_{n=1}^{\infty}$ converges to $a$

c) $\lim\limits_{x \to a} f(x) = l$.

2) a) State the Completeness Axiom

b) State the Bolzano–Weierstrass Theorem

c) State the Intermediate Value Theorem

3) State and prove the Monotone Convergence Theorem.

4) Decide if the following statements are TRUE or FALSE. If FALSE, give an example showing the statement is FALSE.

a) If $A \subseteq \mathbb{R}$ is bounded and nonempty there is $\alpha \in A$, a least upper bound for $A$.

b) If $f : (a, b) \to \mathbb{R}$ is continuous, then it is bounded.

c) If $f : [a, b] \to \mathbb{R}$ is continuous and $f(x) \neq 0$ for all $x \in [a, b]$, then $g(x) = \frac{1}{f(x)}$ is bounded on $[a, b]$.

d) If $\lim_{x \to a} f(x) = b$ and $\lim_{x \to b} g(x) = c$, then $\lim_{x \to a} g(f(x)) = c$.

e) If $(a_n)_{n=1}^{\infty}$ converges, then it is bounded.

f) If $f : [0, 1] \to [0, 2]$ is continuous, there is $x \in [0, 1]$ with $f(x) = 2x$.

g) If $f : \mathbb{R} \to \mathbb{R}$ is one-to-one, then it is either increasing or decreasing.

5) Prove that

$$\prod_{i=2}^{n} \left(1 - \frac{1}{i^2}\right) = \frac{n+1}{2n}$$

for all $n \geq 2$.

6) Suppose $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are one-to-one and onto. Prove that $g \circ f$ is one-to-one and onto.

7) Suppose $\lim\limits_{x \to a} f(x) = l$ and $(a_n)_{n=1}^{\infty}$ converges to $a$ where no $a_n = a$. Prove that $(f(a_n))_{n=1}^{\infty}$ converges to $l$.

8) **Using the definition of limits** prove that $\lim\limits_{x \to 1} \dfrac{2x+1}{x} = 3$.

9) Prove that there is a real number $x$ such that $\cos x = x$.

10) Let

$$f(x) = \begin{cases} 8x & \text{if } x \in Q \\ 2x^2 + 8 & \text{if } x \notin \mathbb{Q} \end{cases}.$$

Prove that $f$ is continuous at 2, but not at 1.

11) Suppose $f : \mathbb{R} \to \mathbb{R}$ and $A, B \subseteq \mathbb{R}$.

a) Prove that $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

b) Give an example of such an $f$ where $f(A \cup B) \neq f(A) \cup (B)$.