Proof Because $\delta \in S$, $A_\delta = B_\delta$ and $a \notin B_\delta$. Let $a = s^{\mathcal{B}}(x_1, \dots, x_k, y_1, \dots, y_l)$ where s is a Skolem term, $\overline{x} \in J_{<\delta}$, and $\overline{y} \in J \setminus J_{<\delta}$. Note that $l > 0$ because $a \notin B_\delta$. Choose $x \in J_{<\delta}$ and $y \in J_\delta$ such that $x > \sup\{\overline{c}, x_1, \dots, x_k\}$ and $y < y_i$ for $i = 1, \dots, l$. By indiscernibility, if $i_1 < \dots < i_n$ and $j_1 < \dots < j_n$ are two sequences from J with $x < i_1, j_1$ and $i_n, j_n < y$, then $t^{\mathcal{B}}(\overline{c}, \overline{i}) < a$ if and only if $t^{\mathcal{B}}(\overline{c}, \overline{j}) < a$.

Because δ is a limit point of S, we can find $\alpha < \delta$ with $\alpha \in S$ such that $x < d_{\alpha,1}$ and $d_{\alpha,n} < y$. But then $t^{\mathcal{B}}(\overline{c}, \overline{d}_\alpha) = a_\alpha < a$ and hence $t^{\mathcal{B}}(\overline{c}, \overline{j}) < a$ for all $j_1, \dots, j_n \in J$ with $x < j_1 < \dots < j_n < y$.

Finally, we will exploit the fact that because $\delta \in S$, $I_\delta \cong \omega^*$ and $J_\delta \cong \omega_1^*$.

Lemma 5.3.18 *i) If $j_1, \dots, j_n \in J_\delta$ and $j_1 < \dots < j_n$, then $t^{\mathcal{B}}(\overline{c}, \overline{j}) > a_\alpha$ for $\alpha \in S$ with $\alpha < \delta$.*

ii) There are $j_1 < \dots < j_n$ in J_δ such that $t^{\mathcal{B}}(\overline{c}, \overline{j}) < a$ for all $a \in I_\delta$.

Proof

i) Because $\delta \in S$ and $\alpha < \delta$, $\overline{d}_\alpha \in B_\delta$. Because, by Lemma 5.3.11 i), $B_\delta \cap J = J_{<\delta}$, $d_{\delta,1} \notin J_{<\delta}$. Thus $d_{\alpha,n} < d_{\delta,1}$.

Because
$$a_\alpha = t^{\mathcal{B}}(\overline{c}, \overline{d}_\alpha) < t^{\mathcal{B}}(\overline{c}, \overline{d}_\delta),$$
by indiscernibility
$$a_\alpha = t^{\mathcal{B}}(\overline{c}, \overline{d}_\alpha) < t^{\mathcal{B}}(\overline{c}, \overline{j}).$$

ii) Let $z_0 > z_1 > \dots$ be a cofinal descending sequence in I_δ. For each i, we find $x_i \in J_{<\delta}$ and $y_i \in J_\delta$ such that $t^{\mathcal{B}}(\overline{c}, \overline{j}) < z_i$ for all $j_1, \dots, j_n \in J$ with $x_i < j_1 < \dots < j_n < y_n$. Because J_δ has order type ω_1^*, we can find $j_1, \dots, j_n \in J_\delta$ such that $x_i < j_1 < \dots < j_n < y_i$ for all $i < n$. Thus, $t^{\mathcal{B}}(\overline{c}, \overline{j}) < a_i$ for $i = 0, 1, 2, \dots$. Thus, $t^{\mathcal{B}}(\overline{c}, \overline{j}) < a$ for all $a \in I_\delta$.

Thus, there is an element of $\mathcal{M}$ that is above all of the elements of $I_{<\delta}$ but below all of the elements of I_δ. Because $\mathcal{A}$ is the Skolem hull of I, this violates Lemma 5.3.11 ii). Thus, $\mathcal{M}^A$ and $\mathcal{M}^B$ are not isomorphic as $\mathcal{L}$-structures.

In this proof, we needed $\kappa > \aleph_1$ so we could use the ordering ω_1^* and still have $|A_\alpha| < \kappa$. More care is needed to prove the theorem when $\kappa = \aleph_1$.

5.4 An Independence Result in Arithmetic

Gödel's famous Incompleteness Theorem asserts that there are sentences ϕ in the language of arithmetic such that ϕ is true in the natural numbers but unprovable from the Peano Axioms for arithmetic. Indeed, for any consistent recursive extension T of Peano arithmetic, we can find a sentence that is independent from T. The original independent sentences were self-referential sentences that asserted their own unprovability or metamathematical sentences asserting the consistency of the theory. People wondered whether the independent statements could be made more "mathematical." In the late 1970s, Paris and Harrington [72] showed that a slight variant of the finite version of Ramsey's Theorem is true but unprovable in Peano arithmetic. The proof is an interesting application of indiscernibles.

We begin with the combinatorial statement.

Theorem 5.4.1 (Paris–Harrington Principle) *For all natural numbers n, k, m, there is a number l such that if $f : [l]^n \to k$, then there is $Y \subseteq l$ such that Y is homogeneous for f, $|Y| \geq m$, and if y_0 is the least element of Y, then $|Y| \geq y_0$.*

Proof We argue as in the proof of the finite version of Ramsey's Theorem. Suppose that there is no such l. For $l < \omega$, let $T_l = \{f : [\{0, \ldots, l-1\}]^n \to k :$ there is no Y homogeneous for f with $|Y| \geq m, \min Y\}$. Clearly, each T_l is finite, and if $f \in T_{l+1}$ there is a unique $g \in T_l$ such that $g \subset f$. Thus, if we order $T = \bigcup T_l$ by inclusion, we get a finite branching tree. Because each T_l is nonempty, T is an infinite finite branching tree and by König's Lemma there is $f_0 \subset f_1 \subset f_2 \subset \ldots$ with $f_i \in T_i$.

Let $f = \bigcup f_i$. Then $f : [\mathbb{N}]^n \to k$. By Ramsey's Theorem, there is an infinite $X \subseteq \mathbb{N}$ homogeneous for f. Let x_1 be the least element of X, and choose $s \geq x_1, m$. Let $x_1, \ldots, x_l$ be the first l-elements of X and let $l > x_l$. Then, $Y = \{x_1, \ldots, x_l\}$ is homogeneous for f_s and $|Y| \geq m, \min Y$, a contradiction.

Although the proof above is only a minor variant of the proof of the finite version of Ramsey's Theorem, the use of the infinite version of Ramsey's Theorem is in this case unavoidable. We will show that the Paris–Harrington Principle cannot be proved in Peano arithmetic. The approach we give here is due to Kanamori and McAloon [48].

Definition 5.4.2 Let $X \subseteq \omega$. We say that $f : [X]^n \to \omega$ is *regressive* if $f(A) < \min A$ for all $A \in [X]^n$. We say that $Y \subseteq X$ is *min-homogeneous* for f, if whenever $A, B \in [Y]^n$ and $\min A = \min B$, then $f(A) = f(B)$.

If $a < b$, we let (a, b) and $[a, b]$ denote $\{x : a < x < b\}$ and $\{x : a \leq x \leq b\}$, respectively.

We will consider the combinatorial principle.

$(*)$ For all c, m, n, k, there is d such that if $f_1, \ldots, f_k : [d]^n \to d$ are regressive, then there is $Y \subseteq [c, d]$ such that $|Y| \geq m$ and Y is min-homogeneous for each f_i.

We will show that $(*)$ is true but not provable in Peano arithmetic. We begin by giving a finite combinatorial proof that $(*)$ follows from the Paris–Harrington Principle. This proof can be formalized in Peano arithmetic. This tells us that not only is $(*)$ true but also if it is not provable in Peano arithmetic, then neither is the Paris-Harrington Principle.

Lemma 5.4.3 *For all $c, m, n, k < \omega$, there is $d < \omega$ such that if $g : [d]^n \to k$, then there is a homogeneous set $Y \subseteq [c, d)$ with $|Y| \geq m + 2n$,* $\min Y + n + 1$.

Proof By the Paris–Harrington Principle, there is a d such that for any partition $h : [d]^n \to k+1$ there is a homogeneous set Z with $|Z| \geq c + m + 2n + 1$, $\min Z$. Given $g : [d]^n \to k$, we define $h : [d]^n \to k+1$ by $h(\{a_1, \ldots, a_n\}) = k$ if some $a_i < c + n + 1$; otherwise,

$$h(\{a_1, \ldots, a_n\}) = g(\{a_1 - n - 1, \ldots, a_n - n - 1\}).$$

Let Z be a homogeneous set for h with $|Z| \geq c + m + 2n + 1, \min Z$. Because $|Z| \geq c + m + 2n + 1$, we can find $a_1, \ldots, a_n \in Z$ such that each $a_i \geq c + n + 1$. Then $h(\{a_1, \ldots, a_n\}) \neq k$ and we must have $h(A) \neq k$ for all $A \in [Z]^n$. Thus, every element of Z is greater than or equal to $c+n+1$. Let $Y = \{a - n - 1 : a \in Z\}$. Then $Y \subseteq [c, d)$ is homogeneous for g and $|Y| = |Z| \geq c + m + 2n + 1, \min Z$. But $\min Z = \min Y + n + 1$.

Lemma 5.4.4 *For all c, m, n, k, there is d such that, if $f_1, \ldots, f_k : [d]^n \to d$ are regressive, then there is $X \subseteq [c, d)$ such that $|Y| \geq m$ and X is min-homogeneous for each f_i.*

Proof By Lemma 5.4.3, there is a $d < \omega$ such that for all $g : [d]^{n+1} \to 3^k$, there is $Y \subseteq [c, d)$ homogeneous for g with $|Y| \geq m + n, \min Y + n + 1$.

Suppose that $f_1, \ldots, f_k : [d]^n \to d$ are regressive. For $i \leq l$, define $g_i : [d]^{n+1} \to 3$. Suppose that as follows: if $A = \{a_0, \ldots, a_n\}$ where $a_0 < a_1 < \ldots < a_n$, then

$$g_i(A) = \begin{cases} 0 & \text{if } f_i(a_0, a_1, \ldots, a_{n-1}) < f_i(a_0, a_2, \ldots, a_n) \\ 1 & \text{if } f_i(a_0, a_1, \ldots, a_{n-1}) = f_i(a_0, a_2, \ldots, a_n) \\ 2 & \text{if } f_i(a_0, a_1, \ldots, a_{n-1}) > f_i(a_0, a_2, \ldots, a_n). \end{cases}$$

Let $g : [d]^n \to 3^k$ by $g(A) = (g_1(A), \ldots, g_l(A))$. By Lemma 5.4.3, there is $Y \subseteq (c, d)$ homogeneous for g with $|Y| \geq \min Y + n + 1, m + n$. Clearly, Y is homogeneous for each g_i. Let $y_0 < y_1 < \ldots < y_s$ list Y. For $j = 1, ..., s - n + 1$, let $\overline{a}_j = (y_j, y_{j+1}, \ldots, y_{j+n-1})$. Because f_i is regressive $f_i(y_0, \overline{a}_j) < y_0$ for each $j \leq s - n + 1$. But $s + 1 = |Y| \geq y_0 + n + 1$. Thus $s - n + 1 \geq y_0 + 1$. Thus, we must have $f_i(y_0, \overline{a}_j) = f_i(y_0, \overline{a}_l)$ for some $j \neq l$. Because Y is homogeneous, the sequence $f_i(y_0, \overline{a}_1), f_i(y_0, \overline{a}_2), \ldots, f_i(y_0, \overline{a}_{s-n+1})$ is either increasing, decreasing, or constant. At least two values are equal. Thus, they must all be equal and g_i is constantly zero on $[Y]^{n+1}$.

Let $z_1 < \ldots < z_{n-1}$ be the largest $n-1$ elements of Y, and let $X = Y \setminus \{z_1, \ldots, z_n\}$. Because $|Y| \geq m + n$, $|X| \geq m$. We claim that X is min-homogeneous for each f_i. Suppose that $x_1 < x_2 < \ldots < x_n$. Then

$$\begin{aligned} f_i(x_1, x_2, \ldots, x_{n-1}, x_n) &= f_i(x_1, x_3, \ldots, x_{n-1}, z_1) \\ &= f_i(x_1, x_4, \ldots, z_1, z_2) \\ &\vdots \\ &= f_i(x_1, z_1, \ldots, z_{n-1}). \end{aligned}$$

But the same argument shows that if $y_2, \ldots, y_{n-1} \in X$ with $x_1 < y_2 < \ldots < y_{n-1}$, then

$$f_i(x_1, y_2, \ldots, y_{n-1}) = f_i(x_1, z_1, \ldots, z_{n-1}) = f_i(x_1, x_2, \ldots, x_{n-1}, x_n).$$

Thus, X is min-homogeneous for each f_i.

The independence proof will use a strong form of indiscernibles. Let Γ be a finite set of formulas in the language of arithmetic and $\mathcal{M}$ be a model of Peano arithmetic. We say that $I \subseteq M$ is a sequence of *diagonal indiscernibles* for Γ if whenever $\phi(u_1, \ldots, u_m, v_1, \ldots, v_n) \in \Gamma$ $x_0, \ldots, x_n, y_1, \ldots, y_n \in I$ with $x_0 < x_1 < \ldots < x_n$ and $x_0 < y_1 < \ldots < y_n$ and $a_1, \ldots, a_m < x_0$, then

$$\mathcal{M} \models \phi(\overline{a}, x_1, \ldots, x_n) \leftrightarrow \phi(\overline{a}, y_1, \ldots, y_n).$$

We first show how the combinatorial principle $(*)$ allows us to find sets of diagonal indiscernibles in the standard model $\mathbb{N}$.

Lemma 5.4.5 *For any c, l, m, n and formulas $\phi_1(u_1, \ldots, u_k, v_1, \ldots, v_n), \ldots, \phi_l(u_1, \ldots, u_k, v_1, \ldots, v_n)$ in the language of arithmetic, there is a set I of diagonal indiscernibles for $\phi_1, \ldots, \phi_l$ with $|I| \geq m$ and $\min I > c$.*

Proof We may assume that $m > 2n$. By the Finite Ramsey Theorem, we can find w such that $w \to (m+n)^{2n+1}_{l+1}$. By $(*)$, we can find s such that whenever $f_1, \ldots, f_k : [s]^{2n+1} \to s$ are regressive there is $Y \subseteq [c, s)$ with $|Y| \geq w$ and Y is min-homogeneous for each f_j. We define regressive functions $f_j : [s]^{2n+1} \to l$ for $j = 1, \ldots, k$ and a partition $g : [s]^{2n+1} \to l+1$ as follows. Let $X = \{x_0, \ldots, x_{2n}\}$ where $x_0 < x_1 < \ldots < x_{2n} < l$. If

$$\phi_i(\overline{a}, x_1, \ldots, x_n) \leftrightarrow \phi_i(\overline{a}, x_{n+1}, \ldots, x_{2n})$$

for all $i \leq l$ and $a_1, \ldots, a_m < x_0$, then let $f_j(X) = 0$ for all j and let $g(X) = 0$. Otherwise, let $g(X) = i$ and $(f_1(X), \ldots, f_k(X)) = \overline{a}$ be such that

$$\phi_{g(X)}(\overline{a}, x_1, \ldots, x_n) \not\leftrightarrow \phi_{g(X)}(\overline{a}, x_{n+1}, \ldots, x_{2n}).$$

Because each function f_j is regressive, there is $Y \subseteq [c, s)$ min-homogeneous for each f_j with $|Y| \geq w$. By choice of w there is $X \subseteq Y$ and $i \leq k$ such that $|X| \geq m+n$ and $g(A) = i$ for $A \in [X]^{2n+1}$.

Suppose that $i > 0$. Because $m > 2n$, $|X| > 3n$. Thus, we can find $x_0 < x_1 < \ldots < x_{3n}$ in X. Because X is min-homogeneous for each f_j, we can find $a_j < x_0$ such that

$$\begin{aligned} a_j &= f_j(x_0, x_1, \ldots, x_{2n}) \\ &= f_j(x_0, x_1, \ldots, x_n, x_{2n+1}, \ldots, x_{3n}) \\ &= f_j(x_0, x_{n+1}, \ldots, x_{2n}). \end{aligned}$$

Let $\overline{a} = (a_1, \ldots, a_k)$. But then,

$$\phi_i(\overline{a}, x_1, \ldots, x_n) \not\leftrightarrow \phi_i(\overline{a}, x_{n+1}, \ldots, x_{2n}),$$

$$\phi_i(\overline{a}, x_1, \ldots, x_n) \not\leftrightarrow \phi_i(\overline{a}, x_{2n+1}, \ldots, x_{3n})$$

and

$$\phi_i(\overline{a}, x_{n+1}, \ldots, x_{2n}) \not\leftrightarrow \phi_i(\overline{a}, x_{2n+1}, \ldots, x_{3n}).$$

But this is impossible because at least two of the formulas must have the same truth value. Thus $i = 0$.

Let $z_1 < \ldots < z_n$ be the n-largest elements of X and let $I = X \setminus \{z_1, \ldots, z_n\}$. Then, $|I| \geq m$ and we claim that I is the desired sequence of diagonal indiscernibles. If $x_0 < x_1 < \ldots < x_n$ and $y_1 < \ldots < y_n$ are sequences from I with $x_0 < y_1$ and $a < x_0$, then for any $i \leq k$,

$$\phi_i(\overline{a}, x_1, \ldots, x_n) \leftrightarrow \phi_i(\overline{a}, z_1, \ldots, z_n)$$

and

$$\phi_i(\overline{a}, y_1, \ldots, y_n) \leftrightarrow \phi_i(\overline{a}, z_1, \ldots, z_n).$$

Thus

$$\phi_i(\overline{a}, x_1, \ldots, x_n) \leftrightarrow \phi_i(\overline{a}, y_1, \ldots, y_n)$$

and I is a set of diagonal indiscernibles.

Note that aside from appealing to the Paris–Harrington Principle in the proof of Lemma 5.4.3, the three proofs above are straightforward finite combinatorics that could easily be formalized in Peano arithmetic.

We will look for diagonal indiscernibles for a rather simple class of formulas.

Definition 5.4.6 The set of Δ_0-formulas is the smallest set D of formulas in the language of arithmetic such that:
i) every quantifier-free formula is in D;
ii) if $\phi, \psi \in D$, then $\phi \wedge \psi$, $\phi \vee \psi$, and $\neg\phi$ are in D;
iii) if $\phi \in D$ and t is any term, then $\exists v < t\ \phi$ and $\forall v < t\ \phi$ are in D.

For example, if $\phi(x)$ is $\forall v < x\ \forall w < x\ vw \neq x$ is a Δ_0-formula defining the set of prime numbers. The next lemma is an easy induction on formulas that we leave to exercise 5.5.12.

Lemma 5.4.7 *Suppose that $\mathcal{M}$ is a model of Peano arithmetic and $\mathcal{N} \subseteq \mathcal{M}$ is an initial segment of $\mathcal{N}$ (i.e., if $a \in M$, $b \in N$, and $a < b$, then $a \in N$). If $\phi(\overline{v})$ is a Δ_0-formula and $\overline{a} \in N$, then $\mathcal{M} \models \phi(\overline{a})$ if and only if $\mathcal{N} \models \phi(\overline{a})$.*

Diagonal indiscernibles can be used to find initial segments that are models of Peano arithmetic.

Lemma 5.4.8 *Suppose that $\mathcal{M}$ is a model of Peano arithmetic and $x_0 < x_1 < \ldots$ is a sequence of diagonal indiscernibles for all Δ_0-formulas. Let $N = \{y \in M : y < x_i \text{ for some } i < \omega\}$. Then, N is closed under addition and multiplication, and if $\mathcal{N}$ is the substructure of $\mathcal{M}$ with underlying set N, then $\mathcal{N}$ is a model of Peano arithmetic.*

Proof Suppose that $i < j < k < l$ and $a < x_i$. If $a + x_j \geq x_k$, then we can find $b \leq a$ such that $b + x_j = x_k$. By indiscernibility, $b + x_j = x_l$, so $x_k = x_l$, a contradiction. Thus $a + x_j < x_k$. It follows that N is closed under addition. Indeed $x_i + x_j \leq x_k$.

Suppose that $i < j < k < l$. We claim that $ax_j < x_k$ for all $a < x_i$. If not, then, by induction, we can find $a < x_i$ such that $ax_j < x_k \leq (a+1)x_j$. By indiscernibility, $x_l \leq (a+1)x_j$. But, adding x_j to the first two terms, we see that $(a+1)x_j < x_k + x_j$. By the remarks above, $x_k + x_j \leq x_l$. Thus, $x_l \leq (a+1)x_j < x_l$, a contradiction. Thus $ax_j < x_k$. It follows that $\mathcal{N}$ is closed under multiplication.

Next, we show that truth of arbitrary formulas in $\mathcal{N}$ can be reduced to the truth of Δ_0-formulas in $\mathcal{M}$.

Suppose that $\phi(\overline{w})$ is the formula $\exists v_1 \forall v_2 \exists v_3 \ldots \exists v_n \psi(\overline{w}, v_1, \ldots, v_n)$, where $\psi(\overline{w}, \overline{v})$ is quantifier-free. By adding dummy variables, every formula can be put in this form. Let $\overline{a} < x_i$.

Because the sequence $x_0 < x_1 < \ldots$ is unbounded in I, then $\mathcal{N} \models \phi(\overline{a})$ if and only if $\exists i_1 > i \forall i_2 > i_1 \ldots \exists i_n > i_{n-1}$:

$$\mathcal{N} \models \exists v_1 < x_{i_1} \forall v_2 < x_{i_2} \ldots \exists v_n < x_{i_n} \ \psi(\overline{a}, v_1, \ldots, v_n).$$

By Lemma 5.4.7, $\mathcal{N} \models \phi(\overline{a})$ if and only if $\exists i_1 > i \ \forall i_2 > i_1 \ldots \exists i_n > i_{n-1}$:

$$\mathcal{M} \models \exists v_1 < x_{i_1} \forall v_2 < x_{i_2} \ldots \exists v_n < x_{i_n} \ \psi(\overline{a}, v_1, \ldots, v_n).$$

By diagonal indiscernibility, $\mathcal{N} \models \phi(\overline{a})$ if and only if

$$\mathcal{M} \models \exists v_1 < x_{i+1} \forall v_2 < x_{i+2} \ldots \exists v_n < x_{i+n} \ \psi(\overline{a}, v_1, \ldots, v_n).$$

Next, we show that induction holds in $\mathcal{N}$. Let $\phi(u, \overline{w})$ be a formula in the language of arithmetic. Suppose that $\overline{a}, b \in N$ and $\mathcal{N} \models \phi(b, \overline{a})$. Choose i_0 such that $\overline{a}, b < x_{i_0}$. If ϕ is $\exists v_1 \forall v_2 \ldots \exists v_n \ \psi(u, \overline{w}, \overline{v})$ where ψ is Δ_0, then, by the analysis above, if $i < i_1 < \ldots < i_n$, then for $c < x_i$

$$\mathcal{N} \models \phi(c, \overline{a}) \Leftrightarrow \mathcal{M} \models \exists v_1 < x_{i_1} \forall v_2 < x_{i_2} \ldots \exists v_n < x_{i_n} \ \psi(c, \overline{a}, v_1, \ldots, v_n).$$

Because induction holds in $\mathcal{M}$, there is a least $c < x_{i_0}$ such that $\mathcal{N} \models \phi(c, \overline{a})$. Thus, $\mathcal{N}$ is a model of Peano Arithmetic.

To prove the independence of the Paris–Harrington Principle from Peano arithmetic, we will assume familiarity with formalizing finite combinatorics and syntactic manipulations in arithmetic via coding. We summarize what we will need and refer the reader to [50] §9 for more complete details.

There are formulas $S(u), l(u,v), e(u,x)$ in the language of arithmetic such that in the standard model $S(u)$ defines the set of codes for finite sequence, $l(u,v)$ if u codes a set of length v, and $e(v,u,i)$ if v is the ith element of the sequence coded by v. All basic properties of finite sets and sequences are provable in Peano arithmetic. Using these predicates, we can formalize the Paris–Harrington Principle and $(*)$ as sentences in the language of arithmetic. We can pick our coding of finite sets such that if $X \subseteq \{0, \ldots, a-1\}$, then the code for X is less than 2^{2^a}.

Next, we use some basic facts about coding syntax in the language of arithmetic. For each formula ϕ, we let $\lceil \phi \rceil$ be the Gödel code for ϕ. There is a formula $Form_0(v)$ that defines the set of Gödel codes for Δ_0-formulas, and there is a formula $Sat_0(u,v,w)$ such that $Sat_0(u,v,w)$ asserts that u is a code for a Δ_0-formula with free variables from $v_1, \ldots, v_w$, v codes a sequence $\overline{a}$ of length w, and the formula with code u holds of the sequence $\overline{a}$. We call Sat_0 a *truth-definition* for Δ_0-formulas. All basic metamathematical properties of formulas and satisfaction for Δ_0-formulas are provable in Peano arithmetic.

Theorem 5.4.9 *The combinatorial principle* $(*)$ *and the Paris–Harrington Principle are not provable in Peano arithmetic.*

Proof By the remarks after Lemma 5.4.4, it suffices to show that $(*)$ is unprovable. Suppose that $\mathcal{M}$ is a nonstandard model of Peano arithmetic and c is a nonstandard element of $\mathcal{M}$. Suppose that $\mathcal{M} \models (*)$. We will use Lemma 5.4.8 to construct an initial segment of $\mathcal{M}$ where $(*)$ fails.

Because the Finite Ramsey Theorem is provable in Peano arithmetic, there is a least $w \in M$ such that $\mathcal{M} \models w \to (3c+1)_c^{2c+1}$. Let $d \in M$ be least such that if $f_1, \ldots, f_c : [d]^{2c+1} \to d$ are regressive, then there is $Y \subseteq (c,d)$ with $|Y| \geq w$ and Y min-homogeneous for each f_i.

Using the truth predicate for Δ_0-sets, we can follow the proof of Lemma 5.4.5 inside $\mathcal{M}$ and obtain $I \subset (c,d)$ with $|I| \geq c$ such that $\mathcal{M}$ believes I is a set of diagonal indiscernibles for all Δ_0-formulas from $\mathcal{M}$ with Gödel code at most c, free variables from $v_1, \ldots, v_c$, and parameter variables from $w_1, \ldots, w_c$. In particular, I is a set of diagonal indiscernibles for all standard Δ_0-formulas.

Let $x_0 < x_1 < \ldots$ be an initial segment of I, and let $\mathcal{N}$ be the initial segment of $\mathcal{M}$ with universe $N = \{y \in M : y < x_i \text{ for some } i = 1, 2, \ldots\}$. By Lemma 5.4.8, $\mathcal{N}$ is a model of Peano arithmetic. Clearly, $c \in N$ and $d \notin N$. We claim that $w \in N$. Because the finite version of Ramsey's

Theorem is provable in Peano arithmetic, there is $w' \in N$ such that $\mathcal{N} \models w' \to (3c+1)^{2c+1}_c$. Because all functions from $[w']^{2c+1} \to c$ and all subsets of w' that are coded in $\mathcal{M}$ are coded in $\mathcal{N}$, $\mathcal{M} \models w' \to (3c+1)^{2c+1}_c$. Because w was minimal, $w \leq w'$ and $w \in N$. By a similar argument, if $d' \in N$ and $\mathcal{N} \models \forall f_1, \ldots, f_c : [d']^{2c+1} \to d'$ is regressive, there is $Y \subseteq (c, d')$ min-homogeneous for each f_i with $|Y| \geq w$. Then, this is also true in $\mathcal{M}$; thus, by choice of d, $d \leq d'$. Because $d \notin N$, this is a contradiction. Thus, $(*)$ fails in $\mathcal{N}$ and $(*)$ is not provable from Peano arithmetic.

5.5 Exercises and Remarks

Exercise 5.5.1 Show that $6 \to (3)^2_2$ (i.e., if there are six people at a party, you can either find three mutual acquaintances or three mutual non-acquaintances).

Exercise 5.5.2 Let $\mathcal{L} = \{E\}$, where E is a binary relation symbol, and let T be the theory of an equivalence relation with infinitely many classes each of which is infinite. Show that in any $\mathcal{M} \models T$ we can find infinite sets of indiscernibles I_0 and I_1 such that $\mathrm{tp}(I_0) \neq \mathrm{tp}(I_1)$, but if J is any other infinite set of indiscernibles, then $\mathrm{tp}(J) = \mathrm{tp}(I_i)$ for $i = 0$ or 1.

Exercise 5.5.3 Let G be the free group on generators X. Show that X is a set of indiscernibles in G.

Exercise 5.5.4 Show that if $\mathcal{M}$ is κ-saturated, then there is $I \subseteq M$, a sequence of order indiscernibles with $|I| = \kappa$.

Exercise 5.5.5 Show that, for any countably infinite $\mathcal{L}$-structure $\mathcal{M}$, we can find $(\mathcal{N}_n : n < \omega)$, a descending elementary chain (i.e., $\mathcal{N}_{n+1} \prec \mathcal{N}_n$ for each n) of elementary extensions of $\mathcal{M}$, such that $\mathcal{M} = \bigcap_{n<\omega} \mathcal{N}$. [Hint: Let $\mathcal{N}_0$ be the Skolem hull of M and an infinite set of indiscernibles.]

Exercise 5.5.6 We say that a theory T has the *order property* if and only if there is a formula $\phi(v_1, \ldots, v_n, w_1, \ldots, w_n)$ and $\mathcal{M} \models T$ with $\overline{x}_1, \overline{x}_2, \ldots$ in M^n such that $\mathcal{M} \models \phi(\overline{x}_i, \overline{x}_j)$ if and only if $i < j$.

a) Show that if ϕ has the order property in T, then T is not κ-stable for any infinite κ. [Hint: Let $(A, <)$ and B be as in Lemma 5.2.12. Find $\mathcal{N} \models T$ containing $(\overline{x}_a : a \in A)$ such that $\mathcal{N} \models \phi(\overline{x}_a, \overline{x}_b)$ if and only if $a < b$. Argue as in Theorem 5.2.13 that $|S_n(\{\overline{x}_b : b \in B\})| > |B|$.]

b) Show that T has the order property if and only if there is a formula $\psi(\overline{v}, \overline{w})$ and $\mathcal{M} \models T$ with $\overline{a}_1, \overline{b}_1 \overline{a}_2, \overline{b}_n \ldots$ such that $T \models \psi(\overline{a}_i, \overline{b}_j)$ if and only if $i < j$.[Hint: $(\Rightarrow)$ Let $\phi(\overline{v}_1, \overline{v}_2, \overline{w}_1, \overline{w}_2)$ be $\psi(\overline{v}_1, \overline{w}_2)$. Let $\overline{c}_i = (\overline{a}_i, \overline{b}_i)$. Show that $\phi(\overline{c}_i, \overline{c}_j)$ if and only if $i < j$. The other direction is even easier.]

Exercise 5.5.7 Let $\mathcal{L} = \{U_0, U_1, \ldots, U_n, E_1, \ldots, E_n\}$, where each U_i is unary and E_i is binary, and let T be the $\mathcal{L}$-theory: