Itereated Deletion and Nash Equilibria

We consider finite two player games–though all of these will generalize to any finite game. We let A denote the set of strategies for Player 1 and B denote the strategies for Player 2.

Recall IDSDS is Iterated Deletion of Strictly Dominated Strategies and IDWDS is Iterated Deletion of Weakly Dominated Strategies

Proposition 1 *Any game as at most one weakly dominant solution.*

Proof It is impossible for a to weakly dominate a_1 and a_1 to weakly dominate a. $\square$

Proposition 2 *If (a^*, b^*) is a weakly dominant solution, then (a^*, b^*) is a Nash equilibrium.*

Proof The strategy a^* weakly dominates every other strategy in A. Thus

$$v_1(a^*, b^*) \geq v(a, b^*)$$

for all $a \in A$ and a^* is the unique best response to b^*. Similarly, b^* is the unique best response to a^*. Thus (a^*, b^*) is a Nash equilibrium. $\square$

The game *Battle of the Sexes* shows that the coverse fails as there are games with Nash equilibria which are not weakly dominant.

Proposition 3 *If (a^*, b^*) is a Nash equilibrium, then (a^*, b^*) is not eliminated by IDSDS.*

Proof We prove this by contradiction. Suppose (a^*, b^*) is eliminated during IDSDS. Then one of the strategies is removed at some stage of the construction. Let's suppose that a^* is removed before b^* (the other case is similar). Consider the stage when a^* is eliminated. At this stage of the construction, we have a game where a^* and b^* are possible strategies and, because it is about to be eliminated, there is a strategy $a' \in A_1$ such that a' strictly dominates a^*. But then

$$v_1(a', b^*) > v_1(a^*, b^*)$$

and a^* is not a best response for Player 1 to b^*. This contradicts our assumption that (a^*, b^*) is a Nash equilibrium. $\square$

The converse fails. For example, in Battle of the Sexes, (F, O) and (O, F) are not eliminated by IDSDS (or even IEDWDS) but are not Nash equilibria. Similarly, in *Mathching Coins* no strategies are elimated by IDSDS (or IDWDS) but no strategy profile is a Nash equilibrium.

The converse is true in the special case when there is IDWDS solution–i.e. some IDWDS procedure ends in a unique solution.

Proposition 4 *If (a^*, b^*) is an IDWDS solution, then (a^*, b^*) is a Nash equilibrium.*

Proof For purposes of contradiction suppose a^* is not a best response to b^*–the other case is similar. Let $X = \{a \in A : v_1(a, b^*) > v_1(a^*, b)\}$. By assumption $A \neq \emptyset$. All of the strategies in X must be eliminated in the process. Look at the last stage where a strategy $a \in X$ is elimiated. For it to be eliminated, there must be a strategy $a' \in A$ such that a' weakly dominates a. But then

$$v_1(a', b^*) \geq v_1(a, b^*) > v_1(a^*, b^*)$$

and $a' \in X$. But we must eventually eliminate a' and this contradicts the fact that a was the last element of X eliminated. $\square$

Corollary 5 *If there is an IEDSDS solution, it is the unique pure strategy Nash equilibrium.*

Proof By Proposition **??** the unique IDSDS equilibrium is a Nash equilibrium. By Proposition **??**, if there was a second Nash equilibrium it would also be an IDSDS equilibrium. $\square$

Corollary 6 *If there is a striclty dominant strategy equilibrium, it is the unique Nash equilibrium.*

Proof If (a^*, b^*) is a strictly dominant strategy equilibrium, then in the IESDS process at stage 1 would eliminate all strategies except a^* and b^*, so (a^*, b^*) is the unique IESDS-equilibrium and hence the unique Nash-equilibrium. $\square$

Example 2 below shows that a game may have a weakly dominant solution and several Nash equilibria.

Corollary 7 *There can only be at most one IDSDS solution. In particular, in IDSDS the order that we eliminate strategies does not matter.*

Proof If there were two IDSDS solutions, then would both be Nash equilibria contradicting Corollary 4. $\square$

More generally, the set of strategies that survive IEDSDS elimination does not depend on the order of elimination.

Example 1 In IDWDS the order of elmination may matter. We also note that this is a game solvable by IDWDS with two Nash equilibria.

Consider the game:

	L	R
T	1,1	0,0
M	3,2	2,2
B	0,0	1,1

Process 1: Since M dominates T, we can eliminate T to get

	L	R
M	3,2	2,2
B	0,0	1,1

Now R dominates L. Eliminating L we get

	R
M	2,2
B	1,1

and (M,R) is an IDWDS solution.

Process 2:

M dominate B

	L	R
T	1,1	0,0
M	3,2	2,2

Now L dominates R

	L
T	1,1
M	3,2

Thus (M,L) is an IDWDS solution.

Example 2 The coordination game below is a game with two Nash equilibria only one of which is an IDDS solution–and no IDSDS solution

	L	R
T	1,1	0,0
B	0,0	0,0

In this game (T,L) is the unique IDWDS solution, indeed it is a dominant solution, but (B,R) is also a Nash equilibrium.

IDSDS does not simplify this game at all.

Example 3 The following game has a unique pure strategy Nash equilibrium but can not be simplified by IDWDS

	L	C	R
T	2,2	-1,1	1,0
M	1,-1	0,0	1,-2
B	0,1	2,-1	-2,3