Auctions

We will look at auctions under the assumption of *Independent Private Values*. We assume there are N bidders. Bidder i has a value v_i and there is a probability distribution F_i such that $Pr(v_i < r) = F_i(r)$. We assume that $v_1, \ldots, v_n$ are independent random variables.

For simplicity we will consider only the case where each $v_i \in [0,1]$ with uniform distribution, i.e., $Pr(v_i < r) = r$ for $r \in [0,1]$

First Price Sealed Bid Auctions

We consider a first price sealed bid auction where there are N players with independent private values v_i uniformly distributed in $[0,1]$.

A strategy for Player i will be of the form $b_i : [0,1] \to [0,1]$ where Player i bids $b_i(v_i)$ with value v_i. We will look for a symmetric equilibrium where each player uses the same strategy $b(v)$. We make two additional reasonable assumptions

- $b(0) = 0$, if my value is 0, I should not bid more that 0.
- b is increasing, if $v < w$, then $b(v) < b(w)$.

We look at Player 1's strategy. We consider the auxiliary function $u(r,v)$ which the expected value for a Player with value v if they decided to bid as if they had value r. The Player will only win if all other players bid less than $b(r)$. Since b is increasing, this is only true if all of the other players value the object less that r. Thus

$$u(r,v) = \Pr(v_2 < r, v_3 < r \ldots, v_N < r)(v - b(r)) = r^{N-1}(v - b(r)).$$

For a fixed v we can maximize $u(r,v)$ by setting $\frac{\partial u}{\partial r} = 0$. Of course, $b(v)$ is the optimal bid if we have value v. Thus, we would maximize $u(r,v)$ by setting $r = v$.

$$\frac{\partial u}{\partial r} = (N-1)r^{N-2}(v - b(r)) - r^{N-1}b'(r) = 0$$

Thus

$$(N-1)r^{N-2}v = (N-1)r^{N-2}b(r) + r^{N-1}b'(r).$$

Substituting $v = r$,

$$\begin{aligned}
(N-1)v^{N-1}v &= (N-1)v^{N-2}b(v) + v^{N-1}b'(v) \\
(N-1)v^{N-1} &= \frac{d}{dv}\left[v^{N-1}b(v)\right]
\end{aligned}$$

then, by the Fundamental Theorem of Calculus

$$\begin{aligned}
v^{N-1}b(v) &= \int_0^v (N-1)x^{N-1}\,dx \\
v^{N-1}b(v) &= \frac{N-1}{N}v^N \\
b(v) &= \frac{N-1}{N}v
\end{aligned}$$

Thus there is a Bayesian-Nash equilibrium where Player i bids $\frac{N-1}{N} v_i$.

Revenue Equivallence

First let's consider the expected revenue for the seller in two types of auctions.

$N = 2$ We break this into two cases. $v_2 \leq v_1$ and $v_1 \leq v_2$ On the first region the expected maximal bid is $\frac{v_1}{2}$ and the expectation on this region is

$$\int_0^1 \int_0^{v_1} \frac{v_1}{2}\, dv_2 dv_1.$$

Since the other region is symmetric, the total expectation is

$$2\int_0^1 \int_0^{v_1} \frac{v_1 2}{2}\, dv_2 dv_1 = 2\int_0^1 \frac{v_1^2}{2}\, dv_1 = 2\left(\frac{1}{6}\right) = 1/3.$$

If $N = 3$ there are 6 cases depending on the 6 possible ordering of $v_1, \ldots, v_3$. If we assume $v_1 > v_2 > v_3$, we get

$$\begin{aligned} 6\int_0^1 \int_0^{v_1} \int_0^{v_2} \frac{2}{3} v_1\, dv_3 dv_2 dv_1 &= 6\int_0^1 \int_0^{v_1} \frac{2}{3} v_1 v_2\, dv_2 dv_1 \\ &= 6\int_0^1 \frac{v_1^3}{3}\, dv_1 \\ &= 6\left(\frac{1}{12}\right) = 1/2 \end{aligned}$$

For general N we need to consider all $N!$ orderings of $x_1, \ldots, x_N$. if $v_1 > \ldots > v_N$ we get the general expression

$$N!\int_0^1 \int_0^{v_1} \cdots \int_0^{v_{N-1}} \frac{N-1}{N} v_1\, dv_N dv_{N-1} \ldots dv_1 = \frac{N-1}{N+1}$$

Second Price Sealed Bid Auctions

We've argued before that bidding your value is a weakly dominant strategy in a second price auction. Thus the equilibrium price will be the second highest of the values. If $v_1 > v_2 > \ldots > v_N$, the equilibrium price is v_2. There are $N!$ possible orderings of the v_i, thus the expected equilibrium price is

$$N!\int_0^1 \int_0^{v_2} \cdots \int_0^{v_{N-1}} v_2\, dv_N dv_{N-1} \ldots dv_1 = \frac{N-1}{N+1}$$

We show this for $N = 2, 3$

Let $N = 2$

$$\begin{aligned} 2\int_0^1 \int_0^{v_1} v_2\, dv_2 dv_1 &= 2\int_0^1 \frac{v_1^2}{2}\, dv_1 \\ &= 2\left(\frac{1}{6}\right) = 1/3 \end{aligned}$$

For $N = 3$

$$
\begin{aligned}
6\int_0^1\int_0^{v_1}\int_0^{v_2} v_2\, dv_3 dv_2 dv_1 &= 6\int_0^1\int_0^{v_1} v_2^2\, dv_1 dv_2 \\
&= 6\int_0^1 \frac{v_1^3}{3}\, dv_1 \\
&= 6\left(\frac{1}{12}\right) = 1/2
\end{aligned}
$$

It may seem surprising that first price sealed bid auction and second price sealed bid auctions give rise to the same expected revenue, but in fact this is always the case! Vickery and Myerson received a Nobel prize for, among other things, the following theorem (which we state vaugely).

Theorem 1 (Revenue Equivalence Theorem) *Suppose we have N Players with independent private values where Player i has values in $[L_i, U_i]$ with probability distribution $Pr(x \leq r) = F_i(r)$. Then, under reasonable assumptions, any two auction procedures will lead to the same expected revenue.*