1 Two Player Zero Sum Games

We consider a two player finite game, i.e., the sets of possible actions A_1 and A_2 are finite. We say that the game is *zero sum* if

$$u_1(a_1, a_2) + u_2(a_2, a_1) = 0$$

Player 1 is trying to maximize $u_1(a_1, a_2)$ and Player 2 is trying to maximize $u_2(a_1, a_2) = -u_1(a_1, a_2)$. Equivalently, we can think of player 2 as trying to *minimize* $u_1(a_1, a_2)$.

We can represent the zero-sum game

	b_1	$\dots$	b_n
a_1	$x_{1,1}, -x_{1.1}$	$\dots$	$x_{1,n}, -x_{1,n}$
$\vdots$	$\vdots$	$\ddots$	$\vdots$
a_m	$x_{m,1}, -x_{m,1}$	$\dots$	$x_{m,n}, -x_{m,n}$

by the matrix

$$\begin{pmatrix} x_{1,1} & \dots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{m,1} & \dots & x_{m,n} \end{pmatrix}$$

Player 1 choose the row while Player 2 chooses the column. Player 1 is trying to maximize the payoff while Player 2 is trying to minimize it.

1.1 Pure Maxmin Strategies

We describe the *maxmin* strategy for Player 1. Roughly, Player 1 looks at each possible row and determines what her worst outcome would be if she played that row. Then she plays the row where her worst outcome is best.

Let $v_i^1 = \min\{x_{i,1}, \dots, x_{i,n}\}$. Then v_i^1 is the worst payoff that Player 1 will get playing row i. Player 1 then chooses $v^1 = \max\{v_1^1, \dots, v_m^1\}$ and choose $\widehat{i}$ such that $v_{\widehat{i}}^1 = v^1$. Playing $a_{\widehat{i}}$ is a maxmin strategy for Player 1. By playing $a_{\widehat{i}}$ Player 1 guarantees doing no worse than v^1.

For example, consider the game

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 4 & -3 & 2 \\ 1 & -2 & -2 \end{pmatrix}.$$

Then $v_1^1 = 0$, $v_2^1 = -3$, and $v_3^1 = -2$. Thus $v^1 = 0$ and Player 1's maxmin strategy is a_1.

Similarly Player 2 has a *minmax* strategy. Player 2 looks for the largest entry in each column and then chooses the column to minimize the largest entry. We let $v_j^2 = \max\{x_{1,j}, \dots, x_{n,j}\}$ be the largest value in the j^{th} column. Then $v^2 = \min\{v_1^2, \dots, v_n^2\}$ and Playing $a_{\widehat{j}}$ is a minmax strategy for Player 2.

In the example A above $v_1^2 = 4$, $v_2^2 = 0$ and $v_3^2 = 2$. In this case $v^2 = 0$ and b_2 is the minmax strategy.

Suppose $v^1 = v^2$. We call the common value v. We call v the *value* of the game. Pick i such that $v_i^1 = v$ and j such that $v_j^2 = v$. We claim that $x_{i,j} = v$ and (a_i, b_j) is a maxmin solution.

If $x_{i,\ j} < v$, then $v_i^1 \leq x_{i,j} < v$, a contradiction. If $x_{i,j} > v$, then $v_j^2 \geq x_{i,j} > v$, a contradiction. Thus $v = x_{i,j} = v$ and (a_i, b_j) is maxmin solution.

Note that a zerosum game need not have a maxmin solution. For example, consider the game

$$B = \begin{pmatrix} 2 & 0 & 1 \\ 4 & -1 & 2 \\ 1 & 3 & -2 \end{pmatrix}.$$

In this game $v^1 = 0$ but $v^2 = 2$.

Lemma 1.1 *If (a_i, b_j) is a maxmin solution to a zero sum game, then (a_i, b_j) is a Nash equilibrium.*

Proof Since we have a maxmin solution $x_{i,j} = v_i^1 = v_j^2$. If Player 1 changes his move to a_s, then, since v_j^2 is maximal in column j, $x_{s,j} \leq x_{i,j}$. Thus Player 1 can not gain by changing her move.

If Player 2 changes his move to b_t, then since v_j^1 is minimal in row j, $x_{i,t} \geq x_{i,j}$. Thus Player 2 can not gain by changing his move. It follows that (a_i, b_j) is a Nash equilibrium.

Lemma 1.2 *Suppose (a_i, b_j) is a pure strategy Nash equilibrium in a zero sum game. Then (a_i, b_j) is a maxmin solution.*

Proof Since Player 2 can not improve his payoff by moving along the i^{th} row $x_{i,j} = v_i^1$, the minimum element in the row. We next show that $v_i^1 = v^1$. For purposes of contradiction, suppose $v_s^1 > v_i^1$ for some s. But then, $v_s^1 \leq x_{s,j}$. Thus Player 1 can improve her payoff my moving to row s, contradicting the fact that (a_i, b_j) is a Nash equilibrium. Thus $v = v^1 = v_i^1$

A similar argument shows $v = v^2 = v_j^2$. Namely, Player 1 can not improve his payment by moving along the j^{th}-column. Thus $v_j^2 = x_{i,j}$. Suppose for contradiction that $v_j^2 > v^2$. Then there is t such that $v_t^2 < v_j^2$. But $x_{i,t} \leq v_t^2 < v_j^2$. Thus Player 2 can improve his payoff by moving to column s, a contradiction.

Putting these results together we have proved

Theorem 1.3 (Maxmin Theorem) *A zero sum game has a pure strategy Nash equilibrium if and only if $v^1 = v^2$. In this case, the pure strategy Nash equilibria are exactly the maxmin solutions.*

Corollary 1.4 *If (a_i, b_j) and (a_s, b_t) are pure strategy Nash equilibria in a zero sum game, then $x_{i,j} = v = x_{s,t}$. In particular, all Nash equilibria have the same payoff.*

1.2 Mixed Strategies

A zero sum game like Matching Pennies

$$C = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

has no pure strategy equilibria, indeed $v^1 = -1$ and $v^2 = +1$. But we can extend the minmax idea to mixed strategies as well. [1]

We consider the zero sum game with payoff matrix

$$\begin{pmatrix} x_{1,1} & \dots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{m,1} & \dots & x_{m,n} \end{pmatrix}$$

Let S_i be the set of mixed strategies for Player i. If α is a mixed strategy for Player 1 and β is a mixed strategy for Player 2, we let $U(\alpha, \beta)$ be the expected payoff to Player 1 if Player 1 plays α and Player 2 plays β.

If α is a mixed strategy for Player 1, we let

$$w^1_\alpha = \min\{U(\alpha, \beta) : \beta \in S_2\}$$

and if β is a mixed strategy for Player 2, we let

$$w^2_\beta = \min\{U(\alpha, \beta) : \alpha \in S_1\}.$$

Note that w^1_α and w^2_β exist because we are taking extreme values of continuous functions on compact sets. Moreover the functions $\alpha \mapsto w^1_\alpha$ and $\beta \mapsto w^2_\beta$ are continuous as well, so we can find

$$w^1 = \max_{\alpha \in S_1} w^1_\alpha, w^2 = \max_{\beta \in S_2} w^2_\beta$$

.

Let's illustrate with Matching Pennies with payoff matrix

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

We describe a mixed strategy for Player 1 by p, the probability Player 1 plays the Heads and describe a mixed strategy for Player 2 by q, the probability, the probability Player 2 plays Heads.

We know the best response function for Player 2. Player 2 plays Heads if $p < \frac{1}{2}$, plays Tails if $p > \frac{1}{2}$ and is indifferent if $p = \frac{1}{2}$. Thus

$$w^1_p = \begin{cases} U_1(p, T) = 2p - 1 & \text{if } p < 1/2 \\ U_1(1/2, q) = 0 & \text{if } p = 1/2 \\ U_1(p, H) = 1 - 2p & \text{if } p > 1/2 \end{cases}.$$

[1] Though the mathematic analysis is much more difficult.

If $p \neq 1/2$, then $w_p^1 < 0$. Thus $w^1 = 0$.

A similar argument shows that $w^2 = 0$ and $p^* = q^* = \frac{1}{2}$ is the maximin solution.

The next theorem of von Neumann shows this is always the case.

Theorem 1.5 (Maxmin Theorem for Mixed Strategies) *$w^1 = w^2$ and (α, β) is a mixed strategy Nash equilibrium if and only if $w_\alpha^1 = w_\beta^2$. In particular, all mixed strategy Nash equilibria have the same payoff.*

1.3 Conservative Play in Nonzero sum games

The notion of a minmax strategy makes sense in any two player game. It may not be an optimal or reasonable strategy because it may not be true that the opposing player is trying to minimize your payoff. For example consider the game

	b_1	b_2
a_1	3,3	0,0
a_2	1,1	2,2

The maxmin strategy for Player 1 is a_2 as her minimum payoff in this case is 1 which is better than her minimum payoff playing a_1.

The maxmin strategy for Player 2 is b_1. In this case both players do worse than if they played one of the pure strategy equilibria.