Evolutionarily Stable Strategies

Consider the game

	Hawk	Dove
Hawk	-2,-2	2,0
Dove	0,2	1,1

Let σ_p be the strategy: play Hawk with probability p and Dove with probability $(1-p)$.

Recall that $F(p,q)$ is the expected payoff to Player 1 when Player uses σ_p and Player 2 uses σ_q.

Definition 0.1 We say that σ_p is an *Evolutionarily Stable Strategy* (ESS) if either:
i) $F(p,p) \geq F(q,p)$ for all $q \neq p$, or
ii) $F(p,p) = F(q,p)$ and $F(p,q) > F(q,q)$ for all $q \neq p$.
In case i) we say p is a strong ESS and in case ii) it is a mild ESS.

Remarks:

- If p is an ESS, then $F(p,p) \geq F(q,p)$ for all q. Thus (σ_p, σ_p) must be a symmetric Nash equilibrium.
- Any strong ESS, must be a pure strategy equilibrium.
- A *strict* pure strategy symmetric equilibrium is a strong ESS. [Here **strict** means that if Player 1 changes her move while Player 2 does not, Player 1's payoff will go down.]

In the Hawk–Dove game there is a unique symmetric equilibrium where each player uses the mixed strategy $\sigma_{1/3}$, i.e., play H with probability 1/3. We show this is a mild ESS.

$$F(\frac{1}{3}, q) = \frac{1}{3}[q(-2) + (1-q)(2)] + \frac{2}{3}[q(0) + (1-q)(1)] = \frac{4}{3} - q$$
$$F(q,q) = q[q(-2) + (1-q)(2)] + (1-q)[q(0) + (1-q)(1)] = 1 - 4q - 3q^2$$

We need $F(1/3, q) > F(q,q)$ for all $q \neq p$.

$$\begin{aligned}
F(1/3,q) &> F(q,q) \\
\frac{4}{3} - 2q &> 1 - 4q - 3q^2 \\
3q^2 - 2q - \frac{1}{3} &> 0 \\
9q^2 - 6q - 1 &> 0 \\
(3q-1)^2 &> 0
\end{aligned}$$

Which is always true as long as $q \neq 1/3$.

Thus we have a mild ESS when $p = 1/3$. In other words, it is evolutionarily stable to have a population of 1/3 Hawks and 2/3 Doves.

Consider next the game

	Slow	Fast
Slow	6,6	0,2
Fast	2,0	1,1

In this case both (Fast, Fast) and (Slow, Slow) are symmetric pure strategy equilibria and both are strict equilibria thus both are strong ESS.

There is a third symmetric equilibria where both players are Slow with probability $p = 1/5$. Since this is a mixed strategy equilibrium $F(1/5, 1/5) = F(q, 1/5)$ for all q, so it can not be a strong ESS. Is it a mild ESS?

$$F(\frac{1}{5}, q) = \frac{1}{5}[q(6) + (1-q)(0)] + \frac{4}{5}[q(2) + (1-q)(1)] = \frac{7}{5}q + 1$$

$$F(q, q) = q[q(6) + (1-q)(0)] + (1-q)[q(2) + (1-q)(1)] = 1 + q + 4q^2$$

We need $F(1/5, q) > F(q, q)$ for all $q \neq p$, i.e., we need

$$\frac{7}{5}q + 1 > 1 + q + 4q^2$$

for all $q \neq 1/5$, but it easy to see this is false if, say $q = 1$. Then the left hand side is 12/5 and the right hand side is 6. Thus 1/5 is not anevolutionarily stable solution.

Thus in an evolutionarily stable population either everone is Slow or everyone is Fast.

Finally consider the game

	A	B
A	1,1	0,0
B	0,0	0,0

(A,A) is a symmetric strict Nash equilibrium and hence a strong ESS.

(B,B) is a Nash symmetric equilibrium, but $F(B, A) = 0$ while $F(A, A) = 2$ so $F(B, A) \not> F(A, A)$, so B is not evolutionarily stable.

There are no mixed strategy equilibria in this game (if there is any positive probability your opponent plays A you should play A, in which case your oppoent should play A). Thus (A,A) is the only ESS.