

Iterated Elimination and Nash Equilibria

We consider finite two player games—though all of these will generalize to any finite game. We let A denote the set of strategies for Player 1 and B denote the strategies for Player 2.

Recall that when we say a *dominates* a' we mean that it weakly dominates a' .

Proposition 1 *Any game has at most one dominant solution.*

Proof It is impossible for a to dominate a_1 and a_1 to dominate a . □

Proposition 2 *If (a^*, b^*) is a dominant solution, then (a^*, b^*) is a Nash equilibrium.*

Proof The strategy a^* dominates every other strategy in A . Thus

$$v_1(a^*, b^*) \geq v(a, b^*)$$

for all $a \in A$ and a^* is the unique best response to b^* . Similarly, b^* is the unique best response to a^* . Thus (a^*, b^*) is a Nash equilibrium. □

The game *Battle of the Sexes* shows that the converse fails as there are games with Nash equilibria which are not strictly dominant.

Proposition 3 *If (a^*, b^*) is a Nash equilibrium, then (a^*, b^*) is not eliminated by IESDS.*

Proof We prove this by contradiction. Suppose (a^*, b^*) is eliminated during IESDS. Then one of the strategies is removed at some stage of the construction. Let's suppose that a^* is removed before b^* (the other case is similar). Consider the stage when a^* is eliminated. At this stage of the construction, we have a game where a^* and b^* are possible strategies and, because it is about to be eliminated, there is a strategy $a' \in A_1$ such that a' strictly dominates a^* . But then

$$v_1(a', b^*) > v_1(a^*, b^*)$$

and a^* is not a best response for Player 1 to b^* . This contradicts our assumption that (a^*, b^*) is a Nash equilibrium. □

The converse fails. For example, in Battle of the Sexes, (F, O) and (O, F) are not eliminated by IESDS (or even IEDS) but are not Nash equilibria. Similarly, in *Matching Coins* no strategies are eliminated by IESDS (or IEDS) but no strategy profile is a Nash equilibrium.

The converse is true in the special case when there is IEDS solution—i.e. some IESDS procedure ends in a unique solution.

Proposition 4 *If (a^*, b^*) is an IEDS solution, then (a^*, b^*) is a Nash equilibrium.*

Proof For purposes of contradiction suppose a^* is not a best response to b^* —the other case is similar. Let $X = \{a \in A : v_1(a, b^*) > v_1(a^*, b^*)\}$. By assumption $A \neq \emptyset$. All of the strategies in X must be eliminated in the process. Look at the last stage where a strategy $a \in X$ is eliminated. For it to be eliminated, there must be a strategy $a' \in A$ such that a' dominates a . But then

$$v_1(a', b^*) \geq v_1(a, b^*) > v_1(a^*, b^*)$$

and $a' \in X$. But we must eventually eliminate a' and this contradicts the fact that a was the last element of X eliminated. \square

Corollary 5 *If there is an IESDS solution, it is the unique pure strategy Nash equilibrium.*

Proof By Proposition 4 the unique IESDS equilibrium is a Nash equilibrium. By Proposition 3, if there was a second Nash equilibrium it would also be an IESDS equilibrium. \square

Corollary 6 *If there is a strongly dominant strategy equilibrium, it is the unique Nash equilibrium.*

Proof If (a^*, b^*) is a strictly dominant strategy equilibrium, then in the IESDS process at stage 1 would eliminate all strategies except a^* and b^* , so (a^*, b^*) is the unique IESDS-equilibrium and hence the unique Nash-equilibrium. \square

Example 2 below shows that a game may have a dominant solution and several Nash equilibria.

Corollary 7 *There can only be at most one IESDS solution. In particular, in IESDS the order that we eliminate strategies does not matter.*

Proof If there were two IESDS solutions, then would both be Nash equilibria contradicting Corollary 4. \square

More generally, the set of strategies that survive IESDS elimination does not depend on the order of elimination.

Example 1 In IEDS the order of elimination may matter. We also note that this is a game solvable by IEDS with two Nash equilibria.

Consider the game:

	L	R
T	1,1	0,0
M	3,2	2,2
B	0,0	1,1

Process 1: Since M dominates T, we can eliminate T to get

	L	R
M	3,2	2,2
B	0,0	1,1

Now R dominates L. Eliminating L we get

	R
M	2,2
B	1,1

and (M,R) is an IEDS solution.

Process 2:

M dominate B

	L	R
T	1,1	0,0
M	3,2	2,2

Now L dominates R

	L
T	1,1
M	3,2

Thus (M,L) is an IEDS solution.

Example 2 The coordination game below is a game with two Nash equilibria only one of which is an IEDS solution—and no IESDS solution

	L	R
T	1,1	0,0
B	0,0	0,0

In this game (T,L) is the unique IEDS solution, indeed it is a dominant solution, but (B,R) is also a Nash equilibrium.

IESDS does not simplify this game at all.

Example 3 The following game has a unique pure strategy Nash equilibrium but can not be simplified by IEDS

	L	C	R
T	2,2	-1,1	1,0
M	1,-1	0,0	1,-2
B	0,1	2,-1	-2,3