Notes on Cournot Oligopoly

We consider an economy where there are N firms. Firm i chooses q_i the amount that they will produce and a cost function $c_i(q)$ so that producing q costs $c_i(q)$. The total amount produced will be

$$Q = q_1 + \ldots + q_N.$$

The unit price of each item will be given by an inverse demand function $p(Q)$.[1]

The profit for each firm will be

$$\pi_i(q_1, \ldots, q_N) = p(Q)q_i - c_i(q_i).$$

At Nash-equilibrium $(q_1^*, \ldots, q_N^*)$ Firm i choose q to maximize profit assuming that for $j \neq i$ Firm j produces q_j^*. Under reasonable assumptions this will happen when

$$\frac{\partial \pi_1}{\partial q_1}(q_1^*, \ldots, q_N^*) = \ldots = \frac{\partial \pi_N}{\partial q_N}(q_1^*, \ldots, q_N^*) = 0$$

Identical Firms

We will work out the details under the following assumptions:

- The inverse demand function is $p(Q) = \alpha - bQ$.
- Each firm has production function $c_i(q_i) = cq_i$.

Monopoly

We first consider the case of a monopoly where there is a unique firm with profit function

$$\pi(q) = (\alpha - bq)q - cq.$$

A monopoly will maximize profits by choosing q such that $\pi'(q) = 0$. Then

$$0 = \pi'(q) = \alpha - 2bq - c$$

or $q^* = \frac{\alpha - c}{2b}$. In this case

- The total production Q is $\frac{\alpha - c}{2b}$
- The price is

$$p(q^*) = \alpha - b\left(\frac{\alpha - c}{2b}\right) = \frac{\alpha + c}{2}$$

[1] It might be more natural to think of the economy as having a demand function $D(p)$, where if the price is p, then $D(p)$ items will be sold. In this case the inverse demand function is $p(Q) = D^{-1}(Q)$. This tells us if the economy produces Q items the price will be $p(Q)$.

- The firm's profit is

$$\pi(q^*) = \frac{\alpha + c}{2}\left(\frac{\alpha - c}{2b}\right) - c\left(\frac{\alpha - c}{2b}\right) = \frac{(\alpha - c)^2}{4b}.$$

Cournot Duopoly

We next consider the case $N = 2$ where there are two identical firms. In this case Firm i has profit function

$$\pi_i(q_1, q_2) = (\alpha - b(q_1 + q_2))q_i - cq_i,$$

$$\frac{\partial \pi_1}{\partial q_1}(q_1, q_2) = \alpha - 2bq_1 - bq_2 - c$$

and

$$\frac{\partial \pi_2}{\partial q_2}(q_1, q_2) = \alpha - bq_1 - 2bq_2 - c.$$

The best response function for Firm 1 is found by setting $\frac{\partial \pi_1}{\partial q_1} = 0$ and solving for q_1. Thus

$$BR_1(q_2) = \frac{\alpha + c - bq_2}{2b}$$

and similarly

$$BR_2(q_1) = \frac{\alpha + c - bq_1}{2b}.$$

We must find q_1^*, q_2^* to solve the linear equations

$$\begin{aligned} 2bq_1^* + bq_2^* &= \alpha - c \\ bq_1^* + 2bq_2^* &= \alpha - c \end{aligned}$$

Solving we find that

$$q_1^* = q_2^* = \frac{\alpha - c}{3b}$$

- The total production Q is $\frac{2(\alpha - c)}{3b}$
- The price is

$$p(q_1^*, q_2^*) = \alpha - b\left(\frac{2(\alpha - c)}{3b}\right) = \frac{\alpha + 2c}{3}$$

- The firm's profit is

$$\pi(q_1^*, q_2^*) = \frac{\alpha + 2c}{3}\left(\frac{2(\alpha - c)}{3b}\right) - c\left(\frac{2(\alpha - c)}{3b}\right) = \frac{2(\alpha - c)^2}{9b}.$$

Note that in equillibrium the consumer is better off then they would be under monopoly. Production is higher and prices are lower. The Firms however are worse off as profits have decreased.

Cartel

Alternatively, two firms could form a Cartel to produce Q where $q_1 = q_2 = \frac{Q}{2}$ and choose Q to maximize profits. They would choose the monopoly solution of $Q = \frac{\alpha - c}{2b}$ and split the profits to get $\frac{(\alpha-c)^2}{8b}$. This is a higher profit than under duopoly but neither firm is using a best response so both could increase profits by breaking the agreement.

$N = 3$ **Firms**

In this case Firm i has profit function

$$\pi_i(q_1, q_2, q_3) = (\alpha - b(q_1 + q_2 + q_3))q_i - cq_i,$$

$$\frac{\partial \pi_1}{\partial q_1}(q_1, q_2, q_3) = \alpha - 2bq_1 - bq_2 - bq_3 - c,$$

$$\frac{\partial \pi_2}{\partial q_2}(q_1, q_2, q_3) = \alpha - bq_1 - 2bq_2 - bq_3 - c$$

and

$$\frac{\partial \pi_3}{\partial q_3}(q_1, q_2, q_3) = \alpha - bq_1 - bq_2 - 2bq_3 - c.$$

We need to solve the system of linear equations:

$$\begin{aligned} 2bq_1^* + bq_2^* + bq_3^* &= \alpha - c \\ bq_1^* + 2bq_2^* + bq_3^* &= \alpha - c \\ bq_1^* + bq_2^* + 2bq_3^* &= \alpha - c \end{aligned}$$

The unique solution is $q_1^* = q_2^* = q_3^* = \frac{\alpha - c}{4b}$.

- The total production Q is $\frac{3(\alpha-c)}{4b}$
- The price is
$$p(q_1^*, q_2^*, q_3^*) = \frac{\alpha + 3c}{4}$$
- The firm's profit is
$$\pi(q_1^*, q_2^*, q_3^*) = \frac{3(\alpha - c)^2}{16b}.$$

The General Case

Consider N firms. We need to solve the system of linear equations:

$$\begin{aligned} 2bq_1^* + bq_2^* + \ldots + bq_N^* &= \alpha - c \\ bq_1^* + 2bq_2^* + \ldots + bq_3^* &= \alpha - c \\ &\vdots \\ bq_1^* + bq_2^* + \ldots + 2bq_N^* &= \alpha - c \end{aligned}$$

Solving we get that $q_1^* = \ldots = q_N^* = \frac{\alpha - c}{(N+1)b}$.[2]

- The total production Q is $\frac{N(\alpha-c)}{(N-1)b}$
- The price is

$$p(q_1^*, \ldots, q_N^*) = \frac{\alpha + Nc}{N+1}$$

- The firm's profit is

$$\pi(q_1^*, q_2^*, q_3^*) = \frac{N(\alpha - c)^2}{(N+1)^2 b}.$$

Perfect Competition

What happens as $N \to +\infty$?

- total production

$$\lim_{N\to\infty} \frac{N(\alpha - c)}{(N-1)b} = \frac{\alpha - c}{b}$$

- price

$$\lim_{N\to\infty} \frac{\alpha + Nc}{N+1} = c$$

- profit

$$\lim_{N\to\infty} \frac{N(\alpha - c)^2}{(N+1)^2 b} = 0$$

This agrees with the usual economic models for perfect competition where, in equilibrium, the price is the marginal cost (which in our case is c) and profits are 0.

[2] An easy way to solve this system is to note that it is symmetric so there should be a solution where $q_1 = \ldots = q_N$.