Notes on Auctions

Second Price Sealed Bid Auctions

These are the easiest auctions to analyze.

Theorem 1 *In a second price sealed bid auction bidding your valuation is always a weakly dominant strategy.*

Proof Suppose we value the item v. We need to show that bidding v weakly dominates bidding b for any $b \neq v$.

Case 1: $b < v$.

Let b^* be the maximum bid of the other bidders.

- If $b^* < b$, then bidding either v or b we will win in pay b^* so we are at least as well off bidding b.
- If $b^* = b$, then we win only a portion of the time if we bid b and would have won if we bid v so would be better off bidding v.
- If $b < b* < v$, we lose bidding b and win bidding v so would be better off bidding v.
- If $v \leq b^*$, then bidding b or v we get a payoff of 0 so are at least as well off bidding v.

Since bidding v is never worse and sometimes better than bidding b, bidding v weakly dominates bidding b.

Case 2 $b > v$

Again we let b^* be the maximum bid of the other bidders.

- If $b^* < v$, then bidding either v or b we win and pay b^* so there is no difference
- If $v = b^* < b$, then bidding either v or b we get payoff 0.
- If $v < b^* \leq b$, then bidding v we get payoff 0, while bidding b we get a negative payoff, so bidding v is better.
- If $b < b^*$, then bidding either v or b we get payoff 0.

Since bidding v is never worse and sometimes better than bidding b, bidding v weakly dominates bidding b.

Since this is true for all $v \neq b$, bidding v is a weakly dominant strategy. □

Note that we did not need to know the number of bidders or anything about the distribution of their valuations. We will see this is very different in first price auctions.

Exercise 2 Show that in a first price sealed bid auction bidding $w \geq v$ is weakly dominated by bidding $v - 1$.

Independent Private Values

We will look at auctions under the assumption of *Independent Private Values*. We assume there are N bidders. Bidder i has a value v_i and there is a probability distribution F_i such that $Pr(v_i < r) = F_i(r)$. We assume that $v_1, \ldots, v_n$ are independent random variables.

For simplicity we will consider only the case where each $v_i \in [0,1]$ with uniform distribution, i.e., $Pr(v_i < r) = r$ for $r \in [0,1]$

First Price Sealed Bid Auctions

We consider a first price sealed bid auction where there are N players with independent private values v_i uniformly distributed in $[0,1]$.

A strategy for Player i will be of the form $b_i : [0,1] \to [0,1]$ where Player i bids $b_i(v_i)$ with value v_i. We will look for a symmetric equilibrium where each player uses the same strategy $b(v)$. We make some additional reasonable assumptions

- $b(0) = 0$, if my value is 0, I should not bid more that 0.
- b is increasing, if $v < w$, then $b(v) < b(w)$.
- $b(v) = \alpha v$ where $0 \leq \alpha \leq 1$.[1]

We look at Player 1's strategy. Suppose Player 1 has value is v and bids $b = \alpha v$. Player 1 will only win if all other Players have value $v_i < v$.[2] Thus Player 1 only wins if all other values are less that $v = \frac{b}{\alpha}$. The probability that Player 1 wins is

$$Pr\left(v_2 < \frac{b}{\alpha}, \ldots, v_N < \frac{b}{\alpha}\right) = \left(\frac{b}{\alpha}\right)^{N-1}$$

and Player 1's expected payoff is

$$E(b) = \left(\frac{b}{\alpha}\right)^{N-1}(v-b).$$

Player 1 will choose b to maximize expectation. Since $E(0) = E(b) = 0$, we maximize $E(b)$ be differentiating with respect to b and setting $E'(b) = 0$. Thus

$$\begin{aligned} 0 &= \frac{N-1}{\alpha}\left(\frac{b}{\alpha}\right)^{N-2}(v-b) - \left(\frac{b}{\alpha}\right)^{N-1} \\ &= \frac{b^{N-1}}{\alpha^N}\left[(N-1)(v-b) - b\right] \\ &= (N-1)v - Nb \end{aligned}$$

Thus $b = \frac{N-1}{N}v$.

Thus there is a Bayesian-Nash equilibrium where Player i bids $\frac{N-1}{N}v_i$.

[1] This assumption is not necessary. A slightly more complicated argument will get us to the same conclusion if we do not assume this.

[2] The probability that two players have the same value is 0 so we can ignore that possibility.

Revenue Equivallence

First let's consider the expected revenue for the seller in a first price sealed bid auction.

$N = 2$ We break this into two cases. $v_2 \le v_1$ and $v_1 \le v_2$ On the first region the expected maximal bid is $\frac{v_1}{2}$ and the expectation on this region is

$$\int_0^1 \int_0^{v_1} \frac{v_1}{2}\, dv_2 dv_1.$$

Since the other region is symmetric, the total expectation is

$$2\int_0^1 \int_0^{v_1} \frac{v_1 2}{2}\, dv_2 dv_1 = 2\int_0^1 \frac{v_1^2}{2}\, dv_1 = 2\left(\frac{1}{6}\right) = 1/3.$$

If $N = 3$ there are 6 cases depending on the 6 possible ordering of $v_1, \ldots, v_3$. If we assume $v_1 > v_2 > v_3$, we get

$$\begin{aligned} 6\int_0^1 \int_0^{v_1} \int_0^{v_2} \frac{2}{3} v_1\, dv_3 dv_2 dv_1 &= 6\int_0^1 \int_0^{v_1} \frac{2}{3} v_1 v_2\, dv_2 dv_1 \\ &= 6\int_0^1 \frac{v_1^3}{3}\, dv_1 \\ &= 6\left(\frac{1}{12}\right) = 1/2 \end{aligned}$$

For general N we need to consider all $N!$ orderings of $x_1, \ldots, x_N$. if $v_1 > \ldots > v_N$ we get the general expression

$$N! \int_0^1 \int_0^{v_1} \cdots \int_0^{v_{N-1}} \frac{N-1}{N} v_1\, dv_N dv_{N-1} \ldots dv_1 = \frac{N-1}{N+1}$$

Now we consider second price sealed bid auctions. We've argued before that bidding your value is a weakly dominant strategy in a second price auction. Thus the equilibrium price will be the second highest of the values. If $v_1 > v_2 > \ldots > v_N$, the equilibrium price is v_2. There are $N!$ possible orderings of the v_i, thus the expected equilibrium price is

$$N! \int_0^1 \int_0^{v_2} \cdots \int_0^{v_{N-1}} v_2\, dv_N dv_{N-1} \ldots dv_1 = \frac{N-1}{N+1}$$

We show this for $N = 2, 3$

Let $N = 2$

$$\begin{aligned} 2\int_0^1 \int_0^{v_1} v_2\, dv_2 dv_1 &= 2\int_0^1 \frac{v_1^2}{2}\, dv_1 \\ &= 2\left(\frac{1}{6}\right) = 1/3 \end{aligned}$$

For $N = 3$

$$\begin{aligned}
6\int_0^1 \int_0^{v_1} \int_0^{v_2} v_2\, dv_3 dv_2 dv_1 &= 6\int_0^1 \int_0^{v_1} v_2^2\, dv_1 dv_2 \\
&= 6\int_0^1 \frac{v_1^3}{3}\, dv_1 \\
&= 6\left(\frac{1}{12}\right) = 1/2
\end{aligned}$$

It may seem surprising that first price sealed bid auction and second price sealed bid auctions give rise to the same expected revenue, but in fact, under some reasonable assumptions, this is always the case! Vickery and Myerson received a Nobel prize for, among other things, the following theorem (which we state vaugely).

Theorem 3 (Revenue Equivalence Theorem) *Suppose we have N players with independent identically distributed private values where each player has values in $[\alpha, \beta]$ with probability distribution $Pr(x \leq r) = F(r)$ where F is continuous and strictly increasing.*

Then any two auctions where:

i) the player with the highest valuation will be awarded the item, and

ii) a player with value 0 has expected payoff 0.

have the same expected revenue.

Reserve price auctions

One way to increase the revenue beyond the bounds of the Revenue Equivalence Theorem is to relax the assumption there is always a winner.

For example consider a reserve price auction where the seller sets a reserve price r and then accepts sealed bids. The seller knows r but the bidders do not. If none of the bids is greater than r, there is no winner. If at least one bidder bids more than r the highest bidder wins and plays the larger of the second highest bid and r.

Exercise 4 Show that for each player bidding your valuation is still a weakly dominant strategy.

Let's analyze what happens in an auction with two bidders with private independent values uniformly distributed in [0,1] and a reserve price of r where $0 \leq r \leq 1$. Let x be the value for Player 1 and y the value for Player 2.

There are five possible outcomes (we can ignore the outcomes where $x = y$ has these occur with probability 0).

1) $x, y \leq r$

In this case the revenue is 0.

2) $y \leq r \leq x$.

In this case Player 1 wins and pays r. The contribution to expected payoff is

$$\int_r^1 \int_0^r r\, dydx = r^2(1-r)$$

3) $r \leq y < x$

In this case Player 1 wins and pays y. The contribution to expected payoff is

$$\begin{aligned}\int_r^1 \int_r^x r\, dydx &= \int_r^1 \left(\frac{x^2}{2} - \frac{r^2}{2}\right) dx \\ &= \left(\frac{1}{6} - \frac{r^3}{6}\right) - \left(\frac{r^2}{2} - \frac{r^3}{2}\right) \\ &= \frac{1 + 2r^3 - 3r^2}{6}\end{aligned}$$

Next we consider the cases where $y \leq r \leq x$ and $r \leq x \leq y$ which similarly give $r^2(1-r)$ and $\frac{1+2r^3-3r^2}{6}$

Let

$$\Phi(r) = 2r^2(1-r) + \frac{1 + 2r^3 - 3r^2}{3} = r^2 - \frac{4r^3}{3} + \frac{1}{3}$$

This is the expected revenue if we set the reserve at r.

Note that $\Phi(0) = 1/3$. If we set the reserve price at 0, this is the same things as having a second price sealed bid auction with no reserve so this agrees with our calculation above. Also note that $\Phi(1) = 0$–if we set the reserve at 1 we will never sell the item.

What if we use different values of r? Note $\Phi(.25) = .375$, $\Phi(.5) = .417$. Thus we can do better that a second price auction by setting a reserve. But we don't want to set it too high as $\Phi(.8) = .291$.

We can find an optimal value for r by setting $\Phi'(r) = 0$ and solving for r.

$$\Phi'(r) = 2r - 4r^2$$

So the maximum expected revenue is obtained by setting a reserve of $r = 1/2$.