Vaught's Conjecture for Differentially Closed Fields: Part II

David Marker

http://www.math.uic.edu/∼marker/vcdcf-slides2.pdf

Our Main Goal

Theorem 1 (Hrushovski–Sokolović 1992) *There are* $2^{\aleph_0}$ *countable differentially closed fields of characteristic zero.*

What are we looking for?

For our method of coding graphs using dimensions to work, we will need:

- large family of types $(p_a : a \in A)$, $p_a \in S(a)$, to which we can assign different countable dimensions.

- good notion of independence in A with lots of elements $a, b, c \in A$, pairwise independent but not independent (non-triviality)

- the ability to realize one type in the family while omitting others (orthogonality)

The types p_a will be generic types of strongly minimal sets. Recall

• Hrushovski and Sokolović showed that if X is a nonmodular strongly minimal set then there is a definable finite-to-one $f : X \to C$, where C is the field of constants.

• We can find many trivial strongly minimal sets. For example, if A is a δ-independent set, and

$$X_a = \left\{ x : x' = \frac{ax}{x+1} \right\}$$

then X_a is an infinite set of indiscernibles and $X_a \perp X_b$ for $a \neq b \in A$.

But all known trivial strongly minimal sets are infinite dimensional.

If this is to work we will need to find nontrivial modular strongly minimal sets.

Abelian Varieties

Let K be an algebraically closed field. An *Abelian variety* is a subvariety $A \subseteq \mathbb{P}^n(K)$, such that there is a rational map $\mu : A \times A \to A$ making A into a group.

The simplest example is an elliptic curve

$$Y^2 = X^3 + aX + b$$

together with a point O at infinity.

Proposition 2 *Every Abelian variety is a divisible commutative group.*

If A has dimension d, then there are n^{2d} points of order n.

Definition 3 We say A is *simple* if A has no proper infinite Abelian subvarieties.

Definition 4 Abelian varieties A and B are *isogenous* if there is a rational group homomorphism $f : A \to B$ with finite kernel.

j-invariants

Consider the elliptic curve E

$$Y^2 = X^3 + aX + b.$$

The *j-invariant* of the curve $j(E)$ is $\dfrac{6912a^3}{4a^3 + 27b^2}$.

Theorem 5 *i) Let L be an algebraically closed field. For $j \in L$ there is E with $j(E) = j$.*

ii) $E \cong E_1$ if and only if $j(E) = j(E_1)$.

iii) If E and E_1 are isogenous, then $j(E)$ and $j(E_1)$ are interalgebraic over $\mathbb{Q}$.

Manin Kernels

Theorem 6 (Manin-Buium) *Let K be a differentially closed field. If A is an Abelian variety defined over K, there is a δ-definable homomorphism $\mu : A \to K^n$ such that the kernel of μ is the Kolchin closure of the torsion of A.*

For example, if E is the elliptic curve

$$Y^2 = X^3 + aX + b$$

where $a, b \in C$ then $\mu(x, y) = \dfrac{x'}{y}$.

Let $A^\sharp$ be the Kolchin closure of the torsion.

If A is defined over C, then $A^\sharp = A(C)$.

Theorem 7 (Hrushovski–Sokolović) *If A is a simple Abelian variety that is not isomorphic to an Abelian variety defined over the constants, then $A^\sharp$ is a modular strongly minimal set.*

If A and B are nonisogenous $A^\sharp$ and $B^\sharp$ are orthogonal.

Moreover, if X is any nontrivial modular strongly minimal set, then X is nonorthogonal to $A^\sharp$ for some simple Abelian variety A.

Independence

Definition 8 We say that $\overline{a}$ is *independent* from B over A if

$$\mathrm{RM}(\overline{a}/A \cup B) = \mathrm{RM}(\overline{a}/A).$$

We write $\overline{a} \mathop{\smile\!\!\!\!\!\!|}_A B$.

Example If $a_0, \ldots, a_n$ are δ-independent over k, then a_0 is δ-transcendental over $k\langle a_1, \ldots, a_n \rangle$ (the differential field generated by $k(a_1, \ldots, a_n)$). Thus

$$\mathrm{RM}(a_0/k) = \omega = \mathrm{RM}(a_0/k, a_1, \ldots, a_n)$$

and $a_0 \mathop{\smile\!\!\!\!\!\!|}_k a_1, \ldots, a_n$.

Example Let a be δ-transcendental over k. Then $a \not\mathop{\smile\!\!\!\!\!\!|}_k a'$, since over $k\langle a' \rangle$, a satisfies the rank 1 formula $X' = a'$.

Theorem 9 (Symmetry) *If $\overline{a} \mathop{\smile\!\!\!\!\!\!|}_A \overline{b}$, then $\overline{b} \mathop{\smile\!\!\!\!\!\!|}_A \overline{a}$.*

Algebraic Characterization of Independence in DCF

Definition 10 Let $k \subseteq l_1, l_2$ be fields. l_1 and l_2 are *free* over k if any $a_1, \ldots, a_n \in l_1$ algebraically dependent over l_2 are already algebraically dependent over k.

Theorem 11 *If k is a differential field and $\overline{a}, B \subseteq \mathbb{K} \models$ DCF, then the following are equivalent*

i) $\overline{a} \mathop{\smile\!\!\!\!\!\!\mid}_k B$

ii) $k\langle\overline{a}\rangle$ *and* $k\langle B\rangle$ *are free over* k*.*

Fact 12 *i) If* $\mathrm{td}(k\langle\overline{a}\rangle/k)$ *is finite, then* $\mathrm{RM}(\overline{a}/k) \leq \mathrm{td}(k\langle\overline{a}\rangle/k)$*.*

ii) If $\mathrm{td}(k\langle\overline{a}\rangle/k)$ *is infinite, then* $\mathrm{RM}(\overline{a}/k) \geq \omega$*.*

Lemma 13 *If* a *is* δ*-transcendental over* k *and* $\mathrm{RM}(\overline{b}/k) < \omega$*, then* $a \mathop{\smile\!\!\!\!\!\!|}_k \overline{b}$*.*

Proof If $a \not\mathop{\smile\!\!\!\!\!\!|}_k \overline{b}$, then $k\langle a, \overline{b}\rangle$ has finite transcendence degree over $k\langle\overline{b}\rangle$. But then $k\langle a, \overline{b}\rangle$ has finite transcendence degree over k, a contradiction.

Orthogonality

Definition 14 Let $p \in S(A)$, $q \in S(B)$. We say $p \perp q$ if $\overline{a} \mathop{\smile\!\!\!\!\!\!\,|}_M \overline{b}$ for any $M \supseteq A \cup B$, a realizing p and b realizing q with $a \mathop{\smile\!\!\!\!\!\!\,|}_A M$ and $b \mathop{\smile\!\!\!\!\!\!\,|}_B M$.

Lemma 15 *Suppose X is a strongly minimal set defined over $K \models$ DCF, p is the generic type of X over K and $p \perp q$. Let $\overline{b}$ realize q. Then p is omitted in $K\langle\overline{b}\rangle^{\text{dif}}$.*

Proof Suppose $\overline{a} \in K\langle\overline{b}\rangle^{\text{dif}}$ realizes p. There is $\phi(\overline{v})$ isolating $\text{tp}(\overline{a}/K\langle\overline{b}\rangle)$. Since $p \perp q$, $\text{RM}(\phi) = 1$. Since X is strongly minimal, ϕ holds of some elements of $X(K)$, a contradiction.

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For our method of coding graphs using dimensions to work, we will need:

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- good notion of independence in A with lots of elements $a, b, c \in A$, pairwise independent but not independent (non-triviality)

- the ability to realize one type in the family while omitting others (orthogonality)

For $a \in \mathbb{K}$, let $E(a)$ be the elliptic curve with j-invariant a, let $E(a)^\sharp$ be the δ-closure of the torsion points and let $p_a \in S(a)$ be the generic type of $E(a)^\sharp$.

- $E(a)^\sharp$ is strongly minimal

- p_a is determine by $\overline{x} \in E(a)^\sharp$, $\overline{x} \notin \mathbb{Q}\langle a\rangle^{\mathrm{alg}}$.

- $E(a)^\sharp \cap \mathbb{Q}(a)^{\mathrm{alg}}$ contains the torsion points of $E(a)$ so is infinite.

- $p_a \not\perp p_b$ if and only if $E(a)$ and $E(b)$ are isogenous, in this case $\mathbb{Q}(a)^{\mathrm{alg}} = \mathbb{Q}(b)^{\mathrm{alg}}$.

- $p_a \perp r$ where r is the type of a δ-transcendental

Lemma 16 *p_a is not realized in $\mathbb{Q}\langle a\rangle^{\mathrm{dif}}$.*

Proof Suppose $\overline{b} \in \mathbb{Q}\langle a\rangle^{\mathrm{dif}}$ realizes p_a. Let $\phi(v)$ isolate tp$(b/\mathbb{Q}\langle a\rangle)$. Since $\overline{b} \notin \mathbb{Q}\langle a\rangle^{\mathrm{alg}}$, $\phi(v)$ defines an infinite subset of $E(a)^\sharp$, but then it must contain a torsion point of $E(a)$. But the torsion points are in $\mathbb{Q}(a)^{\mathrm{alg}}$, a contradiction.

Coding Graphs in DCF

Let G be an infinite graph with vertex set A such that for all $a \in A$ there are $b \neq c$ with $(a,b),(a,c) \in G$.

Let K_0 be the differential closure of $\mathbb{Q}\langle A\rangle$ where the elements of A are independent δ-transcendentals.

Let $B = \{a+b : a,b \in A, (a,b) \in G\}$. Note that the elements of B are also δ-transcendental.

Theorem 17 *There is* $K(G) \models \mathrm{DCF}$ *with* $K(G) \supset K_0$, $|K(G)| = |G|$ *where if* $c \in A \cup B$, $\dim(p_c/K(G)) = 0$ *while if* c *is* δ*-transcendental and* $p_c \perp p_a$ *for all* $a \in A \cup B$*, then* $\dim(p_a, K(G)) = \aleph_0$.

Constructing $K(G)$

Proposition 18 *If $a \in A \cup B$, then p_a is omitted in K_0.*

Suppose $a \in A$ (the other case is similar).

- p_a is omitted in $\mathbb{Q}\langle a \rangle^{\mathrm{dif}}$.

- p_a is omitted in $K_0 \cong (\mathbb{Q}\langle a \rangle^{\mathrm{dif}})\langle A \setminus \{a\}\rangle)^{\mathrm{dif}}$, since $r \perp p_a$.

We build $K_0 \subset K_1 \subset K_2 \ldots$. Suppose $c \in K_n$ and $p_c \perp p_a$ for all $a \in A \cup B$. We can build $K_{n+1} \supseteq K_n$ realizing p_c and adding no new realizations of p_a for $a \in A \cup B$. With careful bookkeeping we construct $K(G) = \bigcup K_i$.

Recovering G from $K(G)$

- $\not\smile$ is an equivalence relation on realizations of r.

For a, b realizing r, $a \not\smile b$ if a is differentially algebraic over $k\langle b\rangle$. If $a \not\smile b$ and $b \not\smile a$, $\mathbb{Q}\langle a, b, c\rangle$ is differentially algebraic over $\mathbb{Q}\langle a, b\rangle$ which is differentially algebraic over $\mathbb{Q}\langle a\rangle$. Thus $a \not\smile c$.

Let $[a]$ be the $\not\smile$-class of a.

Let $S = \{[a] : a \text{ realizes } r, \dim(p_a, K(G)) = 0\}$.

• For each $[a] \in S$ there is a unique $c \in A \cup B$ such that $[c] = [a]$.

If $p_c \not\perp p_a$ for some $a \in A \cup B$, then $E(c)$ and $E(a)$ are isogenous and $c \not\smile a$.

We say that $\{[a], [b], [c]\} \in S^3$ is a *triangle* if a, b, c are pairwise independent but not independent.

• This does not depend on choice of representative. If say $a_1 \not\smile b_1$, then $a \not\smile a_1 \not\smile b_1 \not\smile b$, and, since $\not\smile$ is an equivalence relation, $a \not\smile b$.

Since

$$\mathbb{Q}\langle a_1, b_1\rangle \subseteq \mathbb{Q}\langle a_1, b_1, a, b\rangle \subseteq \mathbb{Q}\langle a_1, b_1, a, b, c\rangle \subseteq \mathbb{Q}\langle a_1, b_1, a, b, c, c_1\rangle$$

and each of these extensions is of finite transcendence degree, the transcendence degree of $\mathbb{Q}\langle a_1, b_1, c_1\rangle$ over $\mathbb{Q}\langle a_1, b_1\rangle$ is finite and $c_1 \not\smile a_1, b_1$. Hence a_1, b_1, c_1 are pairwise independent but not independent.

Proposition 19 *Every triangle is of the form* $\{[a],[b],[a+b]\}$ *where* $a, b \in A$.

- Any three elements of A are independent

- Any three elements of B are independent

For example $a+b, a+c, b+c$ are interdefinable with a, b, c (since $2b = (a+b)+(b+c)-(a+c)$), thus they are independent.

- If $a \in A$ and $x, y \in B$ then a, x, y are independent

For example $a, a+b, a+c$ are interdefinable with a, b, c.

- If $a, b \in A$, $x \in B$ and a, b, x are dependent, then $x = a+b$.

For example $a, b, a+c$ are interdefinable with a, b, c.

Recall that every vertex of G has valance at least 2.

Let $V = \{[a] \in S :$ there are at least two triangles containing $[a]\}$. Then $V = \{[a] : a \in A\}$.

Let $E = \{([a],[b]) :$ there is a triangle $\{[a],[b],[c]\}$.

Then $(V,E) \cong G$.

Theorem 20 $\kappa \geq \aleph_0$. *There are* 2^κ *nonisomorphic* DCF *of cardinality* κ.

For $\kappa > \aleph_0$, this was proved by Poizat using trivial strongly minimal instead of $E(a)^\sharp$.

DOP and ENI-DOP

Definition 21 A theory T has the *Dimension Order Property* (DOP) if there are models $\mathcal{M}_0 \subseteq \mathcal{M}_1, \mathcal{M}_2 \subseteq \mathcal{M}$ with $\mathcal{M}$ prime over $M_1 \cup M_2$, $p \in S(M)$ such that $p \perp \mathcal{M}_1$ and $p \perp \mathcal{M}_2$.

In our case we could take K_0 differentially closed, a, b δ-independent over K_0, $K_1 = K_0\langle a\rangle^{\text{dif}}$, $K_2 = K_0\langle b\rangle^{\text{dif}}$, $K = K_0\langle a, b\rangle^{\text{dif}}$ and $p = p_{a+b}$.

We say that T has ENI-DOP if we can choose the type p to be strongly regular, nonisolated (as in our case), or more generally, nonisolated after adding finitely many parameters.

• In DCF, the type p_a is nonisolated over a (since there are infinitely many torsion points algebraic over a), so we have ENI-DOP

• In T_2 (where $\pi^{-1}(a)$ is a model of Th$(\mathbb{Z}, s)$, the generic type is isolated over a, but once we have a realization b it is nonisolated over a, b, so we have ENI-DOP.

• In T_1 (where $\pi^{-1}(a)$ is an infinite set with no structure), even if we add finitely many realizations $\bar{b}$ the type is isolated. In this case we have DOP but not ENI-DOP.

Theorem 22 (Shelah) *Let T be an ω-stable theory with DOP. If $\kappa \geq \aleph_1$, there are 2^κ nonisomorphic models of cardinality κ.*

Further, if T has ENI-DOP, then there are also $2^{\aleph_0}$ countable models.