A yearlong experimental study showed positive effects of a professional development project that involved 19 urban elementary schools, 180 teachers, and 3735 students from one of the lowest performing school districts in California. Algebraic reasoning as generalized arithmetic and the study of relations was used as the centerpiece for work with teachers in Grades 1–5. Participating teachers generated a wider variety of student strategies, including more strategies that reflected the use of relational thinking, than did nonparticipating teachers. Students in participating classes showed significantly better understanding of the equal sign and used significantly more strategies reflecting relational thinking during interviews than did students in classes of nonparticipating teachers.

Key words: Algebra; Arithmetic; Children’s Strategies; Connections in Mathematics; Elementary, K–8; Large Scale Studies; Professional Development; Teacher Knowledge

Algebra has become a focal point for educators and policymakers, and the widely used catchphrase “Algebra for All” has underscored the importance of providing all students access to algebra (Edwards, 1990; Kaput, 1998; Silver, 1997). Moses and Cobb (2001) have argued that algebra is the “key to the future of disenfranchised communities” (p. 3) because it is not only the gatekeeper to higher mathematics but also “the gatekeeper for citizenship; and people who
Enrolling students in algebra courses, however, is not sufficient. The transition from arithmetic to algebra has proven difficult for students, and it is now widely recognized that students need earlier opportunities to engage in algebraic reasoning (Driscoll, 1999; Kaput, 1998, 1999; National Council of Teachers of Mathematics [NCTM], 1997, 1998, 2000). The work of the NCTM Algebra Working Group (NCTM & MSEB, 1998) and the Early Algebra Group (Kaput, Carraher, & Blanton, in press) reflects recent efforts in which researchers collaborate to find ways to integrate algebraic reasoning throughout Grades K–12. The goal, however, is not simply to push the current high school algebra curriculum down into the elementary school. Instead, algebra is conceptualized more broadly so that the emphasis is shifted from learning rules for symbol manipulation toward developing algebraic reasoning.

Research has provided examples of classrooms of students engaged in algebraic reasoning but little systematic evidence of the effects of professional development on teachers and their students. This lack of data relating professional development with teacher knowledge and student achievement was identified as a key limitation of research on algebraic reasoning at the International Commission on Mathematical Instruction (ICMI) Study Conference on *The Future of the Teaching and Learning of Algebra* (Chick, Stacey, Vincent, & Vincent, 2001). In this article, we describe an experimental study in which we investigated the effects of using algebraic reasoning as the centerpiece of professional development for teachers in Grades 1–5 in an urban, low-performing school district. We investigated the performance of both teachers and students on written and interview assessments of algebraic reasoning. Before presenting the study, however, we describe our conceptualization of algebraic reasoning at the elementary school level and our approach to professional development in which children’s mathematical thinking played a central role.

ALGEBRAIC REASONING IN ELEMENTARY SCHOOL

Algebraic reasoning can be manifested in a number of ways. Kaput (1998) identified five interrelated forms of algebraic reasoning, as paraphrased below:

- Algebra as generalizing and formalizing patterns and regularities, in particular, algebra as generalized arithmetic;
- Algebra as syntactically guided manipulations of symbols;
- Algebra as the study of structure and systems abstracted from computations and relations;
- Algebra as the study of functions, relations, and joint variation; and
- Algebra as modeling. (p. 26)

In our research, we have focused on the first and third of these forms of algebraic reasoning. This focus derived from considerations related both to the mathematics and to our goal of engaging teachers in making sense of algebraic ideas and
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students’ thinking about algebraic ideas in ways that were helpful to their work in the classroom. First, because of the central place that arithmetic holds in the current curriculum, any meaningful change in the elementary school curriculum must address how arithmetic is learned. Therefore, we chose to engage teachers in thinking about extending arithmetic to include algebraic ideas. Second, we wanted to engage teachers with ideas that were woven throughout the curriculum so that they viewed algebraic reasoning as pervading the mathematics curriculum rather than as simply one more topic to teach. Third, we wanted to encourage teachers to notice their own students’ thinking, and thus we focused on algebraic ideas that had a research base for children’s reasoning at the elementary school level. Finally, we focused on algebra as generalized arithmetic and the study of relations because learning to reason algebraically in elementary school not only can provide a foundation for smoothing the transition to algebra in later grades but also can deepen understanding of basic arithmetic (Carpenter, Franke, & Levi, 2003).

Specifically, we built on the growing body of research on how elementary school students think about algebra and which ideas they find accessible (Bastable & Schifter, in press; Blanton & Kaput, 2003, 2005; Carraher, Schliemann, & Brizuela, 2000; Davis, 1964; Kaput & Blanton, 2000; NCTM, 1997, 1998; National Research Council, 2001; Schifter, 1999). We also identified ideas that would be most useful to teachers during instruction—in particular, ideas that would connect with students’ current thinking and have immediate face validity to teachers (for more information, see Carpenter et al., 2003; Franke, Carpenter, & Battey, in press).

We have found what we call relational thinking to be a powerful, unifying idea for engaging teachers in conversations that support their use of algebraic reasoning with students. With respect to algebra, we characterize relational thinking as looking at expressions and equations in their entirety, noticing number relations among and within these expressions and equations. Relational thinking represents an approach to working with numbers that is different from carrying out a computational procedure in a single step-by-step sequence. For example, children can solve the number sentence $25 + 58 + 75 = \square$ by calculating from left to right, but they can solve it more easily if they take advantage of a number combination (25 + 75) they may know or be able to calculate easily. To think in this way, students must look at the original equation in its entirety to identify significant number relations before beginning to calculate and have at least implicit awareness of the commutative and associative properties. The critical issue is not that students who take advantage of these number relations have used a more efficient form of calculation. We would not consider students’ learning of a rule to simplify calculations as relational thinking. Relational thinking entails a flexible approach to calculation in which expressions are transformed on the basis of at least implicit use of fundamental properties of number operations. It involves what Vergnaud (1988) has called theorems in action.

Relational thinking represents a fundamental shift from an arithmetic focus (calculating answers) to an algebraic focus (examining relations). In the traditional study of arithmetic, the topics of addition, subtraction, multiplication, and
division generally have been portrayed as processes in which a collection of numbers is operated on in a progression of steps to generate a single number, which is the answer to the calculation. In contrast, solving algebraic equations has a different character. An algebraic equation expresses a relationship between two expressions and is solved by successive transformations of the equation; students need to understand that each legitimate transformation results in an equation that is equivalent to the first.

Relational thinking depends on critical relations that give meaning to both arithmetic and algebra. In arithmetic, however, these relations are often left unarticulated. Thus, by focusing on relational thinking, teachers can effectively integrate the idea of relations into the learning of arithmetic so that the concepts and skills that students acquire during elementary school are both enhanced and better aligned with the concepts and skills that they later need to learn algebra. For example, although students may be only implicitly aware of the distributive property, it serves as the basis for most common procedures for multiplying multi-digit numbers in arithmetic calculations (e.g., $78 \times 5 = (70 + 8) \times 5 = 70 \times 5 + 8 \times 5 = 390$) and for many student-invented addition algorithms (e.g., $57 + 68 = (5 + 6) \times 10 + 15 = 125$). The distributive property is also the basis for addition in algebraic calculations (e.g., $6y + 4y = (6 + 4)y = 10y$); developing an understanding of this relation in learning arithmetic may go a long way toward preventing common algebraic errors (e.g., $6x + 3y = 9xy$ or $5(y + 8) = 5y + 8$) (Matz, 1982).

In short, relational thinking entails an awareness of relations among numbers and the fundamental properties of number operations. Students can use relational thinking to simplify calculations, construct and learn new concepts, extend procedures to new number domains, and generally make sense of arithmetic. We do not draw strict boundaries between purely arithmetic calculations and the use of relational thinking. At the margins, distinguishing what is and what is not relational thinking is difficult, and students exhibit varying levels of sophistication in using relational thinking. Our goal is to call attention to ways of thinking about arithmetic that draw more explicitly on fundamental ideas of number operations so that students will learn arithmetic with understanding, which can then provide a basis for their ongoing learning of algebra.

One could debate whether our characterization of relational thinking in arithmetic represents a way of thinking about arithmetic that provides a foundation for learning algebra or is in itself a form of algebraic reasoning. A case could be made either way. One fundamental goal of integrating relational thinking into the elementary curriculum is to facilitate students’ transition to the formal study of algebra in the later grades so that no distinct boundary exists between arithmetic and algebra. Kaput (1998) proposed,

The key to algebra reform is to integrate algebraic reasoning across all grades and all topics—to “algebrafy” school mathematics. . . . [T]eachers need to be able to identify and nurture these roots of algebraic reasoning in forms that appear very different from what is deemed “algebra” under the auspices of “Algebra the Institution.” (pp. 25–26)
Similarly, in *Principles and Standards for School Mathematics* (NCTM, 2000), the authors asserted, “When students notice that operations seem to have particular properties, they are beginning to think algebraically” (p. 91). Our choice to characterize relational thinking as a form of algebraic reasoning is also consistent with the use of the term in current research dialogue on algebra (Chick et al., 2001; Kaput et al., in press; NCTM & MSEB, 1998).

In this study, we focused on three specific applications of relational thinking: (a) viewing the equal sign as an indicator of a relation; (b) using number relations to simplify calculations; and (c) making explicit general relations, particularly those based on the fundamental properties of number operations. We describe these applications in the next three sections.

**Viewing the Equal Sign as an Indicator of a Relation**

One application of relational thinking is viewing the equal sign as an indicator of a relationship between two expressions. Unfortunately, many students hold an alternative view in which the equal sign is a signal to carry out the calculation that precedes it and the number after the equal sign is the answer to that calculation. For example, students with this view of the equal sign may solve the equation $57 + 36 = \square + 34$ by putting 93 in the box. Although researchers have documented the prevalence of this problematic view of the equal sign (Behr, Erlwanger, & Nichols, 1980; Erlwanger & Berlanger, 1983; Kieran, 1992; Knuth, Stephens, McNeil, & Alibali, 2006), evidence exists that even young children can learn to think of the equal sign as an indicator of a relation (Carpenter & Levi, 2000; Falkner, Levi, & Carpenter, 1999; Kieran, 1981; Saenz-Ludlow & Walgamuth, 1998).

Developing a relational view of the equal sign is critical for learning algebra, and a lack of such understanding is a major stumbling block for students when they move from arithmetic to algebra (Kieran, 1981; Matz, 1982). For example, transforming equations, a key component of high school algebra, requires an understanding that adding the same number to both sides leaves the relationship between two expressions unchanged. If students do not view the equal sign as an indicator of a relation, these types of transformations make little sense and can be learned only as memorized rules (Kieran, 1992; Matz, 1982).

**Using Number Relations to Simplify Calculations**

Another application of relational thinking entails the use of number relations to simplify calculations. For example, students who view the equal sign relationally can solve the equation $57 + 36 = \square + 34$ by straightforward calculation: They calculate $57 + 36 = 93$ and then determine what number added to 34 is 93. Alternatively,
they can consider the relationship between 36 and 34, notice that 34 is 2 less than 36, and reason that the number in the box must be 2 more than 57. In both cases, students hold a relational view of the equal sign. However, the second strategy shows a level of relational thinking in which students also use number relations to simplify their calculations. Identifying when to capitalize on these relations can change potentially tedious calculations such as \(5 \times 499\) and \(1488 + 375 – 373\) into relatively trivial ones (i.e., \(5 \times 500 – 5\) and \(1488 + 2\)).

Having students know that one can simplify a calculation by changing 499 to 500 or initially subtracting 373 from 375 is not our ultimate goal. Instead, we recognize that understanding why these approaches to computation are legitimate entails understanding important ideas about number and the fundamental properties of number operations. Thinking relationally, therefore, is different from applying a collection of tricks or memorizing a set of mathematical properties. It is a way of reasoning that is not linked to particular procedures or number combinations; children who think relationally identify number relations and reason about which transformations make sense in a particular problem. To do so, they need to understand, at least implicitly, relations among numbers and the fundamental properties that can be used to justify transformations of expressions. Students who lack opportunities to explore these ideas often fail to take advantage of the structure of the number system so that the learning of both arithmetic and algebra becomes harder than it needs to be (National Research Council, 2001).

Making Explicit General Relations Based on Fundamental Properties of Number Operations

Our third application of relational thinking entails making explicit general relations that are based on the fundamental properties of number operations. Students have a great deal of implicit knowledge about fundamental number relations, but they generally have not examined these relations explicitly, thought systematically about them, or expressed them as generalizations. Bastable and Schifter (in press) have argued that, through questioning, teachers can help students explore the generality that often exists naturally in discussions of arithmetic, and researchers have provided evidence that even elementary school children are capable of making generalizations (Bastable & Schifter, in press; Blanton & Kaput, 2005; Carpenter & Levi, 2000; Carraher, Schliemann, Brizuela, & Earnest, 2006; Davis, 1964; Schifter, 1999; Tierney & Monk, in press). For example, Carpenter and Levi (2000) reported on the following conversation after asking first and second graders whether \(78 – 49 = 78\) was true or false:

*Children:* False! No, no false! No way!
*Teacher:* Why is that false?
*Jenny:* Because it is the same number as in the beginning, and you already took away some, so it would have to be lower than the number you started with.
*Mike:* Unless it was \(78 – 0 = 78\). That would be right.
*Teacher:* Is that true? Why is that true? We took something away.
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Steve: But that something is, there is nothing. Zero is nothing.

Teacher: Is that always going to work? (pp. 5–6)

After further discussion of several related number sentences, the group proposed the following rule: “Zero subtracted from another number equals that number” (p. 7). In short, the children not only used their understanding of zero to evaluate particular number sentences but also created generalizations (e.g., \( a - 0 = a \)) and discussed whether they held true for all numbers. This study and others have shown that elementary school children can express generalizations in natural language as well as learn to represent them using variables (e.g., \( s + 0 = s \), \( p + r = r + p \)) (see also Carpenter et al., 2003; Davis, 1964).

Without an understanding of generalization, students cannot appreciate the fundamental properties of number operations. Although traditional computational algorithms are based on these properties, most algorithms are designed for efficiency, and the transformations required in the calculations are typically hidden. Therefore, many students complete elementary and middle school having had little opportunity to move beyond following well-specified sequences of procedures. Students in traditional classrooms often are unaware of the fundamental properties of number operations (Chaiklin & Lesgold, 1984; Collis, 1975; Kieran, 1989), and when students do not understand how these properties are applied in their computations, they cannot recognize that arithmetic and algebra are based on the same fundamental ideas.

PROFESSIONAL DEVELOPMENT ON ALGEBRAIC REASONING

We view professional development as a way to offer teachers ongoing opportunities for learning over sustained periods of time. We also believe that professional development must be explicitly connected to teachers’ work with their students (Ball & Cohen, 1999; Darling-Hammond & McLaughlin, 1996), and we have designed our past and current professional development work accordingly. For example, we found that written student work from teachers’ own classrooms could be used effectively to structure teachers’ conversations about not only their existing practices but also possible ways to adapt their existing practices to better support the development of students’ mathematical thinking (Kazemi & Franke, 2004). As part of professional development, we have visited classrooms and used episodes from teachers’ classrooms in our group discussions. When teachers are supported in making practice public, their learning is consistently linked to their practices (Ball, 2000), and they can develop ways to learn from one another (Ashton & Webb, 1986). Our goal then is to create opportunities for teachers to learn within professional communities through activities embedded in teachers’ everyday work (Fullan, 1991; Lieberman & Miller, 1990; McLaughlin & Talbert, 1993; Secada & Adajian, 1997; Tharp & Gallimore, 1988).

In our previous studies of professional development, we have found that a focus on students’ mathematical thinking provided opportunities for teacher learning that
led to changes in classroom practice and improvement in student achievement (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Fennema et al., 1996). Furthermore, teachers who learned how to learn from the thinking of the students in their classrooms were able to continue learning and improving their practice even after the formal professional development support ended (Franke, Carpenter, Levi, & Fennema, 2001). In the current study, we drew on these research and professional development experiences to design professional development that brought teachers together regularly, within schools, to talk about understanding and supporting students’ thinking and, in particular, students’ engagement in algebraic reasoning.

It is not enough, however, to say that we focus on student thinking in our professional development. We take a particular approach to focusing on students’ mathematical thinking (Kazemi & Franke, 2004). We work with teachers to create frameworks that make explicit the similarities and differences in strategies that students use to solve problems. We attend to the detail of students’ strategies in ways that support teachers in seeing the subtle, yet mathematically relevant, differences in the understandings that students bring to their problem solutions. We highlight core mathematical ideas and discuss problems that provide insight into students’ thinking about these ideas. We ask teachers to pose problems to their students and bring the student work back to the group so that, with help from their colleagues, they can make sense of their students’ work in ways that elaborate the frameworks they are developing. In short, we discuss issues of content and pedagogy in relation to artifacts of students’ thinking and the frameworks being developed.

In the algebra work in this study, we adapted our focus on student thinking to accommodate the mathematical content itself; the teachers’ existing identities in relation to the content; and the fact that teachers were working in urban, low-performing elementary schools. The content of algebra can trigger anxiety for many teachers and students because it not only is less familiar than arithmetic but also carries a sense of who can participate (Franke et al., in press). Therefore, supporting teachers in explicitly working through the mathematical problems they were using with their students was an important aspect of our work. We also engaged teachers in telling stories about the mathematics their students could do rather than what they could not do. A prevailing story in some urban schools is one of students without adequate skills to learn higher level mathematics. We engaged teachers in counter-storytelling by helping them understand why students had difficulties with the content, what mathematics students did know, and how teachers could help students learn more mathematics by using what students did know (Decuir & Dixson, 2004; Delgado & Stefanic, 2001). By helping teachers tell counter-stories, we challenged assumptions about the students in these schools, and we addressed what teachers could do to support students’ growth. Through continually encouraging the telling of these counter-stories during the professional development, we worked to help teachers rethink and challenge traditional classroom practices.
METHODS

In our experimental study, we explored the algebraic reasoning of 180 teachers and their 3735 students from 19 elementary schools. We provided professional development to about half the teachers throughout an academic year and then collected end-of-year assessments from all teachers and their students. We compared the responses of teachers who participated in professional development with those who did not. Similarly, we compared the responses of students in classes of teachers who participated in professional development with those in classes of teachers who did not.

Participants

We worked with a large urban school district in which administrators and teachers recognized the value of engaging in algebraic reasoning in elementary school and in which long-term plans for overall school improvement were underway. The district, in its 2nd year of new leadership, also had a history of poor performance and a long-standing sense from outside that it would never do well. According to the state’s ranking system and standardized-test scores, it was one of the lowest performing school districts in California. As in many urban school districts, hiring and retaining qualified teachers was a struggle. Although the district was making progress, at the beginning of the study, only 57% of the teachers in the sample held credentials and 30% of the teachers were in their 1st or 2nd year of teaching. The community served by this district had shifted from being predominately African American to being predominately Latino, and at the time of our work, the schools served students of whom 99% were minority, 52% were classified as English Language Learners, and 93% received free or reduced-cost lunch.

In the year prior to our study, the school district had taken on mathematics reform. The state had provided professional development opportunities to support the adoption of a new mandated textbook and a corresponding pacing plan that met state standards, and the district had hired mathematics coaches to serve each of their K–12 schools. All teachers in these schools were required to administer quarterly benchmark assessments as well as state standardized tests. With this attention to mathematics, opportunities for students were on the rise, but when we began our study, only 16% of the students scored proficient or above on the state standardized mathematics test (vs. 45% statewide) (California Department of Education, 2003).

Originally 24 of the 25 schools in this district volunteered to participate in our study. We committed to work with teachers from 19 of these 24 schools because the other 5 were unique learning situations, such as charter schools. We recruited volunteer teachers in Grades 1–5 through the mathematics coach at each of the 19 schools. We then matched schools on size and teaching calendar (i.e., year-round

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2 Teachers were required to use the California edition of *Harcourt Math 2002.*
vs. traditional) and randomly selected 10 schools to participate in professional development during the study, which lasted 1 academic year. From these 10 schools, 89 participating teachers received professional development during the study. From the other 9 schools, 91 nonparticipating teachers served as a comparison group during the study and received professional development the year after the study. We collected data from teachers and students in both the participating and nonparticipating teachers’ classrooms (see Table 1).

<table>
<thead>
<tr>
<th>Table 1</th>
<th>Number of Participants by Grade Level</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Grade 1</td>
</tr>
<tr>
<td>Participating teachers</td>
<td>15</td>
</tr>
<tr>
<td>(Number of students)</td>
<td>(269)</td>
</tr>
<tr>
<td>Nonparticipating teachers</td>
<td>21.5</td>
</tr>
<tr>
<td>(Number of students)</td>
<td>(375)</td>
</tr>
</tbody>
</table>

Note. Teachers who taught combination classes with students at two grade levels are represented as 0.5 at each grade level.

Professional Development

Participating teachers engaged in professional development to explore the development of students’ algebraic reasoning and, in particular, how that reasoning could support students’ understanding of arithmetic. The professional development included both school-based work-group meetings and on-site support, each described below.

Work-Group Meetings

We met with the participating teachers at each of the 10 schools eight times, approximately once a month, in after-school work-group meetings. Each meeting, held on site at the participating school, was about 1.5 hours in length. At each school, we worked with 4–25 teachers representing Grades 1–5 and the school’s mathematics coach.3 Initially, teachers also participated in a 4.5-hour Saturday meeting in which teachers came together across schools. The professional development was led by an experienced professional development facilitator—one of the authors or a graduate student. In the next sections, we elaborate on how the content, structure, and practices of our professional development supported our focus on students’ development of algebraic reasoning and the teacher’s role in promoting that development.

3 Kindergarten teachers also participated in the work-group meetings, but we did not assess teachers or students at the kindergarten level.
**Professional Development Content**

Our content, drawn from *Thinking Mathematically: Integrating Arithmetic and Algebra in the Elementary School* (Carpenter et al., 2003), highlighted relational thinking, including (a) understanding the equal sign as an indicator of a relation; (b) using number relations to simplify calculations; and (c) generating, representing, and justifying conjectures about fundamental properties of number operations.

Each work-group meeting was planned to focus on one particular algebraic idea, but ideas from earlier sessions were regularly revisited. We began the year focusing on understanding the equal sign and using number relations to simplify calculations, and a substantial amount of time was devoted to addressing these two topics. We found that extended attention to a few topics provided a useful context for highlighting and exploring some of our underlying themes: eliciting, valuing, and building on student thinking; creating conversations around students’ ideas; and using arithmetic to develop algebraic ideas. Therefore, although generating, representing, and justifying conjectures were the main focus for at least two of the work-group meetings, we spent relatively less time on these topics than on understanding the equal sign and using number relations to simplify calculations.

**Professional Development Structure**

We structured each of the school-based meetings similarly in an attempt to provide a consistent set of opportunities across work groups. In any given week, however, the meetings at each site dealt with slightly different content because the work groups were designed to be responsive to the needs and contributions of the specific participants. In general, we provided opportunities for teachers to (a) talk about how previously discussed algebraic ideas were playing out in their practices, (b) explore that meeting’s target algebraic idea in terms of both the mathematics and student thinking, (c) engage in conversations about how to make sense of that algebraic idea in practice, and (d) leave with a task to try with their students so that those classroom experiences could serve as a source of conversation in upcoming work-group meetings and in informal discussions among teachers between meetings.

Central to our professional development work were these tasks that were generated in the work-group meetings, used in teachers’ classrooms, and then brought back to work-group meetings in the form of written student work and stories about classroom interactions. The mathematical goals of these tasks included understanding the equal sign, using number relations to simplify calculations, and seeding conversations to make explicit fundamental properties of number operations as well as solving typical algebraic reasoning problems involving multiple or repeated variables. Although the mathematical goals were selected by the facilitators, the teachers, as a group and individually, generally generated the particular tasks. For example, when the facilitators selected making commutativity explicit as a mathematical goal, the teachers worked to generate specific tasks (e.g., true/false number
sentences, open number sentences, or story problems) that they could use to engage their students with this mathematical idea. Commutativity of addition and multiplication were both discussed, and each teacher determined which operation was more appropriate for his or her students. Using this approach to task generation meant that teachers engaged students in tasks that had enough commonalities to provide a common foundation for future work-group conversations but, at the same time, had enough flexibility for teachers to customize (and recognize the importance of customizing) tasks for their particular group of students. In all cases, the teachers chose how to engage with their students around the tasks, and we asked them to attend to their students’ thinking and the types of conversations that arose.

Regular work-group meetings provided opportunities to engage teachers, mathematics coaches, and professional development facilitators in ongoing learning and to create a community of learners in which all participants supported one another’s learning. As such, each work-group meeting evolved differently and the conversations at the school sites varied, but we worked to hold consistent the types of artifacts used, the mathematics highlighted, and the student thinking addressed.

**Professional Development Practices**

To provide a sense of the ways we engaged with teachers, we highlight three practices that became central to our professional development work. Specifically, we address how we engaged teachers in thinking about students’ thinking, how we supported the development of mathematical conversations, and how we helped teachers notice opportunities for extending arithmetic into algebraic reasoning.

**Student thinking.** Students’ mathematical thinking served as the focus for our interactions in professional development. Our conversations were informed by research on children’s algebraic reasoning, but our goal was not to provide teachers with a completed framework that encapsulated children’s algebraic reasoning. Instead, we supported teachers in making sense of the approaches that students took, the ways these approaches were connected to important mathematical ideas and to one another, and their relationship to the task posed. In each meeting, we asked teachers to generate a range of potential student responses for algebraic problems, discuss what these responses indicated about students’ understandings, and consider how these understandings might be related to other previously discussed aspects of students’ algebraic reasoning.

We specifically worked to focus teachers’ attention on what their students could do while they engaged in algebraic reasoning rather than on what they could not do. We began each work-group meeting by having teachers share stories from their classes. Our discussions celebrated students’ ideas and helped teachers recast students’ struggles by identifying what students did understand and what types of questions teachers could pose to elicit and build on this understanding. Thus, by making teachers’ practice explicit, we worked to help teachers create counter-stories to commonly encountered deficit stories about students, especially low-income students of color (Decuir & Dixson, 2004; Delgado & Stefancic, 2001). These
counter-stories focused on students’ successfully engaging in mathematics, in particular, in algebra.

Our student-thinking work played out initially and quite significantly regarding understanding of the equal sign. For example, we posed a problem like $8 + 4 = \Box + 5$ and asked teachers to generate the range of strategies that students might use to solve the problem. Teachers detailed incorrect solutions such as 12 (ignoring the 5) and 17 (adding all the numbers). They also discussed ways students could arrive at the correct solution of 7. For instance, students could add 8 and 4 and then determine what number added to 5 equals 12, or they could look at the relationship between 4 and 5 and reason that the number in the box must be 1 less than 8. We asked teachers to talk about how these approaches were similar and different and what each might indicate about a child’s understanding of the equal sign. Our goal was to help teachers create for themselves organized ways of understanding and connecting student responses. As teachers noticed more solutions over the year, we continued to support them in ways that helped them develop notions of themselves as individuals and members of communities who could detail solutions, organize knowledge of students’ thinking, and use that information to guide instruction (Franke et al., 2001; Franke, Carpenter, Fennema, Ansell, & Behrend, 1998).

Mathematical conversations. The algebraic-reasoning work depended not only on teachers’ encouraging students to solve problems in their own ways but also on teachers’ engaging their students in conversations to help them explicate their thinking and debate their reasons for thinking as they did. Teachers needed to see their students’ participation in these types of mathematical conversations as feasible and the conversations as valuable, and they had to learn to lead them. For many teachers, thinking about how to both seed and orchestrate conversations was challenging. Within the work groups, then, teachers were able to gain a sense of possibility—a vision for what was possible and details of supporting practices. We engaged teachers in creating tasks that could initiate conversations, viewing conversations on videotape, and discussing what to notice and what questions to ask to move conversations forward. Throughout the year we used true/false number sentences and open number sentences as contexts in which teachers could base attempts to seed and orchestrate conversations with students (Carpenter et al., 2003; Davis, 1964). We addressed the range of student responses to both particular number sentences and sequences of number sentences, and we explored ways to orchestrate discussions around those tasks.

Principles underlying our work were that each task posed should challenge students’ conceptions and that discussion was essential for highlighting how newly expressed ideas were consistent or inconsistent with previously held ideas. Thus, we worked to help teachers develop the expertise necessary to construct number-sentence sequences, on the spot, while simultaneously facilitating class discussions. Our goal was not to provide teachers a series of ideal tasks to use in their classrooms but to help teachers develop the expertise necessary to construct or adapt tasks to be appropriate for their own students.
To support this goal, we helped teachers create a classroom tool: true/false-number-sentence and open-number-sentence index cards. During the work-group meetings, teachers worked both individually and as a group to create these cards, writing one number sentence per card. Each teacher left the meetings with a personalized set of cards reflecting the ideas and number sizes most important for his or her students to explore. The index cards supported teacher learning about student thinking and the mathematics inherent in the number sentences themselves. We also capitalized on the index-card format to encourage teachers to think of the algebraic-reasoning work as engaging in conversations with students rather than as solving a series of problems on a worksheet. Specifically, we discussed various sequences of number sentences related to specific student solutions and understandings with the intent that teachers would think about subsequent number sentences that could challenge their students’ understandings. The idea was that teachers could use their index cards in their classrooms by presenting an initial number sentence, listening to their students’ thinking, shuffling through their cards to determine an appropriate follow-up number sentence, and repeating the process. Throughout the year, teachers brought their cards and their new understandings back to the work-group meetings for ongoing conversation and elaboration of their index-card sets. These types of explicit connections between the teachers’ classroom experiences and the work-group-meeting discussions supported teachers while they worked on making sense of the algebraic-reasoning ideas in the context of their ongoing practices.

Noticing opportunities to extend arithmetic. The algebraic-reasoning work also depended on teachers’ developing “algebra eyes and ears” (Blanton & Kaput, 2003, 2005) for noticing opportunities to extend their ongoing arithmetic work to include algebraic reasoning. These opportunities arose when teachers attended to the curriculum (see Kaput et al., in press) or to the conversations that occurred around mathematics (e.g., when a student described a strategy that implicitly used ideas of commutativity, an opening arose to make that property explicit). To take advantage of these opportunities, most teachers needed to learn to notice different aspects of mathematics and student thinking than they had noticed in their typical practices.

We used our on-site support visits to watch for opportunities for extending arithmetic, and, in particular, we helped teachers to identify the relational thinking embedded in students’ strategies and to consider how different students’ strategies made use of different types of relational thinking. We then brought the curriculum examples and conversation openings we had noticed in classrooms back to the work-group meetings. We discussed both the reasons these examples might be opportunities to extend arithmetic and the range of approaches a teacher might follow to take advantage of these opportunities. We also brainstormed similar opportunities that might arise in teachers’ practices. Through such practices, we again focused on what was already occurring and then created visions for what was possible. By helping teachers see how often opportunities naturally arose in their own classrooms, we supported the traveling of professional development ideas from work-group meetings to classrooms and back again.
On-Site Support

In addition to interacting during work-group meetings, teachers and professional development facilitators worked together at school sites. At each participating school, the project staff member who facilitated the professional development spent about 1/2 day per week on site. Our goal during this on-site time was to engage with the teachers informally, to support teachers in their attempts to engage in algebraic reasoning in their classrooms, to learn more about the teachers’ and students’ successes and struggles, and to explicitly draw from the teachers’ practices when designing work-group meetings. Our visibility at the school site was important, both to remind teachers to explore algebraic reasoning with students and to highlight our desire to help teachers make sense of what they were learning in the work-group meetings. Thus, in a sense, our presence was less about helping teachers with specific classroom lessons and more about building relationships, serving as a reminder to engage in algebra, and collecting authentic scenarios to use as sites of conversation in upcoming work-group meetings.

Measures

During Spring 2003, we assessed both teacher and student learning in relation to the algebraic-reasoning content addressed in the professional development. We created many of the teacher items to parallel the student items. For that reason, we describe the student-achievement measures before the teacher-learning measures.

Student Measures

Student achievement was assessed during the last 3 weeks of the school year in both participating and nonparticipating teachers’ classes in Grades 1–5. The project staff administered written mathematics tests in whole-class settings and conducted individual interviews focused on students’ use of relational thinking.

Written Mathematics Tests

We constructed three written mathematics tests: one for first grade, one for second and third grades, and one for fourth and fifth grades. The three tests assessed similar constructs, but the items generally used larger numbers in the upper grades, and some constructs were not assessed in the lower grades. Each test had two to four subtests (see Table 2 for a summary), and items were scored as correct or incorrect.

Equality subtest. This 4-item subtest was designed to assess whether students held a relational view of the equal sign. Students were asked to solve the following four open number sentences: \( 3 + 4 = \square + 5; 5 + \square = 6 + 2; \square + 2 = 6 + 1; 3 + 6 = \square + 8. \) We constructed these items to assess whether students held the misconception that the equal sign was a signal to carry out the preceding calculation; in each number sentence, the term after the equal sign could represent the sum of the two numbers before the equal sign. Only in this subtest were the same items used for all grade levels.
Targeted-Computation subtest. This subtest was designed to assess students’ abilities to identify and use number relations to simplify calculations. Number sizes were varied to be grade-level appropriate, and we designed items at the boundary of some students’ computational skills. Our reasoning was that these items would be easier for students who were able to think relationally than for those who carried out all calculations.

This subtest had three categories of items. First, students in all grades were asked to solve equations associated with the idea that subtracting a number from itself yields zero. For example, although students could solve equations such as $25 + 59 - 59 =$ or $54 + 37 - 36 =$ by calculating from left to right, their calculations were simplified if they first recognized that $59 - 59 = 0$ or $37 - 36 = 1$.

Second, students in Grades 2–5 were asked to solve equations in which calculations could be simplified by regrouping (on the basis of the commutative and associative properties) or compensating (adjusting numbers). For example, students could more easily solve a problem such as $98 + 69 + 2 =$ by first rearranging the terms to calculate $98 + 2 = 100$. Similarly, they could more easily solve a problem such as $999 + 999 =$ by computing $1000 + 1000 - 2$ after recognizing that $999$ is only 1 less than $1000$.

Third, students in Grades 4–5 were asked to solve equations in which the use of the distributive property could simplify calculations. For example, students who recognized the distributive property could easily solve $(9 \times 57) + 57 =$ by computing $10 \times 57$.

Solving-Equations subtest. This 3-item subtest was also designed to assess students’ abilities to use relations to simplify calculations, but this subtest involved equations in which letters were used as unknowns. It was administered in Grades 2–5, and equation difficulty was varied to be grade-level appropriate. The second and third graders were asked to solve for the unknown in $c + c + 4 = 16$; $y + y - 3 = 11$; and $n + n + n + 2 = 17$. The fourth and fifth graders were asked to solve for the unknown in $y + y - 3 = 11$; $3 \times c + 5 = 23$; and $n + n + n = n + 12$. We selected
these traditional algebraic tasks because, although students were never taught standard solution methods, by thinking relationally, they could transform the equations to simpler number sentences. For example, students could recognize that $y + y$ must be 3 more than 11 and therefore transform the equation $y + y - 3 = 11$ to $y + y = 14$.

**Generalization subtest.** This subtest was designed to assess students’ abilities to identify generalizations about fundamental properties of number operations that were symbolically represented as equations involving letters used as variables. Specifically, we asked students to identify whether a particular number sentence was “always true” or “not always true” for all replacements of the variables. For each property assessed, we wrote a number sentence that was always true and one that had some features of the property but was not always true. For example, to assess the commutative property of addition, we asked students to evaluate $c + b = b + c$ and $a - b = b - a$. Students needed to correctly evaluate the former as always true and the latter as not always true to receive a correct score for commutativity of addition.

This subtest was administered in Grades 2–5, and each student evaluated three properties (six number sentences). All students considered the commutative property for addition ($c + b = b + c$) and the identity property for addition ($0 + b = b$). In Grades 2 and 3, students also considered the idea that subtracting a number from itself yields zero ($c - c = 0$), and in Grades 4 and 5, students evaluated the identity property for multiplication ($1 \times c = c$).

**Relational-Thinking Interviews**

The written mathematics test was designed to provide information about how successful students were in solving various types of problems but not to reveal how students solved these problems. Because we also wanted to know whether students used relational thinking when solving open number sentences, we interviewed a sample of individual students at Grades 1, 3, and 5. We randomly selected 3 boys and 3 girls\(^4\) from 108 classes for a total of 618 students (300 from 51 participating teachers’ classrooms and 318 from 57 nonparticipating teachers’ classrooms).

The interviews were designed to assess students’ relational thinking, including whether they understood the equal sign and looked for number relations to simplify their calculations. At each grade level, we designed the items to assess similar constructs, but we used larger numbers in the upper grades. Through these items, we addressed a subset of the number relations explored in the Equality and Targeted-Computation subtests of the written mathematics test (see Table 3).

We used a structured-interview protocol in which we presented each problem and then elicited the student’s strategy. If a student did not take advantage of number relations to simplify the calculations, we provided a prompt to encourage relational thinking. For example, in Grade 1, for the problem $19 + 6 - 6 = \square$, we looked for

\(^4\) Multigrade classes sometimes made it impossible to achieve this exact boy/girl split from the target grade level, but at least 2 boys and 2 girls from each of the 108 classes were interviewed.
evidence that the student used the idea that \(6 - 6 = 0\). If we did not see evidence of this relational thinking, we asked, “Is there a way to solve this problem without having to add \(19 + 6\)?” Similarly, in Grade 5, for the problem \(37 + 54 = \square + 55\), we looked for evidence that the student used the relationship between 54 and 55 to obviate the need for computing 37 + 54. If we did not see evidence of this relational thinking, we asked, “Is there a way to solve this problem without having to add 37 + 54?” After each problem, we recorded whether the student used relational thinking either before or after the prompt. Because relational thinking is not tied to particular strategies or problems, we chose to look at each student’s responses across problems. Specifically, students scored 1 if they used relational thinking at least once during the interview, and 0 if they did not.

### Teacher Measures

Teachers completed a survey, a written mathematics test, and an interview about their knowledge of students’ thinking. All measures were administered individually. We addressed similar mathematical content in the student and teacher measures so that we could consider teachers’ knowledge as it directly related to their students’ achievement on the same content.

### Survey of Students’ Experience With Algebraic-Reasoning Tasks

We asked teachers to identify (on a 4-point scale) the amount of experience that students in their classes had with representative items from each subtest of the written mathematics test appropriate for students at their grade level. For example, Grade 1 teachers were asked to rank the experience their students had with the problem \(3 + 4 = \square + 5\) (representing the Equality subtest) and \(23 + 17 - 17 = \square\) (representing the Targeted-Computation subtest). A single item was used to represent each subtest with the exception of the Targeted-Computation subtest, which was represented by 1–3 items reflecting the number of item categories in this subtest at a particular grade level. The link between items and subtests was not made explicit for teachers because our goal was to better understand the teachers’ views of the importance of the mathematical tasks and their role in the curriculum independent of how we used these tasks.
Written Mathematics Test

We asked teachers to complete the fourth- and fifth-grade version of the Solving Equations and Generalization subtests. We adopted the same scoring procedures used for the student measures.

Knowledge-of-Student-Thinking Interview

On the basis of the same reasoning underlying the design of our student measures, we recognized that written mathematics tests were insufficient for fully revealing teachers’ knowledge. Therefore, we interviewed teachers to assess their knowledge of students’ strategies and relational thinking. Specifically, we asked teachers to generate strategies students might use to solve five open number sentences (see Table 4).

Table 4
Interview Items to Assess Teachers’ Knowledge of Student Thinking

<table>
<thead>
<tr>
<th>Item</th>
<th>Open Number Sentences</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>8 + 15 = □ + 16 (Grade 1)</td>
</tr>
<tr>
<td></td>
<td>13 + 18 = □ + 19 (Grades 2 and 3)</td>
</tr>
<tr>
<td></td>
<td>37 + 54 = □ + 55 (Grades 4 and 5)</td>
</tr>
<tr>
<td>2</td>
<td>54 + 37 – 36 = □</td>
</tr>
<tr>
<td>3</td>
<td>203 – 105 = □</td>
</tr>
<tr>
<td>4</td>
<td>(9 × 57) + 57 = □</td>
</tr>
<tr>
<td>5</td>
<td>46 + 27 = □ + 28</td>
</tr>
</tbody>
</table>

We used a structured-interview protocol similar to that used in the student interviews. First, we assessed teachers’ knowledge of students’ strategies with Item 1. Teachers were asked to generate as many strategies as they could for an open number sentence that was selected from the student-interview problems appropriate for their grade level. We encouraged strategies reflecting both correct and incorrect conceptions of the equal sign, and we tracked the total number of distinct strategies that each teacher generated.

Second, we assessed teachers’ knowledge of relational thinking by examining the student strategies they generated across all five items. The teachers scored 0 (no strategy reflecting relational thinking) or 1 (at least one strategy reflecting relational thinking) for each item, so the range for their total scores was 0–5. We designed these five items to provide three types of opportunities for the teachers to use relational thinking. For Item 1, as previously described, we asked teachers to generate as many strategies as they could. For Items 2–4, we asked teachers to describe one strategy that students might use to solve each of the open number sentences, and if teachers initially generated a strategy that did not use relational thinking, we provided a general prompt by asking for other strategies that could make the computation easier. These three items were selected from the Targeted-Computation subtest for fourth and fifth graders. For Item 5, we told teachers that
a student gave the correct answer of 45 without doing any computation, and we asked them to generate a strategy that the student might have used. This item was designed to assess whether teachers could generate a strategy that was based on relational thinking when we directly suggested the existence of this type of reasoning.

Analyses

In analyzing the teacher data, we compared the performance of participating and nonparticipating teachers. We treated the teachers as a single multigrade group; the professional development occurred in multigrade work groups, and teachers, regardless of grade level, responded to the same or similar items. One exception was the survey about students’ experiences with algebraic-reasoning tasks. For this measure, we conducted separate analyses for each grade-level group (Grade 1, Grades 2 and 3 combined, and Grades 4 and 5 combined) because teachers in these groups answered different questions corresponding to the grade-group-specific questions on the student written mathematics test. To analyze the teacher data, we used two-tailed, independent-sample $t$ tests except when analyzing the number of strategies generated by teachers in the interview. Because these data were ordered and categorical, we used an ordered logistic regression.

In analyzing the student data, we compared the performance of students in the participating and nonparticipating teachers’ classes. Using class as the unit of analysis, we conducted a series of independent-sample $t$ tests for each subtest of the written mathematics test. To match the written-test design, we conducted separate analyses for each grade-level group (Grade 1, Grades 2 and 3 combined, and Grades 4 and 5 combined). For the student-interview data, we identified whether each student used relational thinking at any point during the interview and then used logistic regression to determine whether students in the participating or nonparticipating teachers’ classes were more likely to use relational thinking. We conducted separate analyses at Grades 1, 3, and 5 because students at these grade levels responded to different interview items. For all teacher and student tests, we used a Type I error rate of .05.

FINDINGS

We collected data on 180 teachers and 3735 students, and we found that the professional development had a positive effect on both teacher learning and student achievement.\footnote{We had missing data on some measures, so the specific number of participants is identified for each analysis.} We present our findings from the teacher data prior to describing the student-achievement results.
Developing Children’s Algebraic Reasoning

Teachers’ Reporting of Students’ Experience With Algebraic-Reasoning Tasks

We found no significant differences between the participating and nonparticipating teachers’ reports of the amount of experience their students had on the types of tasks that appeared on each subtest of our student written mathematics tests; the responses for the participating and nonparticipating teachers were almost identical for each item. Thus the participating teachers did not perceive that they spent more time on relevant algebraic-reasoning tasks than did the nonparticipating teachers. Table 5 provides a summary, by grade-level group, of the teachers’ reports of students’ experience with tasks related to each written subtest. When we used more than one item to represent a subtest (i.e., Targeted-Computation subtest for Grades 2–5), we averaged the amount of experience across items.

Table 5
Teachers’ Reporting of Students’ Experiences on Algebraic-Reasoning Tasks Related to the Written Mathematics Test

<table>
<thead>
<tr>
<th>Subtest</th>
<th>Equality</th>
<th>Targeted Computation</th>
<th>Solving Equations</th>
<th>Generalization</th>
</tr>
</thead>
<tbody>
<tr>
<td>Group</td>
<td>n</td>
<td>M (SD)</td>
<td>M (SD)</td>
<td>M (SD)</td>
</tr>
<tr>
<td>Grade 1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Participating</td>
<td>14</td>
<td>1.79 (0.89)</td>
<td>0.64 (0.84)</td>
<td>—</td>
</tr>
<tr>
<td>Nonparticipating</td>
<td>19</td>
<td>1.47 (0.77)</td>
<td>0.47 (0.84)</td>
<td>—</td>
</tr>
<tr>
<td>Grades 2 and 3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Participating</td>
<td>39</td>
<td>2.26 (0.82)</td>
<td>1.67 (0.85)</td>
<td>0.64 (0.74)</td>
</tr>
<tr>
<td>Nonparticipating</td>
<td>31</td>
<td>2.26 (0.82)</td>
<td>1.74 (0.77)</td>
<td>0.52 (0.72)</td>
</tr>
<tr>
<td>Grades 4 and 5</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Participating</td>
<td>12</td>
<td>2.58 (0.67)</td>
<td>2.33 (0.38)</td>
<td>1.25 (0.87)</td>
</tr>
<tr>
<td>Nonparticipating</td>
<td>15</td>
<td>2.33 (0.72)</td>
<td>1.02 (0.54)</td>
<td>1.40 (0.74)</td>
</tr>
</tbody>
</table>

Note. The scale ranged from 0 (no experience) to 3 (a lot of experience).

Teachers’ Knowledge on the Written Mathematics Test

We compared the performance of 163 teachers (81 participating and 82 nonparticipating) on the two subtests of the written mathematics test. We found no significant differences between the participating teachers’ ($M = 2.23, SD = 1.19$) and nonparticipating teachers’ ($M = 2.14, SD = 1.16$) knowledge as measured by traditional algebra tasks on the Solving-Equations subtest (maximum score = 3). The

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6 We recognize that these data are based on teachers’ perceptions, and, therefore, we do not make claims about teachers’ actual practices.

7 There were 25 teachers (12 participating and 13 nonparticipating) who left all problems for this subtest blank. We scored these items as incorrect even though whether the teachers did not know the answers or chose to skip them for other reasons is unclear. These means may, therefore, be overly conservative.
Generalization subtest (maximum score = 3) assessed teachers’ knowledge of the fundamental properties of number operations, and again results showed no significant differences between the participating teachers ($M = 2.62$, $SD = 0.75$) and nonparticipating teachers ($M = 2.57$, $SD = 0.74$).

**Interview Data on Teachers’ Knowledge of Students’ Thinking**

Although the two groups of teachers did not significantly differ in their performance on the written mathematics test, a different picture emerged in the interviews. When teachers were asked to generate strategies that students might use to solve open number sentences, participating teachers demonstrated more knowledge than nonparticipating teachers. First, we coded the number of distinct strategies that teachers generated for an open number sentence of the form $8 + 4 = \square + 5$ (see Problem 1 in Table 4 for the specific problem presented at each grade level). We grouped the teachers’ responses into four categories: a single strategy, two strategies, three strategies, or four or more strategies (see Table 6). For this analysis, we used interview data from 170 teachers (84 participating and 86 nonparticipating).

Only 12% of the single-strategy responses came from participating teachers. In addition, 48% of the two-strategies responses, 68% of the three-strategies responses, and 90% of the four-or-more-strategies responses were provided by participating teachers. The inverse pattern then existed for the nonparticipating teachers, with 88% of single-strategy responses and only 10% of the four-or-more-strategies responses coming from nonparticipating teachers.

<table>
<thead>
<tr>
<th>Number of strategies generated</th>
<th>Participating teachers</th>
<th>Nonparticipating teachers</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$n$</td>
<td>%</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>6.0</td>
</tr>
<tr>
<td>2</td>
<td>32</td>
<td>38.1</td>
</tr>
<tr>
<td>3</td>
<td>21</td>
<td>25.0</td>
</tr>
<tr>
<td>4 or more</td>
<td>26</td>
<td>31.0</td>
</tr>
</tbody>
</table>

To compare the participating and nonparticipating teachers on the number of strategies generated, we conducted an ordered logistic regression to take advantage of the sequential nature of the data by simultaneously comparing the two groups of teachers on all the adjacent-category pairs of the number of strategies generated.\(^8\) We found that participation in the professional development was

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8 In using ordered logistic regression, one assumes proportional odds across response categories. A likelihood-ratio test of the proportionality of odds across categories was nonsignificant ($\chi^2 = 2.23$, $p > .32$), indicating that the assumption was met.
significantly related to the number of strategies generated \((z = 6.62, p < .05; R^2 = .27)\). For each comparison of adjacent categories of the number of strategies generated, participating teachers were 9.1 times as likely as nonparticipating teachers to provide the higher number of strategies; participating teachers were 9.1 times as likely as nonparticipating teachers to provide two strategies compared to one strategy, three strategies compared to two strategies, and four or more strategies compared to three strategies.

Second, we used a two-tailed, independent-sample \(t\) test to compare the participating and nonparticipating teachers’ use of strategies reflecting relational thinking across the five problems (maximum score = 5). For this analysis, we considered interview data from 162 teachers (79 participating and 83 nonparticipating) and found that significantly more strategies reflecting relational thinking were generated by participating teachers \((M = 3.68, SD = 1.22)\) than by nonparticipating teachers \((M = 2.14, SD = 1.40)\), \(t (160) = 7.46, p < .05\), and the effect size was 1.18. Although almost all teachers (100% of participating teachers and 89% of nonparticipating teachers) were able to use a strategy that reflected relational thinking on at least one problem, 67% of participating teachers (compared to 19% of nonparticipating teachers) generated a strategy that reflected relational thinking on at least 4 of the 5 problems.

Students’ Achievement on the Written Mathematics Test

For the students’ written data, we considered class to be the unit of analysis, and we conducted separate analyses for Grade 1, Grades 2 and 3 combined, and Grades 4 and 5 combined. On the Equality subtest, students in participating teachers’ classrooms scored significantly better than students in classrooms of nonparticipating teachers (see Table 7). Effect sizes for all three grade-level

<table>
<thead>
<tr>
<th>Grade</th>
<th>Participating classes</th>
<th>Nonparticipating classes</th>
<th>Effect size</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(n)</td>
<td>(M)</td>
<td>(SD)</td>
</tr>
<tr>
<td>1</td>
<td>15</td>
<td>0.42*</td>
<td>0.44</td>
</tr>
<tr>
<td>2 and 3</td>
<td>42</td>
<td>1.49*</td>
<td>0.86</td>
</tr>
<tr>
<td>4 and 5</td>
<td>32</td>
<td>2.39*</td>
<td>0.82</td>
</tr>
</tbody>
</table>

Note. Maximum score = 4, and the same items were used at each grade level.

* One nonparticipating teacher was a multigrade teacher, so her students’ data were split between the first- and second-grade data sets according to each student’s grade level.

* Two-tailed \(t\) test significant at .05 level.

\(^9\) Using simulations, Hagel and Mitchell (1992) found that for ordinal outcomes McKelvey and Zavonia’s (1975) \(R^2\) most closely approximates the \(R^2\) obtained by estimating the linear regression model on the underlying latent variable.
groups were above three fourths of a standard deviation. We note, however, that the highest group mean was 2.39 (on a 4-point scale), and thus students in both participating and nonparticipating classes still needed opportunities to make additional progress.

Significant differences also appeared for scores on the Targeted-Computation subtest, but only at Grade 1; on a 4-item scale, students from participating teachers’ classes ($M = 1.11$, $SD = 0.57$) scored significantly better than students from nonparticipating teachers’ classes ($M = 0.73$, $SD = 0.42$), $t(35) = 2.35$, $p < .05$, and the effect size was 0.79.

Our analyses of the Solving-Equations and Generalization subtests yielded no significant differences between students in participating and nonparticipating teachers’ classes. We also noted that students did not perform at ceiling level in either group, and reliability for these subtests was low (ranging, across grade-level groups, from 0.58–0.76 for the Solving-Equations subtest and 0.33–0.35 for the Generalization subtest).

Students’ Use of Relational Thinking in the Interviews

We categorized students according to whether they used relational thinking at least once during the interview (see Table 8). First graders had two opportunities to use relational thinking, and third and fifth graders each had three opportunities. We used a logistic regression to determine whether the numbers of students who could use relational thinking in the interview differed across the two groups. The differences were not significant at first grade ($\chi^2 = 3.73$, $p > .05$); students in the participating teachers’ classes were not significantly more likely than students of nonparticipating teachers to use relational thinking in first grade. However, the groups did differ significantly in third grade ($\chi^2 = 7.70$, $p < .05$) and in fifth grade ($\chi^2 = 3.84$, $p = .05$). Third-grade students in participating teachers’ classes were 2.0 times as likely as students in nonparticipating teachers’ classes to use relational thinking. Similarly, fifth-grade students in participating teachers’ classes were 1.9 times as likely as students in nonparticipating teachers’ classes to use relational thinking.

Table 8
Percentages of Students Using Relational Thinking at Least Once During the Interview

<table>
<thead>
<tr>
<th>Grade</th>
<th>Participating classes</th>
<th>Nonparticipating classes</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>n</td>
<td>Used relational thinking</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(%)</td>
</tr>
<tr>
<td>1</td>
<td>83</td>
<td>31.3</td>
</tr>
<tr>
<td>3</td>
<td>139</td>
<td>49.3*</td>
</tr>
<tr>
<td>5</td>
<td>75</td>
<td>70.6*</td>
</tr>
</tbody>
</table>

* Logistic regression significant at .05 level.
DISCUSSION

For 1 year, 89 teachers from an urban, low-performing school district engaged in professional development focused on algebraic reasoning. Through an experimental study, we found that the professional development made a difference for both teachers and their students. Our work contributes to the limited research on professional development connecting teacher and student learning (National Research Council, 2001; Sykes, 1999; Wilson & Berne, 1999), and although further research is needed to document specific changes in classroom practice, these results are particularly encouraging, given the following factors. First, 1 year represents just a beginning for teachers working to change their practice in fundamental ways. Second, research has shown that increases in student achievement generally lag behind teachers’ development (Fennema et al., 1996), and, thus, even when changes occur in the classroom, they are not always reflected in student assessments during the 1st year of professional development. Finally, we chose to work in an urban educational environment where the prevailing viewpoint is that professional development and students are likely to fail.

Our study also provides further evidence that professional development focused on students’ mathematical thinking is productive for teachers and their students (see also Carpenter et al., 1989; Fennema et al., 1996). In this study, as well as in our previous work, we view a focus on student thinking as more than noticing what students do when they solve a variety of problems (see also Little, 2004), and this study helps us to further unpack what a focus on student thinking entails. Unpacking notions of teachers’ attention to student thinking can also contribute to current efforts to better understand the knowledge needed for teaching (Ball & Cohen, 1999; Ball, Phelps, & Thames, 2005).

In this study, we worked to help teachers develop an understanding of students’ algebraic reasoning in a way that allowed them to make sense of the mathematics and the students’ perspectives in relation to one another. Teachers were able to generate multiple strategies for arithmetic problems in ways that reflected underlying mathematical ideas about number operations as well as the ways students might construct their understandings of these mathematical ideas. Thus, attending to student thinking involved more than knowing traditional mathematical content, more than appreciating the existence of multiple strategies, and more than being able to repeat what children said in solving problems; teachers needed to be able to differentiate the strategies children used in relation to specific mathematical ideas.

This intertwining of student thinking and underlying mathematical ideas is a distinctive feature of our work. Our analysis of student thinking is always grounded in a structured system of mathematical ideas, and it is this analysis that gives coherence to the mathematics that teachers learn by attending to student thinking. Part of our goal in helping teachers attend to student thinking is to highlight how these ideas are structured, but we also work to help teachers appreciate students’ learning trajectories and the tasks, interactions, and classroom norms that provide
a window on and support the development of students’ thinking within a particular mathematical domain.

In this study, our focus on attending to children’s algebraic reasoning led to significant differences in both teachers’ knowledge and students’ achievement. Specifically, participating teachers, although not differing from nonparticipating teachers on a traditional content measure, were able to generate a significantly greater number of strategies as well as more strategies that reflected the use of relational thinking than were nonparticipating teachers. Similarly, students in participating teachers’ classes were more likely than students in nonparticipating teachers’ classes to generate strategies reflecting relational thinking, and although we did not find significant differences across all subtests on our written assessment, students in participating teachers’ classes did demonstrate a significantly stronger understanding of the equal sign than did students in nonparticipating teachers’ classes.

In reflecting on why our strongest results for students occurred with students’ understanding of the equal sign, we found that discussions about the equal sign provided a mathematical and pedagogical foundation for teacher learning in our study. Mathematically, holding a relational view of the equal sign is essential for our other applications of relational thinking (i.e., simplifying calculations and generalizing). Pedagogically, we found that discussions about the equal sign provided a good starting point for developing classroom practices to support mathematical discussions essential for building algebraic reasoning. As in our previous professional development work, we found that many teachers had no history of engaging their students in mathematical discussions, and we wanted to help them learn to elicit and build on their students’ thinking. Discussions about the equal sign were a good starting point because teachers were motivated to discuss students’ views of the equal sign when they discovered that their students lacked understanding of a symbol used since kindergarten. Furthermore, true/false and open number sentences proved to be useful tools for seeding discussions about the equal sign, giving teachers a window into students’ conceptions, and creating ways to challenge those conceptions.

Productive discussions about the equal sign, and algebraic reasoning in general, require a particular type of problem posing, one in which comparing and contrasting tasks are critical. A key idea is that the number sentence one poses should challenge students’ ideas, and although many number sentences and number-sentence sequences can be effective, teachers must think about a sequence of problems, not just a single problem. Focusing on a sequence of problems is challenging because it requires teachers to think ahead in a way that simultaneously maintains the centrality of mathematical ideas and respects children’s ideas. Therefore, in the current study, we worked to help teachers develop the pedagogical skills to orchestrate mathematical conversations that depended on careful sequencing of problems (Franke et al., in press). Because the development of children’s understanding of the equal sign has been more clearly mapped than development in other areas, teachers could grasp children’s understanding of the equal sign and design appropriate sequences of problems to push their thinking to the next level in this area more
readily than in other areas. Discussions about the equal sign, therefore, laid the groundwork for more complex conversations, including those needed to help students use number relations to simplify calculations and to generalize. In many cases, we think that this groundwork was still being laid when the study concluded.

**FINAL THOUGHTS**

The opportunities for teacher and student learning in this study were strongly linked to our decisions about how to focus and structure the content discussed during professional development. In other words, our results were due in part to the way we conceptualized algebraic reasoning. The explicit rationale for our conceptualization follows, highlighting how we simultaneously considered issues related to mathematical content, students’ thinking, and ways that teachers might engage with the content.

First, we purposefully chose to build on arithmetic through the overarching idea of relational thinking, with particular emphasis on understanding and using the fundamental properties of number operations. Relational thinking is consistent with conceptions of understanding as constructing relationships and reflecting on and articulating those relationships (Carpenter & Lehrer, 1999). We also chose to focus on foundational mathematical ideas that were not only implicit in the mathematical work of elementary school but also significant for students’ later mathematics learning. Furthermore, we wanted teachers, across grade levels, to recognize and value our focus as core mathematical work. Relational thinking provided teachers opportunities not only to support students’ learning of algebra but also to more effectively meet the goals of their existing curriculum; helping students understand and use number relations had a direct connection to computational efficiency, the mainstay of the elementary school curriculum.

Second, we drew from the research on students’ thinking to ensure that the mathematical content we selected was accessible to elementary school students and to understand how students made sense of those ideas. We wanted the models of student thinking generated in discussions with teachers to effectively capture the mathematical thinking of the students in these teachers’ classrooms. An additional goal was to help teachers continually build counter-stories to the negative ones prevalent in urban education. By focusing on students’ thinking and, in particular, on what students could (vs. could not) do, teachers regularly saw that their own students were capable of sophisticated reasoning.

Third, we integrated these ideas of mathematical content and students’ thinking with our ideas of teacher learning. Because we believe that teachers learn through participation in practice, we designed our professional development to provide teachers opportunities to learn while they participated with their students as well as with one another (Wenger, 1998). We focused on ideas that were embedded in the entire year’s mathematics curriculum (vs. content limited to one unit at one time during the year) so that teachers had almost daily opportunities to engage themselves and their students with new ideas and practices. We also recognized the importance of conceptualizing algebraic reasoning in ways that would be useful during the
numerous in-the-moment decisions that teachers make every day, and our overar-
ching construct of relational thinking provided a unifying mathematical idea to guide
teachers’ decision making.

Finally, we considered issues of identity and teachers’ relationship to our math-
ematical focus (Holland, Lachicotte, Skinner, & Cain, 2001; Wenger, 1998). Teachers held already established relationships to algebra and the teaching of
algebra that we took into account in both the content and the way it was addressed. Often elementary school teachers do not see themselves as being strong in content
knowledge, and algebra, in particular, raises concerns for them. Because teachers’
perceptions of themselves in relation to content shape their participation in profes-
sional development and in their classrooms, we worked to conceptualize algebraic
reasoning in ways that teachers would perceive as doable. We began with content
with which teachers were most comfortable (i.e., the equal sign), and we created
tools (e.g., index cards for true/false and open number sentences) to support teachers
in holding explicit conversations with their students about algebraic reasoning. Teachers used these tools not only to support the development of algebraic reasoning
but also to create new norms for classroom discussion.

We began this study with the idea that engaging teachers in discussions about alge-
braic reasoning could be a lever for motivating fundamental change in teaching, not
just algebra, but mathematics generally. Through an experimental study, we found
that teachers engaged in discussions of mathematics, teaching, and learning in ways
that made a difference for themselves and for the students in their classrooms. We
contend that the focus and structure of the content discussed in our professional
development were fundamental to the teachers’ and students’ learning. We offered
a rationale for our conceptualization of algebraic reasoning, not because our concep-
tualization is the only one possible but because we believe that it is critical for
researchers to make their rationales explicit. In every professional development
effort, decisions are made about the focus and structure of the content to be
discussed. However, the principles underlying these decisions are often left unar-
ticulated, and we contend that only through their articulation can the conversation
about professional development truly move forward.

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Developing Children’s Algebraic Reasoning


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