4) On any simply connected domain not containing the origin \( f(z) \neq 0 \) which means that \( \log(z) \) is analytic. But \( z^\alpha = e^{\alpha \log(z)} \) is analytic as well as \( z^\alpha = e^{\alpha \log(z)} \).

5) Any closed curve \( \gamma \) in the domain \( \Omega \) that winds around 1 also winds around \( -1 \). Now we start with some point \( z_0 \in \gamma \) and choose some value of \( \theta_1 = \arg(1 - z) \) so that \( (1 - z_0) = r_1 e^{i\theta_1} \). Take a branch of \( \sqrt{1 - z_0} = r_1^{1/2} e^{i\theta_1/2} \).

Now as we travel around \( \gamma \); as we come back to \( z_0 \) we add \( 2\pi \) to the argument of \( (1 - z) \). The same thing happens to the argument of \( (1 + z) \) and so the argument of the product changes by \( 4\pi \). Now when we take square roots we divide the argument by 2 which means that as we move around \( \gamma \) and come back to \( z_0 \) the argument of \( \sqrt{1 - z^2} \) changes by \( 2\pi \).

Now we compute
\[
\int_\gamma \frac{1}{\sqrt{1 - z^2}} \, dz.
\]
By Cauchy’s theorem we can assume \( \gamma \) is a very large circle and so acts as a small disc about infinity. this suggests a change of variables \( z = 1/w \) on the circle. From \( dz = -dw/w^2 \) we get the integral is
\[
-\int_{|w|=\delta} \frac{dw}{w \sqrt{w^2 - 1}}.
\]
Now \( \sqrt{w^2 - 1} \) is analytic in a neighborhood of 0 and so we can compute the integral by the Cauchy integral formula which gives the integral to be \( -2\pi i \sqrt{(-1)} = 2\pi \).

Page 154

1) Let \( f(z) = 6z^3 \) and \( g(z) = z^7 - 2z^5 + 6z^3 - z + 1 \) Then \( |f(z) - g(z)| = |z^7 - 2z^5 - z + 1| \leq |z^7| + 2|z^5| + |z| + 1 = 5 < |f(z)| = |6z^6| = 6 \) on the circle \( |z| = 1 \). Thus they have the same number of zeroes in side the circle and that is 3.

2) Let \( f(z) = z^4 \) and \( g(z) = z^4 - 6z + 3 \). On the circle of radius 2, we have \( |f(z) - g(z)| \leq 6|z| + 3 = 15 < |f(z)| = 16 \). Thus \( g(z) \) has the same number of zeroes as \( f(z) \) inside which is 4. Inside the circle of radius 1 we let \( f(z) = -6z \). Now \( |f(z) - g(z)| = |z^4 + 3| \leq |z^4| + 3 \) which on the circle of radius 1 is 4 which is less than \( f(z) \) on the circle of radius 1. Thus \( g \) has 1 zero inside the circle of radius 1 and so \( g \) has 3 zeroes in the annulus.