We want to prove that if $f(z)$ analytic in a domain $\Omega^+$ and real on the real axis, we can extend to the symmetric domain $\Omega$ by defining $f(z) = \bar{f}(\bar{z})$ in the lower half plane.

The proof is as follows. We extend $v = \text{Im} f$ by $v(\bar{z}) = -v(z)$. Let $-u_0$ be the harmonic conjugate of $v$. It is defined up to an additive constant. We already know that $v$ has a harmonic conjugate $-Re f$ in the upper half plane so we choose the constant so that $-u_0 = -Re f$ in the upper half plane. Let

$$U_0 = u_0(z) - u_0(\bar{z}).$$

On the real axis $\bar{z} = z$ so $\partial U / \partial x = 0$ on the real axis. Also

$$\partial U_0 / \partial y = 2\partial u_0 / \partial y$$
on the real axis. But since $-u_0$ is the harmonic conjugate of $v$ we have the latter is $-\partial v / \partial x = 0$ since $v$ is 0 on the real axis. Thus

$$\partial U_0 / \partial x - i \partial U_0 / \partial y = 0$$
on the real axis. However one easily checks that for any harmonic function $U_0$,

$$\partial U / \partial x - i \partial U_0 / \partial y$$
is an analytic function (Cauchy Riemann equations) so that this analytic function is 0 on the real axis and hence identically 0. This gives $u_0(z) = u_0(\bar{z})$. This says that the analytic function $u_0 + iv$ does what we require.