Asymptotics of Weil-Petersson geodesics I: ending laminations, recurrence, and flows

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Abstract

We define an ending lamination for a Weil-Petersson geodesic ray. Despite the lack of a natural visual boundary for the Weil-Petersson metric [Br2], these ending laminations provide an effective boundary theory that encodes much of its asymptotic CAT(0) geometry. In particular, we prove an *ending lamination theorem* (Theorem 1.1) for the full-measure set of rays that recur to the thick part, and we show that the association of an ending lamination embeds asymptote classes of recurrent rays into the Gromov-boundary of the curve complex $\mathscr{C}(S)$. As an application, we establish fundamentals of the topological dynamics of the Weil-Petersson geodesic flow, showing density of closed orbits and topological transitivity.

Contents

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1 Introduction

This paper is the first in a series considering the asymptotics of geodesics in the Weil-Petersson metric on the Teichmüller space $\text{Teich}(S)$ of a compact surface S with negative Euler characteristic.

In many settings, measured laminations and foliations encode the asymptotic geometry of Teichmüller space. As key examples, one has:

1. Thurston's natural compactification by projective measured laminations [Th3, Bon2],

2. invariant projective measured foliations for Teichmüller rays [Ker], and

3. the parametrization of Bers's compactification by *end-invariants* (see [Min1, BCM]).

In the series, we seek a similar connection with laminations to describe asymptotics of Weil-Petersson geodesics in Teichmüller space: we define a notion of an *ending lamination* for a Weil-Petersson geodesic ray and investigate its role as an invariant for the ray.

The present paper employs ending laminations for geodesic rays to develop a boundary theory for the Weil-Petersson metric. We establish that the ending lamination is a complete invariant for *recurrent* rays, namely, those whose projections to the moduli space $\mathcal{M}(S)$ visit a fixed compact set at a divergent sequence of times. In particular, it follows that any two such rays starting at the same basepoint with the same ending lamination are identical up to parametrization. Despite the lack of naturality described in [Br2], this boundary theory establishes fundamentals of the topological dynamics of the Weil-Petersson geodesic flow on the unit tangent bundle to the moduli space, showing

(I.) closed Weil-Petersson geodesics are dense in $T^1\mathcal{M}(S)$ (Theorem 1.8), and

(II.) there is a dense Weil-Petersson geodesic in $T^1\mathcal{M}(S)$ (Theorem 1.9).

To the extent the ending lamination determines the ray one can employ properties of laminations to understand Weil-Petersson geometry. To this end, we prove

Theorem 1.1. (RECURRENT ENDING LAMINATION THEOREM) *Let* $\mathbf{r}$ *be an infinite length Weil-Petersson geodesic ray that is recurrent, and let* $\lambda(\mathbf{r})$ *be its ending lamination. If* $\mathbf{r}'$ *is any other infinite length geodesic ray with ending lamination* $\lambda(\mathbf{r}') = \lambda(\mathbf{r})$*, then* $\mathbf{r}$ *is strongly asymptotic to* $\mathbf{r}'$.

Here, we say $\mathbf{r}$ and $\mathbf{r}'$ are *strongly asymptotic* if there are parametrizations for which the distance between the rays satisfies

$$\lim_{t\to\infty} d(\mathbf{r}(t), \mathbf{r}'(t)) = 0.$$

In particular, the negative curvature of the Weil-Petersson metric guarantees that if $\mathbf{r}(0) = \mathbf{r}'(0)$, then the rays are identical if parametrized by arclength.

The *ending lamination* $\lambda(\mathbf{r})$ for a ray $\mathbf{r}$ arises out of the asymptotics of simple closed curves with an a priori length bound. Recall that by a theorem of Bers, there is a constant L_S depending only on S so that for each $X \in \mathrm{Teich}(S)$ there is a pants decomposition by geodesics on X so that each such geodesic has length satisfying

$$\ell_X(\gamma) \le L_S.$$

We call such a γ a *Bers curve* for X.

Given a Weil-Petersson geodesic ray $\mathbf{r}$, the *ending lamination* $\lambda(\mathbf{r})$ is a union of limits of Bers curves for surfaces $X_n = \mathbf{r}(t_n)$ along the ray. We give a precise description in section 2 and the proof that $\lambda(\mathbf{r})$ is well defined.

In Proposition 4.3, we show that for a recurrent ray $\mathbf{r}$, the ending lamination $\lambda(\mathbf{r})$ *fills* S. As such $\lambda(\mathbf{r})$ determines a point in $\mathscr{EL}(S)$, the *ending laminations* on S, naturally the Gromov boundary for the curve complex $\mathscr{C}(S)$ (see [MM1, Kla, Ham]). We remark that Theorem 1.1 determines a preferred subset $\mathscr{REL}(S) \subset \mathscr{EL}(S)$ of ending laminations corresponding to recurrent rays in the Weil-Petersson metric. In particular, the ending lamination determines whether or not a ray is recurrent.

We emphasize that in contradistinction with Teichmüller geometry, where Masur shows that Teichmüller geodesics with the same vertical foliation are strongly asymptotic when the foliation is *uniquely ergodic* [Mas2], there is no assumption in Theorem 1.1 of unique ergodicity for $\lambda(\mathbf{r})$. Indeed, in [Br3] examples are presented exhibiting the following behavior.

Theorem 1.2 (Brock)**.** *For each S with $\dim_{\mathbb{C}}(\mathrm{Teich}(S)) \geq 2$, there are recurrent Weil-Petersson geodesic rays whose ending laminations are non-uniquely ergodic.*

Furthermore, the theorem is sharp in the sense that without the assumption of recurrence examples are known of infinite rays with the same filling ending lamination.

Theorem 1.3 (Brock)**.** *There exist distinct Weil-Petersson geodesic rays $\mathbf{r}$ and $\mathbf{r}'$ based at $X \in \mathrm{Teich}(S)$ with a common ending lamination $\lambda(\mathbf{r}) = \lambda(\mathbf{r}')$ that fills the surface.*

(See [Br3], and compare [Br2, §6]).

Visual boundaries. The negative curvature of the Weil-Petersson metric (see [Tro, Wol3]) provides for a compactification of $\mathrm{Teich}(S)$ by geodesic rays emanating from a fixed basepoint X, the *visual sphere* at X. Work of the first author (see [Br2]) demonstrates that the compactification of $\mathrm{Teich}(S)$ is basepoint dependent and, moreover, that the mapping class group fails to extend continuously to the compactification.

Standard arguments for topological transitivity and the density of closed orbits that arise in Riemannian manifolds of negative curvature involve the use of the boundary at infinity for the universal cover and the natural extension of the action of the fundamental group to the boundary.

The principal source of difficulty with carrying out such a line of argument here is precisely the source of the basepoint dependence shown in [Br2]. The lack of completeness of the metric, due to Wolpert [Wol1], gives rise to finite-length geodesic rays that leave every compact subset of Teichmüller space, and these *finite rays* determine a subset of the boundary on which the change of basepoint map is discontinuous. While such finite rays prevent the Weil-Petersson metric from exhibiting the more standard boundary structure arising in the setting of Hadamard manifolds (see [Eb]) we show the infinite length Weil-Petersson geodesic rays determine a natural *boundary at infinity* for the Weil-Petersson completion.

Theorem 1.4. (BOUNDARY AT INFINITY) *Let $X \in \text{Teich}(S)$ be a basepoint.*

1. *For any $Y \in \text{Teich}(S)$ with $Y \neq X$, and any infinite ray $\mathbf{r}$ based at X there is a unique infinite ray $\mathbf{r}'$ based at Y with $\mathbf{r}'(t) \in \text{Teich}(S)$ for each t so that $\mathbf{r}'$ lies in the same asymptote class as $\mathbf{r}$.*

2. *The change of basepoint map restricts to a homeomorphism on the infinite rays.*

Though the Weil-Petersson completion $\overline{\text{Teich}(S)}$ does not satisfy the extendability of geodesics requirement for a standard notion of a CAT(0) boundary to be well defined, one can simply restrict attention to the infinite rays and consider asymptote classes of infinite rays in the completion of the Weil-Petersson metric, where two half-infinite rays are in the same *asymptote class* if they lie in some bounded Hausdorff distance. We denote this boundary at infinity by $\partial_\infty \overline{\text{Teich}(S)}$.

Any flat subspace in a CAT(0) space provides an obstruction to the visibility property exhibited in strict negative curvature, namely, the existence of a single bi-infinite geodesic asymptotic to any two distinct points at infinity. The encoding guaranteed by Theorem 1.1 of recurrent rays via laminations remedies this conclusion to some degree, as it guarantees such a visibility property almost everywhere.

Theorem 1.5. (RECURRENT VISIBILITY) *Let $\mathbf{r}^+$ and $\mathbf{r}^-$ be two distinct infinite rays based at X.*

1. *If $\mathbf{r}^+$ is recurrent, then there is a single bi-infinite geodesic $\mathbf{g}(t)$ so that $\mathbf{g}^+ = \mathbf{g}|_{[0,\infty)}$ is* strongly *asymptotic to $\mathbf{r}^+$ and $\mathbf{g}^- = \mathbf{g}|_{(-\infty,0]}$ is asymptotic to $\mathbf{r}^-$. In particular, if both $\mathbf{r}^+$ and $\mathbf{r}^-$ are recurrent, then $\mathbf{g}$ is strongly asymptotic to both $\mathbf{r}^-$ and $\mathbf{r}^+$.*

2. *If $\mu \in \mathscr{ML}(S)$ has bounded length on $\mathbf{r}^{\pm}$ then it has bounded length on $\mathbf{g}^{\pm}$.*

Theorem 1.4 leads one naturally to the question of whether, as in other compactifications of Teichmüller space, the laminations associated to serve as parameters. Applying Theorem 1.1, we find that such a parametrization holds for the recurrent locus.

Corollary 1.6. *The recurrent rays are parametrized by their ending laminations: the map λ that associates to an equivalence class of recurrent rays their ending lamination is a homeomorphism to the subset $\mathscr{REL}(S)$ in $\mathscr{EL}(S)$.*

We note that as a consequence of Theorem 1.3 this parametrization fails in general, even when the ending lamination is filling; the paper [Br3] takes up the question of their structure.

To describe our strategy further, we review geometric aspects of the Weil-Petersson metric and its completion.

Weil-Petersson geometry. The Weil-Petersson metric g_{WP} on $\mathrm{Teich}(S)$ arises from the norm

$$\|\varphi\|_{\mathrm{WP}} = \left(\int_X \frac{|\varphi|^2}{\rho}\right)^{\frac{1}{2}}$$

on the cotangent space $Q(X) = T^*_X\mathrm{Teich}(S)$ to Teichmüller space, naturally the holomorphic quadratic differentials on X, where ρ is the hyperbolic metric on X.

A fundamental distinction between the Weil-Petersson metric and other metrics on Teichmüller space is its lack of completeness, due to Wolpert and Chu [Wol1, Chu]. It is nevertheless geodesically convex [Wol4], and has negative sectional curvatures [Tro, Wol3].

The failure of completeness corresponds precisely to *pinching paths* in $\mathrm{Teich}(S)$ along which a simple closed geodesic on X is pinched to a cusp. It is due to Masur that the completion $\overline{\mathrm{Teich}(S)}$ is identified with the *augmented Teichmüller space* and is obtained by adjoining noded Riemann surfaces as limits of such pinching paths [Mas1]. Via this identification, then, the completion $\overline{\mathrm{Teich}(S)}$ (with its extended metric) descends to a metric on the Mumford-Deligne compactification $\overline{\mathscr{M}(S)}$ of the moduli space (cf. [Ab, Brs]).

Because of this failure of completeness, the geodesic flow is not everywhere defined on $T^1\mathscr{M}(S)$; some directions meet the compactification within finite Weil-Petersson distance. The situation is remedied by the following.

Theorem 1.7. *The geodesic flow is defined for all time on a full measure subset of $T^1\mathscr{M}(S)$.*

As a consequence, we address the question of the topological dynamics of the geodesic flow on the unit tangent bundle $T^1\mathscr{M}(S)$ to the moduli space of Riemann surfaces.

That the recurrent rays have full measure in the visual sphere allows us to approximate directions in the unit tangent bundle arbitrarily well by recurrent directions. As a consequence, we have

Theorem 1.8. (CLOSED ORBITS DENSE) *Closed Weil-Petersson geodesics are dense in* $T^1\mathscr{M}(S)$.

Applying our parametrization by ending laminations of the boundary at infinity, we may use the stable and unstable laminations for the axes of pseudo-Anosov isometries of $\mathrm{Teich}(S)$ to find based at any X a geodesic ray whose projection to $\mathscr{M}(S)$ has a dense trajectory in the unit tangent bundle.

Theorem 1.9. (DENSE GEODESIC) *There is a dense Weil-Petersson geodesic in* $T^1\mathscr{M}(S)$.

Combinatorics of Weil-Petersson geodesics. While the this paper's focus on recurrence establishes the importance of the ending lamination as a tool to analyze Weil-Petersson geodesics, it does not directly address the connection between the combinatorics of the lamination (in the sense of [MM2]) and the geometry of geodesics. It is our ultimate goal to establish a stronger connection along these lines. In the sequel to this paper, [BMM], we use a typical *bounded combinatorics* condition (cf. [Min2]) to bound the geometry of surfaces along a Weil-Petersson geodesic and conversely. We remark that the main theorem of this paper, Theorem 1.1, is applied in what appears to be a crucial way to establish the following (see [BMM])

Theorem 1.10. (BOUNDED COMBINATORICS GEOMETRICALLY THICK) *For each* $K > 0$ *there is an* $\varepsilon > 0$ *so that if the ending laminations of a bi-infinite Weil-Petersson geodesic* $\mathbf{g}$ *have K-bounded combinatorics then* $\mathbf{g}(t)$ *lies in the* ε*-thick part for each* t.

A similar statement applies when $\mathbf{g}$ is finite or half-infinite. The notion of bounded combinatorics refers to the *subsurface projections* of [MM2] applied to the ending laminations associated to the forward and backward rays determined by $\mathbf{g}$.

A kind of converse we will prove shows that combinatorics remain bounded along any geodesic in the thick part.

Theorem 1.11. (THICK GEODESICS COMBINATORIALLY BOUNDED) *Given* $\varepsilon > 0$ *there is a* $K > 0$ *so that if* $\mathbf{g}$ *is any bi-infinite geodesic in the* ε *thick part of* $\mathrm{Teich}(S)$*, then the combinatorics of the ending laminations associated to its ends are K-bounded.*

These results give good control over the subset of geodesics with bounded type, and imply further dynamical consequences involving the topological entropy of the geodesic flow on compact invariant subsets. We take up this discussion in [BMM].

Questions and Conjectures. We expect in general that the ending lamination should predict extensive information about bounded and short curves along the ray, as in Theorems 1.10 and 1.11. We make the following conjecture.

Conjecture 1.12. (WEIL-PETERSSON COMBINATORICS) *Let* $\mathbf{g}$ *be a bi-infinite geodesic with ending laminations* λ^- *and* λ^+ *that fill the surface* S*. For each simple closed curve* γ*, we have the following*

1. *for each* $K > 0$ *there is a* $\delta > 0$ *so that if* $\inf_t \ell_{\mathbf{g}(t)}(\gamma) < \delta$ *then there is a subsurface* $Y \subset S$ *with* $\gamma \subset \partial Y$ *for which* $d_Y(\lambda^-, \lambda^+) > K$*.*
2. *for each* $\delta' > 0$ *there is a* $K' > 0$ *so that if there is a subsurface* $Y \subset S$ *with* $\gamma \subset \partial Y$*, for which* $d_Y(\lambda^-, \lambda^+) > K'$ *then we have* $\inf_t \ell_{\mathbf{g}(t)}(\gamma) < \delta'$*.*

The quantities

$$d_Y(\lambda^-, \lambda^+)$$

introduced in [MM2] are combinatorial invariants associated to the pair (λ^-, λ^+) representing the distance of their *projections* to the curve complex of the subsurface Y. These invariants play the role of the entries in a kind of "continued fraction expansion" for the pair (λ^-, λ^+) and are the coefficients for the continued fraction expansion in the case of dimension one Teichmüller spaces, where they control the behavior of Weil-Petersson geodesics on the modular surface. We remark that for Teichmüller spaces of complex dimension at most 3, the conjecture follows from the (relative) hyperbolicity theorems of [BF] and [BM], but we defer this discussion to the sequel [BMM].

Conjecture 1.12 suggests a strong connection with hyperbolic 3-manifolds with the homotopy type of a surface, where the same type of combinatorial information controls the model for the hyperbolic structure on $S \times \mathbb{R}$ determined by its end-invariants. See [Min3, BCM].

Connections with ends of hyperbolic 3-manifolds motivate other questions about the structure of the ending lamination $\lambda(\mathbf{r})$ for a ray $\mathbf{r}$.

Conjecture 1.13. *Let* $\mathbf{r}$ *be a Weil-Petersson geodesic ray along which no simple closed curve has length asymptotic to zero. Then the ending lamination* $\lambda(\mathbf{r})$ *fills the surface.*

We establish this conjecture for recurrent rays in Proposition 4.3.

Plan of the paper. In section 2 we set out necessary background, and give the definition of ending lamination for a Weil-Petersson geodesic ray, establishing its basic properties. Section 3 establishes that the geodesic flow is defined for all time on a full measure set and gives the natural application of the Poincaré recurrence theorem in this setting. Section 4 establishes the main theorem, that the ending lamination is a complete invariant for a recurrent ray, as well deriving important topological properties of the ending lamination itself that mirror the behavior of ending laminations for hyperbolic 3-manifolds. Finally, in section 5 we present applications of this boundary theory to the topological dynamics of the Weil-Petersson geodesic flow.

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2 Ending laminations for Weil-Petersson rays

In this section we begin by reviewing some of the notions and results necessary for our discussion, provide references for background, and give the definition of the ending lamination, establishing its basic properties.

Teichmüller space and moduli space. The Teichmüller space of S, $\mathrm{Teich}(S)$, parametrizes the marked complete hyperbolic structures on $\mathrm{int}(S)$, namely, pairs (f,X) where

$$f\colon \mathrm{int}(S) \to X$$

is a *marking homeomorphism* to a hyperbolic surface X and $(f,X) \sim (g,Y)$ if there is an isometry $\phi\colon X \to Y$ for which $\phi \circ f$ is isotopic to g. The *mapping class group* $\mathrm{Mod}(S)$ of orientation preserving homeomorphisms up to isotopy acts naturally on $\mathrm{Teich}(S)$ by precomposition of markings, inducing an action by isometries in the Weil-Petersson metric. The quotient is the *moduli space* $\mathscr{M}(S)$, of hyperbolic structures on $\mathrm{int}(S)$ (without marking), and the Weil-Petersson metric descends to a metric on $\mathscr{M}(S)$.

Hyperbolic geometry of surfaces. For all that follows it will be important to have in place the Theorem of Bers (see [Bus]) that given S, a compact orientable surface of negative Euler characteristic, there is a constant $L_S > 0$ so that for each $X \in \mathrm{Teich}(S)$ there is a pants decomposition P_X determined by simple closed geodesics on X so that

$$\ell_X(\gamma) < L_S$$

for each $\gamma \in P_X$. We call the pants decomposition P_X a *Bers pants decomposition for X* and the curves in such a pants decomposition P_X *Bers curves for X*.

A *geodesic lamination* λ on hyperbolic surface $X \in \mathrm{Teich}(S)$ is a closed subset of X foliated by simple complete geodesics. Employing the natural boundary at infinity for $\widetilde{X}$, a geodesic lamination, like a simple closed curve, has a well defined isotopy class on X, and we may speak of a single geodesic lamination λ as an object associated to S with realizations on each hyperbolic structure $X \in \mathrm{Teich}(S)$ (see [Th1, Ch. 8] or [Bon2]). The realizations of geodesic laminations on X may be given the Hausdorff topology, and the correspondence between realizations of λ on different surfaces X and X' gives a homeomorphism. Hence, we refer to a single *geodesic lamination space* $\mathscr{GL}(S)$.

A geodesic lamination λ equipped with a *transverse measure* μ, namely a measure on each arc transverse to the leaves of λ invariant under isotopy preserving intersections with λ, determines a *measured lamination*. The lamination λ is the called *support* of the measured lamination μ and is denoted by $|\mu|$. The simple closed curves with positive real weights play the role of Dirac measures, and the measured lamination space can be viewed as their completion with respect to the linear extension of the geometric intersection number for simple closed curves on S (see [FLP, Th1, Bon2]).

Curve and arc complexes. The complex of curves $\mathscr{C}(S)$ associated to the surface S is a simplicial complex whose vertices correspond to isotopy classes of distinct essential simple closed curves on S, and whose k-simplices span $k+1$-tuples of vertices whose corresponding isotopy classes can be realized as a pairwise disjoint collection of simple closed curves on S. By convention, we obtain the *augmented curve complex* by adjoining the empty simplex and denote

$$\widehat{\mathscr{C}(S)} = \mathscr{C}(S) \cup \varnothing.$$

When S has boundary, a relevant related complex is the *arc complex* of S, $\mathscr{A}(S)$, which is defined analogously and whose vertices correspond to isotopy classes *mod-boundary* of arcs with endpoints in ∂S (an isotopy *mod-boundary* of an arc with endpoints in the boundary of S allows the endpoints of arcs to move within the boundary of S).

It was shown in [MM1] that the curve complex $\mathscr{C}(S)$ is δ-hyperbolic in the sense of Gromov. Any such space carries a natural *Gromov boundary*, namely, asymptote classes of infinite geodesic rays in the space, where two rays are *asymptotic* if they lie in a uniformly bounded Hausdorff distance. Klarreich showed [Kla] (see also [Ham]) that the Gromov boundary is identified with the space $\mathscr{EL}(S)$ of *ending laminations* on S. This space consists of the supports of the subset of the measured lamination space consisting of laminations that *fill the surface*, namely, laminations $\mu \in \mathscr{ML}(S)$ so that every simple closed curve γ satisfies $i(\mu, \gamma) > 0$. The space $\mathscr{EL}(S)$ inherits the quotient topology from

$\mathscr{ML}(S)$, but it is a Hausdorff subspace of this quotient; this topology is sometimes called the *measure-forgetting topology* or the *Thurston topology* [CEG].

Given a reference hyperbolic structure X on S, for each $\gamma \in \mathscr{C}^0(S)$ there is a $\delta_\gamma > 0$ so that for each η with $i(\eta, \gamma) = 0$, we have disjoint neighborhoods

$$\mathscr{N}_{\delta_\gamma}(\gamma^*) \cap \mathscr{N}_{\delta_\eta}(\eta^*)$$

where γ^* denotes the geodesic representative of γ on X. Given a simplex $\sigma \subset \mathscr{C}(S)$, denote by $\mathbf{collar}(\sigma)$ the union

$$\bigcup_{\gamma \subset \sigma^0} \mathscr{N}_{\delta_\gamma}(\gamma^*).$$

Definition 2.1. *Let λ be a connected geodesic lamination on $X \in \mathrm{Teich}(S)$. The* supporting subsurface *$S(\lambda) \subset S$ is the compact subsurface up to isotopy whose interior is isotopic in X either to*

1. $\mathbf{collar}(\lambda)$ *if λ is a simple closed curve, or*
2. *the minimal connected component containing λ of $X \setminus \mathbf{collar}(\sigma)$ where σ ranges over $\{\sigma \in \widehat{\mathscr{C}(S)} \mid \gamma \cap \lambda = \varnothing,\ \gamma \in \sigma^0\}$ if λ is not a simple closed curve.*

The pants complex. A quasi-isometric model was obtained for the Weil-Petersson metric in [Br1] using pants decompositions of surfaces. We say two pants decompositions P and P' are related by an *elementary move* if P' can be obtained from P by replacing a curve α in P with a curve β so that α and β have minimal intersection among all possible replacements for α that yield a pants decomposition. Let $P(S)$ denote the graph whose vertices represent distinct isotopy classes of pants decompositions of S, or maximal simplices in $\mathscr{C}(S)$, and whose edges join vertices that differ by an elementary move.

Hatcher and Thurston showed that $P(S)$ is connected (see [HLS]) so we may consider the edge metric on $P(S)$ as a distance on the pants decompositions of S. Letting $Q\colon P(S) \to \mathrm{Teich}(S)$ be any map that associates to P a surface X on which P is a Bers pants decomposition.

Theorem 2.2. ([Br1, Thm. 1.1]) *The map Q is a quasi-isometry.*

In other words, the map Q distorts distances by a bounded multiplicative factor and a bounded additive constant.

The Weil-Petersson completion and its strata. Non-completeness of the Weil-Petersson metric corresponds to finite-length paths in Teichmüller space along which length functions for simple closed curves converge to zero (see [Wol1]). In [Mas1], Masur described the completion concretely as the *augmented Teichmüller space* [Brs, Ab] obtained from Teichmüller space by adding strata consisting of spaces $\mathcal{S}_\sigma$ defined by the vanishing of length functions

$$\ell_\alpha = 0$$

for each $\alpha \in \sigma^0$ where σ is a simplex in the augmented curve complex $\widehat{\mathscr{C}(S)}$. Points in the σ-null strata $\mathcal{S}_\sigma$ correspond to *nodal* Riemann surfaces Z, where (paired) cusps are introduced along the curves in σ.

One can describe the topology via extended Fenchel Nielsen coordinates: given a pants decomposition P, the frontier spaces *subordinate to* P represent the union of boundary strata $\mathcal{S}_\sigma$ where $\sigma^0 \subset P$ is a subcollection of pants curves. The topology on the union of $\text{Teich}(S)$ with the boundary stratum $\mathcal{S}_\sigma$ is described by the requirement that the usual Fenchel-Nielsen length-twist functions for P vary continuously where for $\alpha \in \sigma^0$ the twist parameter θ_α is omitted and $\ell_\alpha = 0$ is an allowed value.

Then the strata $\mathcal{S}_\sigma$ are naturally products of lower dimensional Teichmüller spaces corresponding to the complete, finite-area hyperbolic "pieces" of the nodal surface $Z \in \mathcal{S}_\sigma$.

The completion $\overline{\text{Teich}(S)}$ has the structure of a CAT(0) space: it is a length space, satisfying the sub-comparison property for chordal distances in comparison triangles in the Euclidean plane (see [BH, II.1, Defn. 1.1]). Given $(X,Y) \in \overline{\text{Teich}(S)} \times \overline{\text{Teich}(S)}$ we will denote by $g(X,Y)$ the unique geodesic joining X to Y. Then the *main stratum*, $\mathcal{S}_\varnothing$, is simply the full Teichmüller space $\text{Teich}(S)$.

Apropos of this convention, we recall the fundamental non-refraction for geodesics on the Weil-Petersson completion.

Theorem 2.3 ([DW, Wol5])**.** (NON-REFRACTION IN THE COMPLETION) *Let $g(X,Y)$ be the geodesic joining X and Y in $\overline{\text{Teich}(S)}$, and let σ_- and σ_+ be the maximal simplices in the curve complex so that $X \in \mathcal{S}_{\sigma_-}$ and $Y \in \mathcal{S}_{\sigma_+}$. If $\eta = \sigma_- \cap \sigma_+$, then we have*

$$\text{int}(g) \subset \mathcal{S}_\eta.$$

We remark that in the special case that X and Y lie in the interior of Teichmüller space the theorem is simply Wolpert's original geodesic convexity theorem (see [Wol4]).

As a consequence of Theorem 2.3, the authors obtain a classification of elements of $\text{Mod}(S)$ in terms of their action by isometries of the Weil-Petersson completion $\overline{\text{Teich}(S)}$. In particular, a mapping class ψ is *pseudo-Anosov* if no non-zero power of ψ preserves any isotopy class of simple closed curves on S. As in the setting of the Teichmüller metric, ψ preserves an invariant Weil-Petersson *geodesic axis* $A_\psi \subset \text{Teich}(S)$ (see [DW, Wol5]) on which it acts by translation.

Weil-Petersson geodesic rays and ending laminations. Although triangles in a CAT(0) space can fail the stronger thin-triangles condition of Gromov hyperbolicity, the comparison property for triangles suffices to guarantee that there is still a well defined notion of an *asymptote class* for a geodesic ray: two rays $\mathbf{r}$ and $\mathbf{r}'$ lie in the same asymptote class, or are *asymptotic* if there is a $D > 0$ so that

$$d(\mathbf{r}(t), \mathbf{r}'(t)) < D$$

for each t.

Fixing a basepoint $X \in \mathrm{Teich}(S)$, however, it is natural in the setting of negative curvature to consider the sphere of geodesic rays emanating from X, which we denote by $\mathscr{V}_X(S)$ the Weil-Petersson *visual sphere*. Geodesic convexity (see [Wol4]) guarantees that we can compactify Teichmüller space by the visual sphere $\mathscr{V}_X(S)$. By convention, the rays in $\mathscr{V}_X(S)$ will be parametrized by arclength.

Allowing $T = \infty$, let

$$\mathbf{r}\colon [0,T) \to \mathrm{Teich}(S),$$

be a Weil-Petersson geodesic ray parametrized by arclength, with $\mathbf{r}(0) = X$ so that $\mathbf{r}(t)$ leaves every compact subset of Teichmüller space. We call a simple closed curve $\gamma \in \mathscr{C}^0(S)$ a *Bers curve for the ray* $\mathbf{r}$ if there is a $t \in [0,T)$ for which γ is a Bers curve for $\mathbf{r}(t)$.

We associate a geodesic lamination $\lambda(\mathbf{r})$ to a ray $\mathbf{r}$ as follows.

Definition 2.4. *An* ending measure *for a geodesic ray* $\mathbf{r}(t)$ *is any limit* $[\mu]$ *in* $\mathscr{PML}(S)$ *of the projective classes* $[\gamma_n]$ *of any infinite family of distinct Bers curves for* $\mathbf{r}$.

Given $L > 0$ there may be a fixed curve γ that satisfies $\ell_{\mathbf{r}(t)}(\gamma) \leq L$ for each t. Those γ that have no positive lower bound to their length, however, play a special role.

Definition 2.5. *A simple closed curve* γ *is a* pinching curve *for* $\mathbf{r}$ *if* $\ell_{\mathbf{r}(t)}(\gamma) \to 0$ *as* $t \to T$.

A single ray can exhibit both types of behavior, motivating the following definition.

Definition 2.6. *If* $\mathbf{r}(t)$ *is a Weil-Petersson geodesic ray, the* ending lamination $\lambda(\mathbf{r})$ *for* $\mathbf{r}$ *is the union of the pinching curves and the geodesic laminations arising as supports of ending measures for* $\mathbf{r}$.

To justify the definition we must show that pinching curves and ending measures together have the underlying structure of a geodesic lamination. To this end, we establish the following basic property of ending measures.

Lemma 2.7. *Let μ be an ending measure for a ray* $\mathbf{r}$*. Then there is a $K > 0$ so that for each t we have*

$$\ell_{\mathbf{r}(t)}(\mu) \leq K.$$

Proof. We first note that if μ is an ending measure for $\mathbf{r}$, then $\mathbf{r}$ has infinite length. Indeed, if the length of $\mathbf{r}$ were finite, then the path $\mathbf{r}(t)$ converges to a nodal surface in the Weil-Petersson completion of Teich(S), and for each simple closed curve γ on S either

1. the length of γ converges along the ray $\mathbf{r}(t)$ or,
2. there is a pinching curve α for which $i(\alpha, \gamma) > 0$.

It follows that the union of Bers curves over all surfaces $\mathbf{r}(t)$ is finite, and thus there is no infinite family of distinct Bers curves for the ray $\mathbf{r}(t)$.

Let γ_n be a sequence of Bers curves for the ray $\mathbf{r}$ so that the length of γ_n is infimized at $\mathbf{r}(t_n)$, and for which $t_i < t_{i+1}$, $i \in \mathbb{N}$. Let $[\mu]$ be any accumulation point of the sequence of projective classes $[\gamma_n]$ in $\mathscr{PML}(S)$. Then μ is an ending measure for $\mathbf{r}$. We may assume, after rescaling, that μ is the representative in the projective class $[\mu]$ with $\ell_X(\mu) = 1$.

Letting $s_n > 0$ be taken so that

$$s_n = \frac{1}{\ell_X(\gamma_n)},$$

the measured laminations $s_n\gamma_n$ satisfy $\ell_X(s_n\gamma_n) = 1$ for each n, and it follows that $s_n\gamma_n \to \mu$ in $\mathscr{ML}(S)$.

Fixing a value $t' > 0$, there is an N' so that for $n > N'$, we have $t_n > t'$. Applying convexity of the length function $\ell_{\mathbf{r}(t)}(\gamma_n)$ as a function of t, [Wol4], we conclude that

$$\ell_{\mathbf{r}(t')}(s_n\gamma_n) < 1$$

for each $n > N'$. The length of a lamination

$$\ell_{.}(.) \colon \mathrm{Teich}(S) \times \mathscr{ML}(S) \to \mathbb{R}^+$$

is a bi-continuous function, and we conclude that

$$\ell_{\mathbf{r}(t')}(\mu) \leq 1.$$

Since $t' > 0$ was arbitrary, we have that the inequality is satisfied for our normalization of μ, which was by a scalar multiple. The Lemma follows. $\square$

For future reference, we establish the following continuity property for the behavior of bounded length laminations along rays.

Lemma 2.8. *Let $\mathbf{r}_n \to \mathbf{r}$ be a convergent sequence of rays in the visual sphere $\mathscr{V}_X(S)$. Then if μ_n is any sequence of ending measures or weighted pinching curves for $\mathbf{r}_n$, any representative $\mu \in \mathscr{ML}(S)$ of the limit $[\mu]$ of projective classes $[\mu_n]$ in $\mathscr{PML}(S)$ has bounded length along the ray $\mathbf{r}$.*

Proof. After normalizing so that $\ell_X(\mu_n) = 1$ we may assume that

$$\ell_{\mathbf{r}_n(t)}(\mu_n) < 1$$

along $\mathbf{r}_n$. Assume that $\mathbf{r}_n$ and $\mathbf{r}$ are parametrized by arc-length. Then for each surface $Y = \mathbf{r}(s)$ along $\mathbf{r}$ there are surfaces $X_n = \mathbf{r}_n(s)$ with $X_n \to Y$ in $\mathrm{Teich}(S)$. By continuity of length on $\mathrm{Teich}(S) \times \mathscr{ML}(S)$, we have

$$\ell_{X_n}(\mu_n) \to \ell_Y(\mu)$$

and thus that $\ell_Y(\mu) \leq 1$. Since s is arbitrary, we have that the length of μ is bounded above by 1 along the ray $\mathbf{r}$. $\square$

Proposition 2.9. *Given a ray $\mathbf{r}(t)$, the union $\lambda(\mathbf{r})$ is a non-empty geodesic lamination.*

Proof. We first show that given $\mathbf{r}$, there exists either a pinching curve or an ending measure for $\mathbf{r}$. If $\mathbf{r}$ is a ray of finite length, then it terminates in the completion at a nodal surface Z in a boundary stratum $\mathscr{S}_\sigma$. It follows that each curve γ associated to a vertex of σ has length tending to zero along $\mathbf{r}$ and is thus a pinching curve for $\mathbf{r}$.

Assume there are no pinching curves along $\mathbf{r}$. Then, since $\mathbf{r}$ leaves every compact subset of $\mathrm{Teich}(S)$, and it does not terminate in the completion, it follows that it has infinite Weil-Petersson length. Then we claim there is a non-trivial ending measure μ for $\mathbf{r}$. It suffices to show that there are infinitely many distinct Bers curves γ_n for surfaces $\mathbf{r}(t_n)$, with $t_n \to \infty$. But otherwise, the set of all Bers pants decompositions along the ray is also bounded. By Theorem 2.2, such a bound also determines a bound for the length of the ray $\mathbf{r}$ via the quasi-isometry Q, contradicting the assumption that $\mathbf{r}$ was infinite.

As in the definition of the ending lamination for hyperbolic 3-manifolds [Th1, Ch. 8], it suffices to show that for any pair of μ_1 and μ_2 of weighted pinching curves or ending measures, that the intersection number satisfies

$$i(\mu_1, \mu_2) = 0.$$

We note first that by the collar lemma any two pinching curves for $\mathbf{r}$ must be disjoint. Furthermore, if $\ell_{\mathbf{r}(t)}(\gamma) \to 0$ as $t \to T$, then γ is disjoint from each Bers curve on $\mathbf{r}(t)$ for t sufficiently large. Thus, if μ is an ending measure for $\mathbf{r}(t)$, then we have $i(\gamma, \mu) = 0$ as well. Thus we reduce to the case that μ_1 and μ_2 are both ending measures.

Assume that $i(\mu_1, \mu_2) > 0$. We note in particular that if μ_1 and μ_2 fill the surface, Lemma 2.7 guarantees that the ray $\mathbf{r}(t)$ defines a path of surfaces that range in a compact family in Teich(S) by Thurston's *Binding Confinement* (see [Th2, Prop. 2.4]). This contradicts the assumption that $\mathbf{r}$ leaves every compact subset of Teich(S).

More generally, let μ_1 and μ_2 fill a proper essential subsurface $Y \subset S$. Then a more general version of binding confinement, *Converge on Subsurface* (see [Th2, Thm. 6.2]), ensures that representations $\rho_t \colon \pi_1(S) \to \mathrm{PSL}_2(\mathbb{R})$ for which $\mathbf{r}(t) = \mathbb{H}^2/\rho_t(\pi_1(S))$ have restrictions to $\pi_1(Y)$ that converge up to conjugacy after passing to a subsequence.

Let Y_t denote the minimal subsurface with geodesic boundary in $\mathbb{H}^2/\rho_t(\pi_1(S))$ isotopic to Y. Then the length function for each simple closed curve $\alpha \in \mathscr{C}(Y)$ and each arc mod boundary $\beta \in \mathscr{A}(Y)$ converges along the convergent subsequence to a positive number. As a result, the set of Bers curves for $\mathbf{r}$ intersects Y in a finite collection of isotopy classes mod-boundary. Furthermore, since the marked hyperbolic structure on Y_t is converging, no such curve can be a pinching curve for $\mathbf{r}$. Each ending measure μ is a limit of weighted Bers curves $s_n\gamma_n$, with $s_n \to 0$, so it follows that for each $\eta \in \mathscr{C}(Y)$ we have

$$i(\eta, s_n\gamma_n) \to 0$$

which contradicts that the support of μ_i intersects Y.

We conclude that $i(\mu_1, \mu_2) = 0$, and thus that the set of complete geodesics in the support of all pinching curves and ending measures forms a closed subset consisting of disjoint complete geodesics, namely, a geodesic lamination. $\square$

We note the following corollary of the proof.

Corollary 2.10. *Let $\mu \in \mathscr{ML}(S)$ be any lamination whose length is bounded along the ray $\mathbf{r} \in \mathscr{V}_X(S)$. Then if μ' is an ending measure for $\mathbf{r}$ or a measure on any pinching curve for $\mathbf{r}$, we have*

$$i(\mu, \mu') = 0.$$

Proof. The proof of Proposition 2.9 employs only the bound on the length of μ_1 and μ_2 along the ray to show the vanishing of their intersection number. The argument applies equally well under the assumption that μ_1 is a simple closed curve of bounded length, and μ_2 is an ending measure, or a weighted pinching curve. Letting μ play the role of μ_1 and μ' play the role of μ_2, the Corollary follows. $\square$

By Thurston's classification of elements of $\mathrm{Mod}(S)$, a pseudo-Anosov element $\psi \in \mathrm{Mod}(S)$ determines laminations μ^+ and μ^- in $\mathscr{ML}(S)$, invariant by ψ up to scale [Th3]. Each determines an unique projective class in $\mathscr{PML}(S)$, the so-called *stable* and *unstable* laminations for ψ, and arises as a limit of iteration of ψ on $\mathscr{PML}(S)$. Specifically, given a simple closed curve γ, we have

$$[\mu^+] = \lim \psi^n([\gamma]) \quad \text{and} \quad [\mu^-] = \lim \psi^{-n}([\gamma])$$

in $\mathscr{PML}(S)$. Similarly, each $X \in A_\psi$, determines a *forward ray* $\mathbf{r}^+$ based at X so that $\psi(\mathbf{r}^+) \subset \mathbf{r}^+$ and a *backward ray* $\mathbf{r}^-$ at X so that $\mathbf{r}^- \subset \psi(\mathbf{r}^-)$. Invariance of the axis A_ψ, then, immediately gives the following relationship between the stable and unstable laminations for ψ and the ending laminations for the forward and backward rays at X for the invariant axis A_ψ.

Proposition 2.11. *Let $\psi \in \mathrm{Mod}(S)$ be a pseudo-Anosov element with invariant axis A_ψ. Let $X \in A_\psi$, and let $\mathbf{r}^+$ and $\mathbf{r}^-$ be the forward and backward geodesic rays at X determined by A_ψ. Then we have*

$$|\mu^+| = \lambda(\mathbf{r}^+) \quad \textit{and} \quad |\mu^-| = \lambda(\mathbf{r}^-)$$

where μ^+ is the stable lamination for ψ and μ^- is the is the unstable lamination.

Proof. Letting γ be a Bers curve for the surface X, the projective class $[\mu^+]$ of μ^+ is the limit of the projective classes $[\gamma_n]$ where $\gamma_n = \psi^n(\gamma)$ and likewise, $[\mu^-]$ is the limit of $[\gamma_{-n}]$. Since γ_n is a Bers curve for $\psi^n(X)$, it follows that μ^+ and μ^- are ending measures $\mathbf{r}^+$ and $\mathbf{r}^-$, respectively. Since μ^+ fills the surface, any other ending measure μ for $\mathbf{r}^+$ has intersection number $i(\mu^+, \mu) = 0$, so we have $\lambda(\mathbf{r}^+) = |\mu^+|$ and likewise $\lambda(\mathbf{r}^-) = |\mu^-|$. $\square$

3 Density, recurrence, and flows

In [Br2], we employ the CAT(0) geometry of the Weil-Petersson completion to show the following.

Theorem 3.1. ([Br2, Thm. 1.5]) *The finite rays are dense in the visual sphere.*

To obtain a simplified proof of the Masur-Wolf theorem that the orientation-preserving isometries of the Weil-Petersson metric are identified with mapping classes, Wolpert observed that one obtains the following generalization (see [Wol5, Sec. 5]).

Theorem 3.2 (Wolpert)**.** *Restrictions to* $\mathrm{Teich}(S)$ *of Weil-Petersson geodesics in* $\overline{\mathrm{Teich}(S)}$ *joining pairs of maximally noded surfaces are dense in the unit tangent bundle* $T^1\mathrm{Teich}(S)$.

We recall a key element of the proof.

Lemma 3.3 (Wolpert)**.** *The finite rays have measure zero in the visual sphere.*

(See [Wol5, Wol6]).

Proof. Given a simplex σ in $\mathscr{C}(S)$, consider the natural geodesic retraction map from a given null-stratum $\mathscr{S}_\sigma$ onto the unit tangent sphere at $X \in \mathrm{Teich}(S)$, sending each point $Z \in \mathscr{S}_\sigma$ to the unit tangent at X in the direction of the unique geodesic from X to Z. Wolpert observes this map is Lipschitz from the intrinsic metric on $\mathscr{S}_\sigma$ to the standard metric on the unit tangent sphere. As each stratum has positive complex co-dimension, the image of $\overline{\mathrm{Teich}(S)} \setminus \mathrm{Teich}(S)$ has Hausdorff measure zero in the (real co-dimension 1) visual sphere. It follows that *infinite* directions have full measure. $\square$

Theorem 1.7 follows as an immediate corollary.

Theorem 1.7. *The geodesic flow is defined for all time on the full-measure subset consisting of lifts of bi-infinite Weil-Petersson geodesics on* $\mathscr{M}(S)$ *to its unit tangent bundle* $T^1\mathscr{M}(S)$.

Proof. That the infinite rays have full-measure in the unit tangent bundle $T^1_X\mathscr{M}(S)$ at $X \in \mathscr{M}(S)$ implies that the directions determining bi-infinite geodesics have full measure in $T^1_X\mathscr{M}(S)$ as well. The union of these over all $X \in \mathscr{M}(S)$ is a flow-invariant set of full measure, by Fubini's theorem. $\square$

A geodesic ray $\mathbf{r}$ based at $X \in \mathscr{M}(S)$ is *divergent* if for each compact set $K \subset \mathscr{M}(S)$, there is a T for which $\mathbf{r}(t) \cap K = \varnothing$ for each $t > T$. A ray $\mathbf{r}$ is called *recurrent* if it is not divergent. Alternatively, Mumford's compactness theorem [Mum], guarantees that given $\varepsilon > 0$ the "ε-thick-part"

$$\mathrm{Teich}_{\geq\varepsilon}(S) = \{X \in \mathrm{Teich}(S) \mid \ell_X(\gamma) \geq \varepsilon,\ \gamma \in \mathscr{C}^0(S)\}$$

of Teichmüller space projects to a compact subset of $\mathscr{M}(S)$. Thus we may characterize recurrent rays equivalently by the condition that there is an $\varepsilon > 0$ and a sequence of times $t_n \to \infty$ so that $\mathbf{r}(t_n) \subset \mathrm{Teich}_{\geq\varepsilon}(S)$.

Taking Theorem 1.7 together with the Poincaré recurrence theorem, we have the following.

Theorem 3.4. *The recurrent geodesics in $T^1\mathscr{M}(S)$ determine a full-measure invariant subset.*

Proof. The geodesic flow is volume-preserving, by Liouville's theorem (see [CFS, §2, Thm. 2]), and thus finiteness of the Weil-Petersson volume of moduli space ([Mas1, Wol2]), and hence its unit tangent bundle, guarantees that no positive measure set of geodesics can be divergent by Poincaré recurrence. $\square$

The construction of an infinite ray at $Y \in \text{Teich}(S)$ asymptotic to a given ray at $X \in \text{Teich}(S)$ is an essential tool in our discussion. This is a general feature of complete CAT(0) spaces, as shown in [BH, II.8, 8.3], and thus applies to the completion $\overline{\text{Teich}(S)}$. More care is required, however, to show that the resulting infinite ray in $\overline{\text{Teich}(S)}$ actually determines an infinite ray in $\text{Teich}(S)$. Indeed, the possibility that a limit of unbounded or even infinite geodesics might be finite cannot be ruled out a priori, as was shown in [Br2] (see also [Wol5]). This is also a consequence of Theorem 1.7. Theorem 1.4 follows from a key application of Theorem 2.3, the non-refraction of geodesics in the Weil-Petersson completion.

Theorem 1.4. (BOUNDARY AT INFINITY) *Let $X \in \text{Teich}(S)$ be a basepoint.*

1. *For any $Y \in \text{Teich}(S)$ with $Y \neq X$, and any infinite ray $\mathbf{r}$ based at X there is a unique infinite ray $\mathbf{r}'$ based at Y with $\mathbf{r}'(t) \in \text{Teich}(S)$ for each t so that $\mathbf{r}'$ lies in the same asymptote class as $\mathbf{r}$.*

2. *The change of basepoint map restricts to a homeomorphism on the infinite rays.*

Remark. Because of totally geodesic flats in the completion arising from product strata, the condition that rays be merely asymptotic, namely, that they remain a bounded distance apart, cannot be improved to the condition that they be strongly asymptotic, though we will see this follows for recurrent rays (Theorem 4.1).

Proof. It is a general consequence of [BH, II.8, 8.3] applied to the complete CAT(0) space $\overline{\text{Teich}(S)}$ that we have an infinite geodesic ray $\mathbf{r}'(t)$ in $\overline{\text{Teich}(S)}$ based at Y in the asymptote class of $\mathbf{r}$ based at X. Indeed, the ray $\mathbf{r}'(t)$ is the limit of finite-length geodesics $g(Y, \mathbf{r}(t))$ joining Y to points along the ray $\mathbf{r}$ with their parametrizations by arclength, a fact we note for future reference.

It remains only to conclude that $\mathbf{r}'(t) \in \text{Teich}(S)$ for each $t > 0$. But by Theorem 2.3, for each $T > 0$ the geodesic $\mathbf{r}'([0,T])$ has interior $\mathbf{r}'((0,T))$ in the stratum $\mathscr{S}_{\sigma_0 \cap \sigma_T}$ where $\mathbf{r}'(0) \in \mathscr{S}_{\sigma_0}$ and $\mathbf{r}'(T) \in \mathscr{S}_{\sigma_T}$. But since $X \in \text{Teich}(S)$ we have $\sigma_0 = \varnothing$, so $\mathbf{r}'(t)$ lies in the main stratum $\mathscr{S}_{\varnothing} = \text{Teich}(S)$ for each $t < T$. Since T is arbitrary, the conclusion follows.

It is general for a CAT(0) space that given a basepoint X, and an infinite ray $\mathbf{r}$ at X, the ray $\mathbf{r}$ is the unique representative of its asymptote class that is based at X. Thus, we have a unique infinite ray based at a fixed X in each asymptote class. Applying the CAT(0)-geometry of $\overline{\mathrm{Teich}(S)}$, it follows that if $\mathbf{r}_n$ is a sequence of rays based at X with convergent initial tangents to the initial tangent of the infinite ray $\mathbf{r}_\infty$, then the corresponding infinite rays $\mathbf{r}'_n$ based at Y in the same asymptote class converge to the ray $\mathbf{r}'_\infty$ based at Y in the same asymptote class as $\mathbf{r}_\infty$. Thus the change of basepoint map is a homeomorphism on the infinite rays. $\square$

We remark that the assumption that Y lies in the interior of $\mathrm{Teich}(S)$ is just for simplicity: the same argument may be carried out to prove the following stronger statement.

Theorem 3.5. *Let σ and σ' be simplices in $\widehat{\mathscr{C}(S)}$. Let Y lie in the interior of a boundary stratum $\mathscr{S}_\sigma$. Then given an infinite ray $\mathbf{r}$ in $\overline{\mathrm{Teich}(S)}$ based at $X \in \mathscr{S}_{\sigma'}$, there is a unique infinite ray $\mathbf{r}'$ based at Y with $\mathbf{r}'(t) \in \mathrm{Teich}(S) \cup \mathscr{S}_\sigma$ for each t so that $\mathbf{r}'$ lies in the same asymptote class as $\mathbf{r}$.*

Proof. The proof goes through as before with the additional observation that for each s the limit $g_\infty([0,s))$ lies in $\mathrm{Teich}(S) \cup \mathscr{S}_\sigma$ by Theorem 2.3. $\square$

4 Ending laminations and recurrent geodesics

The primary goal of this section is to establish Theorem 1.1.

Theorem 1.1. (RECURRENT ENDING LAMINATION THEOREM) *Let $\mathbf{r}$ be an infinite length Weil-Petersson geodesic ray that is recurrent, and let $\lambda(\mathbf{r})$ be its ending lamination. If $\mathbf{r}'$ is any other infinite length geodesic ray with ending lamination $\lambda(\mathbf{r}') = \lambda(\mathbf{r})$, then $\mathbf{r}$ is strongly asymptotic to $\mathbf{r}'$.*

The main technical tool in this section will be the following application of the Gauss-Bonnet theorem.

Theorem 4.1. *Let $\mathbf{r}$ be a recurrent Weil-Petersson geodesic ray. Then if $\mathbf{r}'$ is a ray asymptotic to $\mathbf{r}$ then $\mathbf{r}$ is strongly asymptotic to $\mathbf{r}'$.*

Proof. We wish to harness the fact that the recurrent ray $\mathbf{r}$ returns to a portion of $\mathscr{M}(S)$ where the sectional curvatures are definitely bounded away from 0 by a negative number. To do this we employ the following *simplicial ruled quadrilateral*, a special case of the simplicial ruled surfaces employed initially by Canary in his thesis [Can1] (see also [Bon1, Can2, Sou]).

Given $T > 0$, we construct a simplicial ruled quadrilateral Q_T out of two ruled triangles given by

$$\Delta = g(\mathbf{r}'(0), \mathbf{r}(0)) \cdot \mathbf{r}([0,T]) \cdot g(\mathbf{r}(T), \mathbf{r}'(0))$$

ruled by geodesics γ_t from $\mathbf{r}'(0)$ to $\mathbf{r}(t)$, where $t \in [0,T]$, and

$$\Delta' = \mathbf{r}'([0,T]) \cdot g(\mathbf{r}'(T), \mathbf{r}(T)) \cdot g(\mathbf{r}(T), \mathbf{r}'(0))$$

ruled by geodesics γ'_t from $\mathbf{r}(T)$ to $\mathbf{r}'(t)$, where $t \in [0,T]$.

As in [Can1, Sou] the ambient Riemannian metric induces a smooth metric on Q_T whose curvature is pointwise bounded from above by the upper bound on the ambient sectional curvatures.

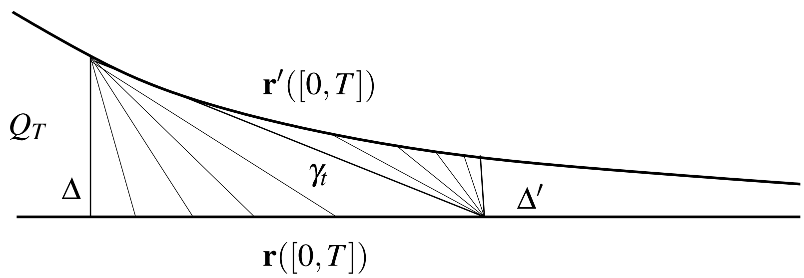

Figure 1. A simplicial ruled quadrilateral Q_T.

By recurrence of $\mathbf{r}(t)$, there is a sequence of times t_n, for which $\mathbf{r}(t_n)$ lies in the ε-thick part, and thus the curvatures at $\mathbf{r}(t_n)$ are bounded above by $\kappa_\varepsilon < 0$.

Assume there is a $\delta > 0$ so that the distance from $\mathbf{r}(t)$ to $\mathbf{r}'(t)$ remains bounded below by δ. Then for each $t_n < T$ there is a definite contribution to the integral

$$-\int_{Q_T} \kappa$$

of the curvature over the quadrilateral Q_T. The exterior angles of Q_T are uniformly bounded above, so by Gauss-Bonnet there is a universal upper bound to the total integral, contradicting the existence of δ. $\square$

Remark: This "simplicial ruled surface" argument, which goes back to Bonahon and Canary has been observed independently by Bestvina and Fujiwara to have applicability to the study of action of $\mathrm{Mod}(S)$ on $\overline{\mathrm{Teich}(S)}$ as the isometry group of a CAT(0)-space (cf. [BeFu]).

We employ the fact that recurrent rays exhibit such strongly asymptotic behavior to conclude Theorem 1.5.

Theorem 1.5. (RECURRENT VISIBILITY) *Let $\mathbf{r}^+$ and $\mathbf{r}^-$ be two distinct infinite rays based at X.*

1. *If $\mathbf{r}^+$ is recurrent, then there is a single bi-infinite geodesic $\mathbf{g}(t)$ so that $\mathbf{g}^+ = \mathbf{g}|_{[0,\infty)}$ is* strongly *asymptotic to $\mathbf{r}^+$ and $\mathbf{g}^- = \mathbf{g}|_{(-\infty,0]}$ is asymptotic to $\mathbf{r}^-$. In particular, if both $\mathbf{r}^+$ and $\mathbf{r}^-$ are recurrent, then $\mathbf{g}$ is strongly asymptotic to both $\mathbf{r}^-$ and $\mathbf{r}^+$.*

2. *If $\mu \in \mathscr{ML}(S)$ has bounded length on $\mathbf{r}^\pm$ then it has bounded length on $\mathbf{g}^\pm$.*

Proof. We seek to exhibit a bi-infinite Weil-Petersson geodesic $\mathbf{g}\colon \mathbb{R} \to \mathrm{Teich}(S)$ with the property that $\mathbf{g}$ is strongly asymptotic to the recurrent ray $\mathbf{r}^+$, in positive time and asymptotic to $\mathbf{r}^-$ in negative time. In other words there is a reparametrization $t \mapsto s(t) > 0$ so that we have

$$d(\mathbf{g}(s(t)), \mathbf{r}^+(t)) \to 0$$

as $t \to \infty$, and

$$d(\mathbf{g}(t), \mathbf{r}^-(-t))$$

is bounded for $t < 0$.

Applying Theorem 1.4 and Theorem 4.1 we may consider geodesic rays $\mathbf{g}_n(t)$ with $\mathbf{g}_n(0) = \mathbf{r}^-(n)$ that are strongly asymptotic to $\mathbf{r}^+(t)$. Given n we may again consider a simplicial ruled quadrilateral $Q_T(n)$ with corners at $\mathbf{r}^+(0)$, $\mathbf{r}^+(T)$, $\mathbf{g}_n(0)$ and $\mathbf{g}_n(T)$ for each $T > 0$. Once again, the integral of the curvature is uniformly bounded in absolute value as above, and thus there is a uniform upper bound for the value $d > 0$ for which each point in $\mathbf{r}^+([0,d])$ lies at distance at least δ from $\mathbf{g}_n(t)$, where d is independent of n. Then $\mathbf{r}^+(d)$ lies within δ of some point $\mathbf{g}_n(t_{d,n})$ for each $n > 0$. Taking δ small enough so that $\mathbf{r}^+(d)$ has a precompact δ-neighborhood in $\mathrm{Teich}(S)$, the points $\mathbf{g}_n(t_{d,n})$ converge within $\mathrm{Teich}(S)$ to a limit Z after passing to a subsequence.

By a further application of Theorem 1.4, there are infinite rays $h^+(t)$ and $h^-(t)$ based at Z so that $h^-(t)$ is asymptotic to $\mathbf{r}^-(t)$ and $h^+(t)$ is strongly asymptotic to $\mathbf{r}^+(t)$. The convergence of $\mathbf{g}_n(t_d)$ to Z guarantees that the limiting directions of $\mathbf{g}_n$ at Z converge to the initial tangent directions of h^+ and h^- up to sign, so their concatenation is the desired bi-infinite geodesic $\mathbf{g}(t)$.

For statement (2), we note that the forward ray $\mathbf{g}^+$ is also the limit of geodesics $\mathbf{g}_n^+$ joining $\mathbf{g}^+(0)$ to $\mathbf{r}^+(n)$, so if μ has bounded length along $\mathbf{r}^+$, then convexity of the length of μ guarantees that the length of μ is uniformly bounded on $\mathbf{g}_n^+$. Each point on the ray $\mathbf{g}^+$ is a limit of $\mathbf{g}_n^+(t_n)$ for some collection $\{t_n\}$, so by continuity of the length of μ on $\mathrm{Teich}(S)$ we have a length bound for μ along all of $\mathbf{g}^+$. $\square$

In Section 2, we employed the boundedness of ending measures along a ray to establish that the ending lamination is well defined. For a recurrent ray, however, we can guarantee that the length of any lamination with bounded length decays to zero.

Lemma 4.2. *Let $\mathbf{r}(t)$ be a recurrent ray, and let $\mu \in \mathscr{ML}(S)$ be any lamination with $\ell_{\mathbf{r}(t)}(\mu) < K$ along $\mathbf{r}(t)$. Then we have*

$$\ell_{\mathbf{r}(t)}(\mu) \to 0$$

as $t \to \infty$.

Proof. Assume $\mathbf{r}(t)$ recurs to the ε-thick part at times $t_n \to \infty$. Wolpert's extension of his convexity theorem for geodesic length functions guarantees that the length of $\mu \in \mathscr{ML}(S)$, in addition to being convex along geodesics [Wol4], satisfies the following stronger convexity property: given $\varepsilon > 0$, there is a $c > 0$ so that at each t for which $\mathbf{r}(t)$ lies in the ε-thick part, we have

$$\ell''_{\mathbf{r}(t)}(\mu) > c\,\ell_{\mathbf{r}(t)}(\mu)$$

(see [Wol6]). The proof of the Lemma then follows from the observation that if the bounded convex function $\ell_{\mathbf{r}(t)}(\mu)$ does not tend to zero, then we nevertheless have $\ell_{\mathbf{r}(t)}(\mu) \to C > 0$ as $t \to \infty$, which guarantees that $\ell''_{\mathbf{r}(t)}(\mu) \to 0$ by convexity. This contradicts the above inequality at the times t_n for n sufficiently large. $\square$

Though the ending lamination need not fill the surface in general, the recurrent rays provide a class of rays where μ fills S.

Proposition 4.3. *Let μ be any measured lamination with bounded length along the recurrent ray $\mathbf{r}(t)$. Then μ is a filling lamination.*

Proof. Assume μ does not fill, and let $S(\mu)$ be the supporting subsurface for its support $|\mu|$. Let $\gamma_n \in \mathscr{C}(S(\mu))$ be a sequence of simple closed curves whose projective classes $[\gamma_n]$ converge to $[\mu]$ in $\mathscr{PML}(S)$. Note in particular that

$$i(\partial S(\mu), \gamma_n) = 0$$

for each n.

By an application of Lemma 2.8 given any $Z \in \text{Teich}(S(\mu))$ there is a Weil-Petersson ray $\hat{\mathbf{r}}$ in $\text{Teich}(S(\mu))$ based at Z along which μ has bounded length: to see this, note that that the limit of rays joining Z to nodal surfaces Z_n in $\overline{\text{Teich}(S(\mu))}$ with γ_n pinched has the property that $[\mu] = \lim_{n\to\infty}[\gamma_n]$ is the projective class of a lamination with bounded length on any limit $\hat{\mathbf{r}}$, of the finite segments $g(Z, Z_n)$ in $\overline{\text{Teich}(S(\mu))}$, by Lemma 2.8, and the fact that μ fills $S(\mu)$ guarantees that $\hat{\mathbf{r}}$ has no pinching curves. Thus $\hat{\mathbf{r}}$ has infinite length.

Letting $\sigma_\mu \in \widehat{\mathscr{C}(S)}$ be the simplex spanned by the curves in $\partial S(\mu)$ we note that the stratum $\mathscr{S}_{\sigma_\mu}$ is the metric product of Weil-Petersson metrics on $\text{Teich}(S(\mu))$ and the Weil-Petersson metrics on $\text{Teich}(Y)$ where Y is the disjoint union of non-annular components of $S \setminus S(\mu)$.

Together with the basepoint X, then, the ray $\hat{\mathbf{r}}$ naturally determines a ray $\overline{\mathbf{r}}$ in the stratum $\mathscr{S}_{\sigma_\mu}$ by taking the projection of $\overline{\mathbf{r}(t)}$ to $\text{Teich}(S(\mu))$ to be $\hat{\mathbf{r}}(t)$ and identifying each other coordinate of $\overline{\mathbf{r}}(t)$ in the product decomposition of $\mathscr{S}_{\sigma_\mu}$ with the (constant) coordinate function of the nearest point projection of X to $\mathscr{S}_{\sigma_\mu}$.

Applying Theorem 3.5, there is a unique ray $\mathbf{r}'$ based at X asymptotic to $\overline{\mathbf{r}}$. The ray $\mathbf{r}'$ is constructed as a limit of segments $g_t = g(X, \overline{\mathbf{r}}(t))$ joining X to points along $\overline{\mathbf{r}}$. The length of μ and each curve $\gamma \subset \partial S(\mu)$ is uniformly bounded on the segments $g(X, \overline{\mathbf{r}}(t))$, by convexity of length functions. Applying continuity of length, then, that we have a $K > 0$ so that

$$\ell_{\mathbf{r}'(t)}(\mu) < K \quad \text{and} \quad \ell_{\mathbf{r}'(t)}(\gamma) < K$$

for each $\gamma \in \partial S(\mu)$.

If $\mathbf{r}'$ is distinct from $\mathbf{r}$, however, Theorem 1.5 guarantees that we may find a bi-infinite geodesic $\mathbf{g}(t)$ that is strongly asymptotic to $\mathbf{r}(t)$, for $t > 0$, by recurrence, and so that $\mathbf{g}|_{(-\infty,0]}$ stays a bounded distance from $\mathbf{r}'(t)$. Once again, the length of μ is uniformly bounded over the entire bi-infinite geodesic $\mathbf{g}$, a contradiction. We conclude that $\mathbf{r} = \mathbf{r}'$ and thus that γ has bounded length along the ray $\mathbf{r}(t)$. But by Lemma 4.2, this boundedness implies that the length of γ tends to zero along $\mathbf{r}(t)$, violating recurrence of $\mathbf{r}(t)$.

We conclude that μ fills S. $\square$

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let $\mathbf{r}$ be based at $X \in \text{Teich}(S)$ with ending lamination $\lambda = \lambda(\mathbf{r})$. Let Σ denote the simplex of projective classes of measures on λ in $\mathscr{PML}(S)$, and let $\mu \in \mathscr{ML}(S)$ be a representative of the projective class determined by a point in the interior of the top dimensional face. Then μ is a positive linear combination of all ergodic measures on λ.

Let γ_n be a sequence of simple closed curves for which the projective classes $[\gamma_n]$ converge to $[\mu]$, and let $\mathbf{r}_n$ be a sequence of finite rays based at X limiting to points Z_n in the strata $\mathscr{S}_{\gamma_n}$. Since γ_n are pinching curves for $\mathbf{r}_n$, Lemma 2.8 guarantees that any limit $\mathbf{r}_\infty$ of $\mathbf{r}_n$ has the property that μ has bounded length along $\mathbf{r}_\infty$. Since μ is a positive linear combination of all the ergodic measures on λ, it follows that each ergodic measure on λ has bounded length along $\mathbf{r}_\infty$. Hence, any measured lamination representing a projective class in Σ has bounded length along $\mathbf{r}_\infty$ since each is a linear combination of ergodic measures.

Since $\lambda(\mathbf{r})$ is filling, by Proposition 4.3, we have that μ is filling. This guarantees that $\mathbf{r}_\infty$ has infinite length, since otherwise $\mathbf{r}_\infty$ would have a pinching curve γ with $i(\gamma,\mu) > 0$, violating the length bound on μ along $\mathbf{r}_\infty$.

If $\hat{\mu}$ is any ending measure for $\mathbf{r}$, then $\hat{\mu}$ represents a projective class in Σ, and thus has bounded length along $\mathbf{r}_\infty$. If $\mathbf{r}$ and $\mathbf{r}_\infty$ are distinct rays, then Theorem 1.5 guarantees that we have a bi-infinite geodesic $\mathbf{g}(t)$ asymptotic to $\mathbf{r}$ and $\mathbf{r}_\infty$ along which $\hat{\mu}$ has bounded length, a contradiction. It follows that $\mathbf{r} = \mathbf{r}_\infty$.

If $\mathbf{r}'$ is another ray based at $Y \in \mathrm{Teich}(S)$ with ending lamination λ, the same argument applied to finite rays $\mathbf{r}'_n$ joining Y to Z_n shows that $\mathbf{r}'$ is the limit of $\mathbf{r}'_n$. But if $D = d(X,Y)$, then $\mathbf{r}_n$ and $\mathbf{r}'_n$ have the property that

$$d(\mathbf{r}_n(t), \mathbf{r}'_n(t)) \le D$$

by the CAT(0) comparison property. It follows that the limits $\mathbf{r}_\infty$ and $\mathbf{r}'_\infty$ are asymptotic, and thus that $\mathbf{r}$ and $\mathbf{r}'$ are as well.

Applying Theorem 4.1, we conclude that $\mathbf{r}$ and $\mathbf{r}'$ are strongly asymptotic, concluding the proof. $\square$

As a further consequence, we note the following.

Corollary 4.4. *Let $\mathbf{g}(t)$ be a bi-infinite Weil-Petersson geodesic whose forward trajectory is recurrent. Then the ending laminations λ^+ and λ^- for the rays $\mathbf{g}^+ = \{\mathbf{g}(t)\}_{t=0}^{\infty}$ and $\mathbf{g}^- = \{\mathbf{g}(t)\}_{t=0}^{-\infty}$ bind the surface S.*

Proof. The ending lamination λ^+ for the forward trajectory fills the surface, so the ending lamination for the backward trajectory must intersect it, since otherwise the laminations λ^- and λ^+ would be identical therefore we would have $\mathbf{g}^+ = \mathbf{g}^-$ by Theorem 1.1, a contradiction. $\square$

To derive Corollary 1.6, we establish a final further continuity property for ending measures when the limit is recurrent.

Proposition 4.5. *If $\mathbf{r}_n$ is a convergent sequence of rays with a recurrent limit $\mathbf{r}$, any sequence μ_n of ending measures or pinching curves for $\mathbf{r}_n$ converges in $\mathscr{PML}(S)$ up to subsequence to a measure on $\lambda(\mathbf{r})$.*

Proof. Let μ be any limit of μ_n in $\mathscr{PML}(S)$ after passing to a subsequence. Then by Lemma 2.8, the length $\ell_{\mathbf{r}(t)}(\mu)$ is bounded. Since $\mathbf{r}$ is recurrent, any ending measure μ' for $\mathbf{r}$ fills S by Proposition 4.3. But by Corollary 2.10, we have

$$i(\mu, \mu') = 0$$

so μ and μ' have identical support since μ' is filling. Hence, μ is a measure on $\lambda(\mathbf{r})$. $\square$

Restricting to the recurrent rays, we obtain Corollary 1.6.

Corollary 1.6. *The recurrent rays are parametrized by their ending laminations: the map λ that associates to an equivalence class of recurrent rays their ending lamination is a homeomorphism to the subset $\mathscr{REL}(S)$ in $\mathscr{EL}(S)$.*

Proof. That the map is an bijection follows from the fact that $\mathscr{REL}(S)$ is defined as its image and from Theorem 1.1.

To show continuity in each direction, we begin by noting that although the topology induced by forgetting the measure on a measure lamination is not a Hausdorff topology on the geodesic laminations admitting measures, it is Hausdorff when one restricts to those that fill the surface, namely, the subset $\mathscr{EL}(S)$. As such it suffices to consider sequential limits to establish continuity.

Let $\mathbf{r}_n$ be a sequence of recurrent rays with recurrent limit $\mathbf{r}$. By Proposition 4.3, their ending laminations λ_n are filling laminations and thus determine points in $\mathscr{REL}(S)$. Their recurrent limit $\mathbf{r}$ has ending lamination $\lambda(\mathbf{r})$, with support identified with the support of a limiting measure of measures on λ_n by Proposition 4.5, so λ is the limit of λ_n in $\mathscr{REL}(S)$, by the definition of the topology on $\mathscr{EL}(S)$.

For continuity in the other direction, compactness of the visual sphere guarantees that any convergent family of laminations λ_n converging to λ_∞ in $\mathscr{REL}(S)$ determine a sequence of rays $\mathbf{r}_n$ with limit $\mathbf{r}_\infty$ after passing to a subsequence. A convergent family of measures μ_n on λ_n has limit μ_∞, a measure on λ_∞, with bounded length on the limiting ray $\mathbf{r}_\infty$ by Lemma 2.8. Since μ_∞ is filling, and any ending measure or weighted pinching curve μ' for $\mathbf{r}_\infty$ satisfies $i(\mu, \mu') = 0$, we conclude that μ' has the same support as μ_∞, namely λ_∞. Thus $\mathbf{r}_\infty$ is the recurrent ray determined (uniquely) by λ_∞. Since any accumulation point of the rays $\mathbf{r}_n$ has this property, the original sequence of rays itself was convergent to $\mathbf{r}_\infty$, obviating passage to subsequences. $\square$

Finally, we address the asymptotic behavior of length functions along a recurrent ray.

Theorem 4.6. *Let* $\mathbf{r}(t)$ *be a recurrent ray with ending lamination* $\lambda(\mathbf{r})$*. Then every measure* μ *on* $\lambda(\mathbf{r})$ *has the property*

$$\ell_{\mathbf{r}(t)}(\mu) \to 0.$$

Proof. By the proof of Theorem 1.1, every measure on the ending lamination $\lambda(\mathbf{r})$ is bounded along the ray. The theorem follows as an application of Lemma 4.2. $\square$

5 The topological dynamics of the geodesic flow

We now relate the preceding results to the study of the Weil-Petersson geodesic flow on the unit tangent bundle of the Moduli space.

Though it is seen in [Br2] that the change of basepoint map is discontinuous on the visual sphere, the visibility property for recurrent rays (Theorem 1.5) is sufficient to remedy the situation for considerations of topological dynamics, yielding Theorem 1.8, whose proof we now supply.

Theorem 1.8. (CLOSED ORBITS DENSE) *The closed orbits of the geodesic flow are dense in* $T^1\mathcal{M}(S)$*.*

Proof. Because of the density of bi-recurrent geodesics in the unit tangent bundle to moduli space, it suffices by a diagonal argument to approximate a bi-recurrent direction with closed geodesics.

To this end, let $\{\mathbf{g}(t)\}_{t=-\infty}^{\infty}$ be a bi-infinite geodesic that is bi-recurrent. Let $X = \mathbf{g}(0)$ be a basepoint, and let λ^+ be the ending lamination for the forward ray $\mathbf{g}^+(t) = \{\mathbf{g}(t)\}_{t=0}^{\infty}$ and likewise let λ^- denote the ending lamination for the backward ray $\mathbf{g}^-(t) = \{\mathbf{g}(-t)\}_{t=0}^{\infty}$.

By Corollary 4.4, λ^+ and λ^- bind the surface S, so letting μ^+ and μ^- be measures on λ^+ and λ^-, respectively, any pair of simple closed curves γ^+ and γ^- very close to μ^+ and μ^- in $\mathcal{PML}(S)$ also bind S.

Letting τ_+ be a Dehn twist about γ^+ and τ_- be a Dehn twist about γ^-, the composition

$$\psi_k = \tau_+^k \circ \tau_-^k$$

is pseudo-Anosov for all k sufficiently large [Th3]. Furthermore, the stable and unstable laminations for ψ_k converge to γ^+ and γ^- in $\mathcal{PML}(S)$ as $k \to \infty$. Diagonalizing, then, we obtain a sequence of pseudo-Anosov mapping classes φ_n whose unstable and stable laminations μ_n^+ and μ_n^- converge to μ^+ and μ^- in $\mathcal{PML}(S)$. Since the supports $|\mu_n^{\pm}|$ and $|\mu^{\pm}|$ lie in $\mathcal{REL}(S)$, we have convergence of $|\mu_n^{\pm}|$ to $\lambda^{\pm}$ in $\mathcal{REL}(S)$ by the definition of the topology on $\mathcal{EL}(S)$.

Letting A_n be the axis for φ_n, we claim A_n is arbitrarily close to $\mathbf{g}$ at $\mathbf{g}(0)$ in the unit tangent bundle for n sufficiently large.

To see this, we apply Theorem 1.4 to obtain a ray $\mathbf{r}_n^+$ in $\mathscr{V}_X(S)$ asymptotic to A_n in the forward direction. We note that, as A_n is itself bi-recurrent, the ray $\mathbf{r}_n^+$ is strongly asymptotic to A_n, by Theorem 4.1, and that the ending lamination λ_n^+ for $\mathbf{r}_n^+$ is equal to the support of μ_n^+. It follows that λ_n^+ converges to λ^+ in $\mathscr{REL}(S)$. Likewise, if $\mathbf{r}_n^-$ denotes the ray in $\mathscr{V}_X(S)$ asymptotic to A_n in the negative direction, then $\lambda_n^- = \lambda(\mathbf{r}_n^-)$ converges to λ^- in $\mathscr{REL}(S)$.

The parametrization of recurrent rays by their ending laminations in $\mathscr{EL}(S)$, Corollary 1.6, guarantees that $\mathbf{r}_n^+$ and $\mathbf{r}_n^-$ converge to $\mathbf{g}^+$ and $\mathbf{g}^-$ respectively.

We claim that the bi-recurrence of $\mathbf{g}$ guarantees that for n sufficiently large $\mathbf{r}_n^+$ and $\mathbf{r}_n^-$ themselves recur to the thick part sufficiently so that the axis A_n lies within uniformly bounded distance of the basepoint X. To see this, we note that because the rays $\mathbf{r}_n^+$ and $\mathbf{r}_n^-$ are converging to the recurrent rays $\mathbf{g}^+$ and $\mathbf{g}^-$, for each segment along $\mathbf{g}^+$, say, there is a segment along $\mathbf{r}_n^+$ close to it for all n sufficiently large. Thus, if $\mathbf{g}^+$ enters the 2ε-thick part at some point $\mathbf{g}^+(t)$, then $\mathbf{r}_n^+$ will eventually encounter the ε-thick part for each n sufficiently large (and likewise for $\mathbf{g}^-$ and $\mathbf{r}_n^-$). In particular, by recurrence of $\mathbf{g}^+$ and $\mathbf{g}^-$ we have positive $\varepsilon > 0$ and δ so that for each integer $m > 0$ there is a $T > 0$, and an integer N so that for all $n > N$, both rays $\mathbf{r}_n^+$ and $\mathbf{r}_n^-$ recur to the ε-thick part for m segments of length at least δ within time T of X.

Applying the ruled quadrilateral argument of Theorem 4.1 to the two quadrilaterals made up of the segments $g(X, \mathbf{r}_n^+(T))$, $g(X, \mathbf{r}_n^-(T))$, the nearest point projection paths joining X, $\mathbf{r}_n^+(T)$, and $\mathbf{r}_n^-(T)$ to their nearest points Z_n^0, Z_n^+ and Z_n^- on A_n, and the segments $g(Z_n^-, Z_n^0)$ and $g(Z_n^0, Z_n^+)$ (see figure 2), we again conclude that there are times $t_d^\pm$ so that

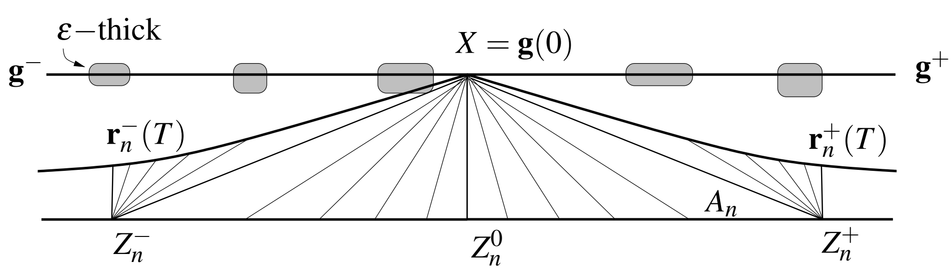

Figure 2. The axis A_n converges to $\mathbf{g}$.

$\mathbf{r}_n^+(t_d^+)$ and $\mathbf{r}_n^-(t_d^-)$ lie within a uniformly bounded distance of A_n for all n sufficiently large.

It follows that we may extract a limit A_∞ of the axes A_n. Since A_n is strongly asymptotic to $\mathbf{r}_n^+$ and $\mathbf{r}_n^-$, the limit A_∞ is strongly asymptotic to $\mathbf{g}^+$ and $\mathbf{g}^-$, guaranteeing that $A_\infty = \mathbf{g}$.

Thus, the projections of A_n to $\mathscr{M}(S)$ are closed geodesic approximating the bi-recurrent projection of $\mathbf{g}$ to $\mathscr{M}(S)$, as was desired. $\square$

Using the boundary theory for the recurrent rays and its connection with measured laminations, we can harness the north-south dynamics of pseudo-Anosov elements on $\mathscr{PML}(S)$ to establish Theorem 1.9 as a consequence of Theorem 1.8.

Theorem 1.9. (DENSE GEODESICS) *Given any $X \in \mathrm{Teich}(S)$, there is a Weil-Petersson geodesic ray based at X whose projection to $T^1\mathscr{M}(S)$ is dense.*

Remark: With a tractable boundary theory in place, arguments for topological transitivity from the density of closed orbits are standard. The proof, which follows these general lines, shows no complications arise in our setting.

Proof. Given $X \in \mathrm{Teich}(S)$, a positive δ, and a pseudo-Anosov mapping class ψ, we have from Corollary 1.6 that there is a neighborhood $U_\delta(\psi) \subset \mathscr{REL}(S)$ of the support λ^+ of the attracting lamination μ^+ for ψ so that if $\lambda' \in U_\delta(\psi)$ is the support of any other pseudo-Anosov fixed point in $\mathscr{PML}(S)$, then the ray $\mathbf{r}$ in $\mathscr{V}_X(S)$ with ending lamination λ' comes within δ of the axis A_ψ of ψ for a full period $g_{\psi,Z} = g(Z, \psi(Z))$, for some $Z \in A_\psi$, of the action of ψ on A_ψ.

Thus we may argue by induction. Let $\{\psi_n\}_{n=1}^\infty \subset \mathrm{Mod}(S)$ be a family of pseudo-Anosov elements whose corresponding closed geodesics on $\mathscr{M}(S)$ form a dense family in $T^1\mathscr{M}(S)$, and let $X \in \mathrm{Teich}(S)$ be a basepoint. Let $\delta_n \to 0$ be given so that the δ_n neighborhood of the axis A_n for ψ_n lies entirely within $\mathrm{Teich}(S)$. It suffices to find a geodesic ray $\mathbf{r}$ based at X so that for each n there is a segment along $\mathbf{r}$ lies within δ_n of the axis of some conjugate in $\mathrm{Mod}(S)$ of ψ_n for a full period g_n along the axis.

Assume that for $k > 1$ we have a ray $\mathbf{r}_k$ based at X forward asymptotic to the axis of a suitable conjugate $\hat{\psi}_k$ of ψ_k so that the support $\hat{\lambda}_k^+$ of the attracting lamination of $\hat{\psi}_k$ lies in the intersection

$$V_k = U_{\delta_1}(\hat{\psi}_1) \cap \ldots \cap U_{\delta_{k-1}}(\hat{\psi}_{k-1}).$$

Then for a sufficiently large power p_{k+1}, the support λ_{k+1} of the attracting lamination for ψ_{k+1} has image $\hat{\psi}_k{}^{p_{k+1}}(\lambda_{k+1})$ within V_k. Taking $\mathbf{r}_{k+1}$ to be the ray asymptotic to the axis of the pseudo-Anosov conjugate

$$\hat{\psi}_{k+1} = \hat{\psi}_k^{p_{k+1}} \circ \psi_{k+1} \circ \hat{\psi}_k^{-p_{k+1}}$$

of ψ_{k+1}, we have a ray asymptotic to the axis of a pseudo-Anosov element with attracting lamination in the intersection

$$V_{k+1} = U_{\delta_1}(\hat{\psi}_1) \cap \ldots \cap U_{\delta_k}(\hat{\psi}_k).$$

Thus, $\mathbf{r}_{k+1}$ lies within δ_n of the axis of the conjugate $\hat{\psi}_n$ of ψ_n, $n = 1, \ldots, k+1$, for a full period g_n along the axis of each. This completes the induction.

Thus any limit $\mathbf{r}_\infty$ of $\mathbf{r}_k$ as $k \to \infty$ in the visual sphere at X will have a dense trajectory in its projection $T^1\mathscr{M}(S)$, provided, once again, that it is an infinite ray. But $\mathbf{r}_k$ passes within δ_n of the axis $\hat{A}_n$ of $\hat{\psi}_n$ at the segment $g_n \subset \hat{A}_n$, for each $k > n$, so the closest points $\mathbf{r}_k(t_n)$ to g_n range in a compact neighborhood of a bounded interval along $\hat{A}_n$. Thus, given $T > 0$, and n so that $t_n > T$, the segments $\mathbf{r}_k([0,T])$ sit as a subsegments in a family of segments $\mathbf{r}_k(t_n)$ whose endpoints converge in $\mathrm{Teich}(S)$ as $k \to \infty$. Thus, the sequence of geodesics $\mathbf{r}_k([0,T])$ converges to a geodesic in Teichmüller space for each T, by geodesic convexity of $\mathrm{Teich}(S)$ [Wol4]. It follows that the limit $\mathbf{r}_\infty$ is infinite and projects to a dense subset of $T^1\mathscr{M}(S)$ as was claimed. $\square$

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