PRACTICE PROBLEMS
MATH 320
11/20/14

Consider the following linear transformations.

(1) $T(x, y) = (-x, y)$.
(2) $T(x, y) = (y, x)$.
(3) $T(x, y) = (-y, x)$.
(4) $T(x, y) = (0, y)$.
(5) $T(x, y) = (x - y, x + y)$.
(6) $T(x, y) = (x + y, y)$.
(7) $T(x, y) = (2x, 2y)$.
(8) $T(x, y) = (2x, \frac{1}{2}y)$.

For each of these linear transformations, complete the following:

(a) Describe the transformation geometrically; draw the image of the unit square (i.e. the square with vertices (0, 0), (1, 0), (0, 1), and (1, 1)) under the transformation.
(b) Compute the matrix M of T relative to the standard basis $\mathcal{S}$ and the matrix N of T relative to the basis $\mathcal{B} = \{(-1, -1), (2, -1))\}$.
(c) Find a matrix P such that $P^{-1}MP = N$.
(d) For a vector $\alpha = (a, b)$, compute $[\alpha]_{\mathcal{B}}$ and $[T(\alpha)]_{\mathcal{B}}$.
(e) Find all eigenvalues and the corresponding eigenvectors for each linear transformation.
(f) Compute the determinant of each linear transformation and decide whether each T is invertible. If so compute T^{-1} and describe it geometrically as in (a). For additional practice, complete the rest of the above steps for each T^{-1}.

Solutions

First, $(a,b) = \frac{-a-2b}{3}(-1,-1) + \frac{a-b}{3}(2,-1)$, so $[\alpha]_{\mathcal{B}} = (\frac{-a-2b}{3}, \frac{a-b}{3})$. Also the matrix P is the change of basis matrix and is the same for each problem. So P is the matrix with columns $[(-1,-1)]_{\mathcal{S}}$ and $[(2,-1)]_{\mathcal{S}}$, that is $P = \begin{pmatrix} -1 & 2 \\ -1 & -1 \end{pmatrix}$. Note that $P[\alpha]_{\mathcal{B}} = (a,b)$. Also P^{-1} is the matrix with columns $[(1,0)]_{\mathcal{B}}$ and $[(0,1)]_{\mathcal{B}}$, so $P^{-1} = \begin{pmatrix} -\frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{pmatrix}$ and $P^{-1}\begin{pmatrix} a \\ b \end{pmatrix} = [\alpha]_{\mathcal{B}}$.

$T(x,y) = (0,y)$ is not invertible, the rest of these linear transformations are invertible.

(1) T is a reflection across the y-axis. $M = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $N = \begin{pmatrix} \frac{1}{3} & \frac{4}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{pmatrix}$. $[T(\alpha)]_{\mathcal{B}} = N[\alpha]_{\mathcal{B}} = (\frac{a-2b}{3}, \frac{-a-b}{3})$.

(2) T is a reflection across the line $y = x$. $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $N = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$. $[T(\alpha)]_{\mathcal{B}} = N[\alpha]_{\mathcal{B}} = (\frac{-2a-b}{3}, \frac{b-a}{3})$.

(3) T is a counter-clockwise rotation by 90°. $M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $N = \begin{pmatrix} \frac{1}{3} & -\frac{5}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{pmatrix}$. $[T(\alpha)]_{\mathcal{B}} = N[\alpha]_{\mathcal{B}} = (\frac{-2a+b}{3}, \frac{-a-b}{3})$.

(4) T is a projection onto the y-axis. $M = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $N = \begin{pmatrix} \frac{2}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}$. $[T(\alpha)]_{\mathcal{B}} = N[\alpha]_{\mathcal{B}} = (\frac{-4b}{3}, \frac{-b}{3})$.

(5) T rotates a vector by 45° and then scales it by $\sqrt{2}$. $M = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, $N = \begin{pmatrix} \frac{4}{3} & \frac{2}{3} \\ -\frac{5}{3} & \frac{2}{3} \end{pmatrix}$. $[T(\alpha)]_{\mathcal{B}} = N[\alpha]_{\mathcal{B}} = (\frac{-6a-10b}{9}, \frac{3a+8b}{9})$.

(6) T is a horizontal shear. $M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $N = \begin{pmatrix} \frac{4}{3} & \frac{1}{3} \\ 1 & 0 \end{pmatrix}$. $[T(\alpha)]_{\mathcal{B}} = N[\alpha]_{\mathcal{B}} = (\frac{-5a-9b}{9}, \frac{-a-2b}{3})$.

(7) T scales each vector by a factor of 2. $M = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, $N = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. $[T(\alpha)]_{\mathcal{B}} = N[\alpha]_{\mathcal{B}} = (\frac{-2a-4b}{3}, \frac{2a-2b}{3})$.

(8) T stretches the horizontal component of a vector by 2 and squeezes the vertical component by $\frac{1}{2}$. $M = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$, $N = \begin{pmatrix} 1 & -1 \\ \frac{5}{6} & -\frac{7}{6} \end{pmatrix}$. $[T(\alpha)]_{\mathcal{B}} = N[\alpha]_{\mathcal{B}} = (\frac{-b}{3}, \frac{2a+21b}{18})$.