MATH 320 **Quiz 1** **Name:**________________________

1. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a linear function, that is for all $x, y, a \in \mathbb{R}$, $f(x+y) = f(x) + f(y)$ and $f(ax) = af(x)$. Prove that there exists $m \in \mathbb{R}$ such that for all $x \in \mathbb{R}$,

$$f(x) = mx$$

Let $f\colon \mathbb{R} \to \mathbb{R}$ be a linear function, and let $m = f(1)$. Then for all $x \in \mathbb{R}$,

$$f(x) = f(x \cdot 1) = xf(1) = mx.$$

2. Solve the system of equations:

$$\begin{aligned} 3x + y &= 1 \\ 2x + 3y &= -1 \end{aligned}$$

Row-reduce the augmented matrix:

$$\begin{pmatrix} 3 & 1 & 1 \\ 2 & 3 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & \frac{4}{7} \\ 0 & 1 & -\frac{5}{7} \end{pmatrix}$$

Hence the only solution is $x = \frac{4}{7}$, $y = -\frac{5}{7}$.

MATH 320 **Quiz 2** **Name:**__________________________

1. Prove that the following two systems of equations are *not* equivalent:

System 1:

$$
\begin{aligned}
3x + y &= 0 \\
2x + 3y &= 0
\end{aligned}
$$

System 2:

$$
\begin{aligned}
5x + 4y &= 0 \\
x + \frac{4}{5}y &= 0
\end{aligned}
$$

$x = 4$, $y = -5$ is a solution to System 2 but not to System 1. Since equavalent systems of equations have the same solutions, these systems are not equivalent.

MATH 320 **Quiz 3** **Name:**__________________________

1. Let

$$A = \begin{pmatrix} 6 & -4 & 0 \\ 4 & -2 & 0 \\ -1 & 0 & 3 \end{pmatrix}$$

Find all solutions of $AX = 3X$.

The corresponding system of equations is

$$\begin{aligned} 6x_1 - 4x_2 &= 3x_1 \\ 4x_1 - 2x_2 &= 3x_2 \\ -x_1 + 3x_3 &= 3x_3 \end{aligned}$$

Which is the same as the homogeneous system

$$\begin{aligned} 3x_1 - 4x_2 &= 0 \\ 4x_1 - 5x_2 &= 0 \\ -x_1 &= 0 \end{aligned}$$

This implies $x_1 = x_2 = 0$ and x_3 is arbitrary. Hence the set of solutions to this system of equations is $\{(0, 0, s) \mid s \in \mathbb{R}\}$.

MATH 320 **Quiz 4** **Name:**____________________________

1. Row-reduce the coefficient matrix of the following system, then find all solutions to the system.

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ 3x_1 - x_2 + x_3 &= 0 \\ x_1 - 3x_2 + 3x_3 &= 0 \end{aligned}$$

The coefficient matrix row-reduces to:

$$\begin{pmatrix} 1 & 1 & 1 \\ 3 & -1 & 1 \\ 1 & -3 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

So this system has only the trivial solutions $x_1 = x_2 = x_3 = 0$.

MATH 320 **Quiz 5** **Name:**____________________________

1. Consider the system of equations:

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 1 \\ 2x_1 \qquad + 2x_3 &= 1 \\ x_1 - 3x_2 + 4x_3 &= 2 \end{aligned}$$

   Does this system have a solution? If so, describe explicitly all solutions.

The augmented matrix row-reduces to:

$$\begin{pmatrix} 1 & -1 & 2 & 1 \\ 2 & 0 & 2 & 1 \\ 1 & -3 & 4 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{2} \end{pmatrix}$$

Hence the system has a solutions, and the only solutions is $x_1 = 0$, $x_2 = 0$, $x_3 = \frac{1}{2}$.

MATH 320 **Quiz 6** **Name:**________________________

1. Give an example of a system of two linear equations in two unknowns which has no solution.

$$\begin{aligned} x_1 + x_2 &= 1 \\ x_1 + x_2 &= 0 \end{aligned}$$

2. Give an example of a 2×2 matrix $A \neq 0$ for which $A^2 = 0$.

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; A^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

MATH 320 **Quiz 7** **Name:**__________________________

1. Let $A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{pmatrix}$. Find elementary matrices $E_1, ..., E_k$ such that

$$E_k ... E_1 A = I$$

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}$$

$$E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$E_4 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$E_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix}$$

$$E_6 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$E_7 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$E_8 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

MATH 320 **Quiz 8** **Name:**__________________________

1. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Prove, using elementary row operations, that A is invertible if and only if $(ad - bc) \neq 0$.

First suppose $a \neq 0$. Then applying elementary row operations, we obtain

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow \begin{pmatrix} a & b \\ ac & ad \end{pmatrix} \longrightarrow \begin{pmatrix} a & b \\ 0 & ad - bc \end{pmatrix}$$

Note that $\begin{pmatrix} a & b \\ 0 & ad - bc \end{pmatrix}$ can be row reduced to the identity matrix I if and only if $ad - bc \neq 0$. Since A is invertible if and only if A is row equivalent to I, we get that A is invertible if and only if $ad - bc \neq 0$.

Now suppose $a = 0$. If $c = 0$, then we get that $ad - bc = 0$ and A is not invertible since it has a column of $0's$. If $c \neq 0$, then we can switch the two rows of A and apply the same argument as above to get that A is invertible if and only if $ad - bc \neq 0$.

MATH 320 **Quiz 9** **Name:**________________________

1. Let A be a non-invertible $n \times n$ matrix. Prove that there exists an $n \times n$ matrix $B \neq 0$ such that $BA = 0$.

Let R be a row-reduced echelon matrix which is row equivalent to A. Since A is not invertible, $R \neq I$, in particular the last row of R contains only 0's. Let $C = (c_{ij})$ be an $n \times n$ matrix where $c_{ij} = 0$ if $j < n$ and $c_{ij} = 1$ if $j = n$. Since the last row of R contains only 0's and every entry outside the last column of C is 0, we get that $CR = 0$. Since A is row equivalent to R, there exist elementary matrices $E_1, ..., E_k$ such that $R = E_k ... E_1 A$. Let $B = CE_k ... E_1$. Then

$$BA = CE_k ... E_1 A = CR = 0$$

Finally, we show that $B \neq 0$. Since elementary matrices are invertible, if $B = 0$, then we get $C = 0E_1^{-1} ... E_k^{-1} = 0$, but this contradicts the fact that $C \neq 0$. Therefore, $B \neq 0$.

MATH 320 **Quiz 10** **Name:**________________________

1. True/False

   (a) $(AB)^{-1} = B^{-1}A^{-1}$.

   True

   (b) If $AB = 0$ and $B \neq 0$, then A is not an invertible matrix.

   True

   (c) $\begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is an elementary matrix.

   False

   (d) $AX = 0$ has non-trivial solutions if and only if A is row-equivalent to the identity matrix.

   False

   (e) Every invertible matrix is equal to a product of elementary matrices.

   True

MATH 320 **Quiz 11** **Name:**________________________

1. On $\mathbb{R}^n$, define the operations

$$\alpha \oplus \beta = \alpha - \beta$$

$$c \cdot \alpha = -c\alpha$$

Below are listed the vector space axioms for vector addition and scalar multiplication. Next to each axiom, write "yes" if the axiom is satisfied by the above operations and "no" otherwise.

Vector additon:

(a) Associative

No

(b) Commutative

No

(c) Identity

Yes

(d) Inverses

Yes

Scalar multiplication:

(a) Identity

Yes

(b) Associative

No

(c) Distributes over vector addition

No

(d) Distributes over scalar addition

Yes

MATH 320 **Quiz 12** **Name:**________________________

1. Which of the following sets of vectors $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ in $\mathbb{R}^3$ are subspaces of $\mathbb{R}^3$? Show why or why not.

   (a) All α such that $\alpha_1 \geq 0$.

   This is not a subspace, since it contains $\alpha = (1, 0, 0)$ but not $-\alpha = (-1, 0, 0)$.

   (b) All α such that $\alpha_1 + 3\alpha_2 = \alpha_3$.

   This is a subspace, since it is equal to the set of solutions of the homegeneous (system of one) equation $\alpha_1 + 3\alpha_2 - \alpha_3 = 0$.

   (c) All α such that $\alpha_2 = \alpha_1^2$.

   This is not a subspace since it contains $\alpha = (1, 1, 0)$ but not $-\alpha = (-1, -1, 0)$.

   (d) All α such that $\alpha_1\alpha_2 = 0$.

   This is not a subspace since it contains $\alpha = (1, 0, 0)$ and $\beta = (0, 1, 0)$, but not $\alpha + \beta = (1, 1, 0)$.

MATH 320 **Quiz 13** **Name:**__________________________

1. Is the vector $(3, -1, 0, -1)$ in the subspace of $\mathbb{R}^4$ spanned by the vectors $(2, -1, 3, 2)$, $(-1, 1, 1, -3)$, $(1, 1, 9, 5)$?

$(3, -1, 0, -1)$ is in the span of $(2, -1, 3, 2)$, $(-1, 1, 1, -3)$, and $(1, 1, 9, 5)$ if and only if there exists a solution to the system of equations

$$\begin{pmatrix} 2 & -1 & 1 \\ -1 & 1 & 1 \\ 3 & 1 & 9 \\ 2 & -3 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 0 \\ -1 \end{pmatrix}$$

Row reducing the augmented matrix gives

$$\begin{pmatrix} 2 & -1 & 1 & 3 \\ -1 & 1 & 1 & -1 \\ 3 & 1 & 9 & 0 \\ 2 & -3 & 5 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & \frac{12}{5} \\ 0 & 1 & 0 & \frac{8}{5} \\ 0 & 0 & 1 & -\frac{1}{5} \\ 0 & 0 & 0 & -7 \end{pmatrix}$$

Which shows that there are no solutions to this system of equations, since there are no solutions to the equation $0 = -7$. Therefore, $(3, -1, 0, -1)$ is not in the span of $(2, -1, 3, 2)$, $(-1, 1, 1, -3)$, and $(1, 1, 9, 5)$.

MATH 320 **Quiz 14** **Name:**________________________

1. Let V be a vector space over F. Given $S \subseteq V$, state the definition of S *is linearly dependent*.

S is linearly dependent if there exist distinct vectors $\alpha_1, ..., \alpha_n \in S$ and scalars $c_1, ..., c_n \in F$ with at least one $c_i \neq 0$, such that

$$c_1\alpha_1 + c_2\alpha_2 + ... + c_n\alpha_n = 0.$$

MATH 320 **Quiz 15** **Name:**________________________

1. Find three vectors in $\mathbb{R}^3$ which are linearly dependent and are such that any two of them are linearly independent.

Let $\alpha = (1, 0, 0)$, $\beta = (0, 1, 0)$, $\gamma = (1, 1, 0)$. Since none of these vectors is equal to a multiple of another, any two are linearly independent. However,

$$\alpha + \beta - \gamma = 0$$

Therefore the set of all three is linearly dependent.

MATH 320 **Quiz 16** **Name:**____________________________

1. Let V be a vector space over a field F. Suppose $\{\alpha, \beta, \gamma\} \subseteq V$ is a linearly independent set of vectors. Prove that $\{\alpha+\beta, \alpha+\gamma, \beta+\gamma\}$ is a linearly independent set of vectors.

Suppose $c_1, c_2, c_3 \in F$ are scalars such that

$$c_1(\alpha + \beta) + c_2(\alpha + \gamma) + c_3(\beta + \gamma) = 0$$

This implies that

$$\begin{aligned} c_1\alpha + c_1\beta + c_2\alpha + c_2\gamma + c_3\beta + c_3\gamma &= 0 \\ (c_1 + c_2)\alpha + (c_1 + c_3)\beta + (c_2 + c_3)\gamma &= 0 \end{aligned}$$

Since $\{\alpha, \beta, \gamma\}$ is linearly independent, this means that

$$\begin{aligned} c_1 + c_2 &= 0 \\ c_1 + c_3 &= 0 \\ c_2 + c_3 &= 0 \end{aligned}$$

Solving this system of equations yields $c_1 = c_2 = c_3 = 0$, and thus $\{\alpha + \beta, \alpha + \gamma, \beta + \gamma\}$ is linearly independent.

MATH 320 **Quiz 17** **Name:**________________________

1. Let V be the vector space over $\mathbb{R}$ of solutions to the homogeneous system of equations:

$$\begin{aligned} x_1 - 2x_2 &= 0 \\ -2x_1 + 4x_2 &= 0 \end{aligned}$$

What is $dim(V)$?

Row-reducing the coefficient matrix gives

$$\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix}$$

Hence x_2 is a free variable and x_1 is a leading variable, so the space of solutions is $\{(2s, s) \mid s \in \mathbb{R}\}$. In particular, a basis for the space of solutions is $\{(2, 1)\}$. Therefore $dim(V) = 1$.

MATH 320 **Quiz 18** **Name:**____________________________

1. Express $(1,0,0)$ as a linear combination of $\alpha_1 = (1,0,-1)$, $\alpha_2 = (1,2,1)$, and $\alpha_3 = (0,-3,2)$.

$$(1,0,0) = \frac{7}{10}(1,0,-1) + \frac{3}{10}(1,2,1) + \frac{1}{5}(0,-3,2).$$

MATH 320 **Quiz 19** **Name:**____________________________

1. Let $\mathcal{B} = \{\alpha_1, \alpha_2\}$ be the basis for $\mathbb{R}^2$ where $\alpha_1 = (0, -1)$, $\alpha_2 = (1, 1)$. What are the coordinates of the vector (a, b) with respect to the basis $\mathcal{B}$?

$$[(a, b)]_{\mathcal{B}} = \begin{pmatrix} a - b \\ a \end{pmatrix}$$

MATH 320 **Quiz 20** **Name:**____________________________

1. Show that the vectors $\alpha_1 = (1,1,0,0)$, $\alpha_2 = (1,0,0,4)$, $\alpha_3 = (0,0,1,1)$, and $\alpha_4 = (0,0,0,2)$ form a basis for $\mathbb{R}^4$.

Let A be the matrix with rows $\alpha_1, \alpha_2, \alpha_3, \alpha_4$. A row-reduces to the identity matrix I; it follows that the dimension of the row space of A is 4. Since $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ spans a 4-dimensional space, this set of vectors must be linearly independent.

MATH 320 **Quiz 21** **Name:**__________________________

1. Let $s < n$, and let A be an $s \times n$ matrix with entries in a field F. Prove that there is an non-zero $n \times 1$ column vector X such that $AX = 0$.

Let $\alpha_1, ..., \alpha_n$ be the columns of A; that is, if $A = (a_{ij})$, $\alpha_1 = (a_{11}, a_{21}, ..., a_{s1}), ..., \alpha_n = (a_{1n}, a_{2n}, ..., a_{sn})$. Since each α_i is a vector in F^s, $\{\alpha_1, ..., \alpha_n\}$ is a set of n vectors in F^s. Since $dim(F^s) = s$ and $s < n$, $\{\alpha_1, ..., \alpha_n\}$ must be linearly dependent. This means that for some $c_1, ..., c_n \in F$, not all zero,

$$c_1\alpha_1 + ... + c_n\alpha_n = 0.$$

This means that for the non-zero column vector $X = \begin{pmatrix} c_1 \\ . \\ . \\ . \\ c_n \end{pmatrix}$,

$$AX = c_1\alpha_1 + ... + c_n\alpha_n = 0.$$

MATH 320 **Quiz 22** **Name:**__________________________

1. Let V and W be vector spaces over a field F. Give the definition of a linear transformation from V to W.

A linear transformation from V to W is a function $T\colon V \to W$ such that for all $\alpha, \beta \in V$ and $c \in F$,

$$T(\alpha + \beta) = T(\alpha) + T(\beta)$$

and

$$T(c\alpha) = cT(\alpha).$$

MATH 320 **Quiz 23** **Name:**__________________________

1. Describe explicitly the linear transformation T from $\mathbb{R}^2$ to $\mathbb{R}^2$ such that $T((1,0)) = (a,b)$ and $T((0,1)) = (c,d)$.

$$\begin{aligned} T((x,y)) &= T(x(1,0) + y(0,1)) = xT((1,0)) + yT((0,1)) \\ &= x(a,b) + y(c,d) = (ax + cy, bx + dy) \end{aligned}$$

MATH 320 **Quiz 24** **Name:**________________________

1. Which of the following functions $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation?

   (a) $T(x, y) = (1 + x, y)$
   (b) $T(x, y) = (y, x)$
   (c) $T(x, y) = (x^2, y)$
   (d) $T(x, y) = (\sin x, y)$
   (e) $T(x, y) = (x - y, 0)$

(b) and (e) are linear transformations, (a), (c), and (d) are not.

MATH 320 **Quiz 25** **Name:**____________________________

1. Let $T\colon V \to W$ be a linear transformation. Prove that $Range(T) = \{\beta \in W \mid \beta = T(\alpha) \text{ for some } \alpha \in V\}$ is a subspace of W.

Let $\beta_1, \beta_2 \in Range(T)$. Then by definition, there exists $\alpha_1, \alpha_2 \in V$ such that $\beta_1 = T(\alpha_1)$ and $\beta_2 = T(\alpha_2)$. Then

$$\beta_1 + \beta_2 = T(\alpha_1) + T(\alpha_2) = T(\alpha_1 + \alpha_2).$$

Hence, $\beta_1 + \beta_2 \in Range(T)$. Also, for any scalar c,

$$c\beta_1 = cT(\alpha_1) = T(c\alpha_1).$$

Thus $c\beta_1 \in Range(T)$. Therefore, $Range(T)$ is a subspace of W.

MATH 320 **Quiz 26** **Name:**____________________________

1. Describe explicitly a linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that $Range(T) = Span(\{(1,0,-1),(1,2,2)\}$.

$$T(x_1, x_2, x_3) = (x_1 + x_2,\ 2x_2,\ 2x_2 - x_1).$$

MATH 320 **Quiz 27** **Name:**__________________________

1. Let T be the linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T((x_1, x_2, x_3)) = (x_1 + x_2, 2x_3 - x_1)$$

Find the matrix of T relative to the standard basis on $\mathbb{R}^3$ and $\mathbb{R}^2$.

$T((1,0,0)) = (1,-1)$, $T((0,1,0)) = (1,0)$, $T((0,0,1)) = (0,2)$. Hence the matrix of T relative to the standard basis is

$$\begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \end{pmatrix}.$$

MATH 320 **Quiz 28** **Name:**__________________________

1. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

$$T(x_1, x_2, x_3) = (x_1 + 2x_2 + x_3,\ x_2 + x_3,\ -x_1 + 3x_2 + 4x_3)$$

Find a basis for $Null(T)$.

The matrix of T relative to the standard basis is

$$M = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 3 & 4 \end{pmatrix}.$$

$Null(T)$ is equal to the solutions to $MX = 0$. M row reduces to the matrix

$$R = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

So the system of equations $RX = 0$ has one free variable x_3 and $x_1 = x_3$, $x_2 = -x_3$. So a basis for the solution space, and hence a basis for $Null(T)$, is $\{(1, -1, 1)\}$.

MATH 320 **Quiz 29** **Name:**________________________

1. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by

$$T(x_1, x_2) = (-x_2,\ x_1)$$

Find the matrix for T relative to the basis $\{(1,1),\ (1,-1)\}$.

$$T((1,1)) = (-1,1) = 0(1,1) + (-1)(1,-1)$$
$$T((1,-1)) = (1,1) = 1(1,1) + 0(1,-1)$$

Hence, the matrix for T relative to this basis is

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

MATH 320 **Quiz 30** **Name:**____________________________

1. Let $T\colon V \to W$ and $S\colon W \to U$ be linear transformations. Prove that $S \circ T\colon V \to U$ is a linear transformation.

For all α, $\beta \in V$ and all scalars c,

$$S\circ T(\alpha+\beta) = S(T(\alpha+\beta)) = S(T(\alpha)+T(\beta)) = S(T(\alpha))+S(T(\beta)) = S\circ T(\alpha)+S\circ T(\beta)$$

and

$$S \circ T(c\alpha) = S(T(c\alpha)) = S(cT(\alpha)) = c(S(T(\alpha))) = c(S \circ T(\alpha))$$

Thus $S \circ T$ is a linear transformation.

MATH 320 **Quiz 31** **Name:**____________________________

1. Let $T\colon V \to W$ be a linear transformation. Prove that T is injective if and only if $Null(T) = \{0\}$.

Suppose T is injective. If $\alpha \in Null(T)$, then $T(\alpha) = 0 = T(0)$, so by injectivity of T $\alpha = 0$. Thus $Null(T) = \{0\}$.

Now Suppose $Null(T) = \{0\}$. If $\alpha, \beta \in V$ such that $T(\alpha) = T(\beta)$, then

$$\begin{aligned} T(\alpha) &= T(\beta) \\ T(\alpha) - T(\beta) &= 0 \\ T(\alpha - \beta) &= 0 \end{aligned}$$

Which means $\alpha - \beta \in Null(T) = \{0\}$. So $\alpha - \beta = 0$, and hence $\alpha = \beta$. Therefore T is injective.

MATH 320 **Quiz 32** **Name:**____________________________

1. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$T(x_1, x_2, x_3) = (3x_1,\ x_1 - x_2,\ 2x_1 + x_2 + x_3)$$

Is T invertible?

Yes; you can compute that $Null(T) = \{0\}$, you can check that $Range(T) = \mathbb{R}^3$, or you can show that the determinent of the matrix which represents T is non-zero.

MATH 320 **Quiz 33** **Name:**____________________________

1. State the definition of an isomorphism between two vector spaces.

An isomorphism beween vector spaces V and W is an invertible (equivalently, bijective) linear transformation $T\colon V \to W$.

MATH 320 **Quiz 34** **Name:**____________________________

1. Compute the determinant of $\begin{pmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{pmatrix}$.

$$det \begin{pmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{pmatrix} = 4$$

MATH 320 **Quiz 35** **Name:**____________________________

1. Given a linear transformation $T\colon V \to V$, state the definition of an eigenvalue and and eigenvector corresponding to T.

A scaler λ is an *eigenvalue* for T if $T(\alpha) = \lambda\alpha$ for some $\alpha \in V$, $\alpha \neq 0$. A vector $\alpha \in V$ is called an *eigenvector* for T if there is some eigenvalue λ such that $T(\alpha) = \lambda\alpha$.

MATH 320 **Quiz 36** **Name:**____________________________

1. Find all eigenvalues for the matrix $\begin{pmatrix} 1 & 2 \\ -2 & -3 \end{pmatrix}$. For each eigenvalue, find the corresponding space of eigenvectors. Is this matrix diagonalizable?

$\lambda = -1$ is the only eigenvalue. The corresponding space of eigenvectors is $Span\{(-1,1)\}$. The matrix is not diagonalizible, since all eigenvectors lie in a one-dimensional subspace of two-dimensional vector space.