HOMEWORK
MATH 330

(1) Let $n > 1$ be an integer which is not prime. Prove that there exists a prime p such that $p|n$ and $p \leq \sqrt{n}$.

(2) pg 76, # 37: Let $\mathbb{E}$ denote the set of positive even integers. An element $p \in \mathbb{E}$ is called an $\mathbb{E}$-*prime* if p cannot be written as a product of two elements of $\mathbb{E}$. Determine a simple criteria for when elements of $\mathbb{E}$ can be *uniquely* factored into a product of $\mathbb{E}$-primes.

(3) pg 73, #15: Suppose today is Monday.
  (a) What day of the week will it be 100 days from now?
  (b) What day of the week will it be 10,000 days from now?
  (c) What day of the week will it be 1,000,000 days from now?

(4) pg 73, #21: Show that if n is a positve odd integer then the sum of any n consecutive integers is divisible by n.

(5) Let a, b, c be integers such that $c|a$ and $c|b$. Prove that c divides any linear combination of a and b.

(6) pg 92, # 7: Show that if $a, b, n, m \in \mathbb{N}$ such that a and b are relatively prime, then a^n and b^m are relatively prime.

(7) pg 93, # 21.

(8) For each of the relations defined on $\mathbb{Q}$, either prove that it is an equivalence relation or show which properties of an equivalence relation fail.
  (a) $x \sim y$ whenever $x - y \in \mathbb{Z}$.
  (b) $x \sim y$ whenever $x \leq y$.
  (c) $x \sim y$ whenever $xy \in \mathbb{Z}$
  (d) $x \sim y$ whenever $|x - y| \leq 1$.

(9) pg 132, #36.

(10) pg 133, #42.

(11) Let a and n be postive integers. Prove that $a^{\frac{1}{n}}$ is either an integer or is irrational.

(12) pg 148, # 7-10.

(13) pg 150, # 39.

(14) Section 5.1, # 43.

(15) Section 5.3, # 53.

(16) Section 7.1, # 12.

(17) Section 7.1, # 56.

(18) Section 7.1, # 57.

(19) Section 7.1, # 60.

(20) pg 248, Section 7.2, #69.

(21) pg 376, Section 9.1, #35.

(22) pg 377, Section 9.1, #38.

(23) pg 377, Section 9.1, #41.

(24) pg 377, Section 9.1, #48.

(25) pg 389, Section 9.2, #33.

(26) Prove that for all $n \geq 1$, there exists a polynomial $f(x) \in \mathbb{Q}[x]$ with $deg(f(x)) = n$ such that $f(x)$ is irreducible in $\mathbb{Q}[x]$.

(27) pg 453, Section 12.1, #2.

(28) pg 455, Section 12.1, #22.

(29) pg 456, Section 12.1, #48.

(30) Let F be a field and $f(x) \in F[x]$. Prove that $\alpha \in F$ is a root of $f(x)$ if and only if $(x - \alpha)|f(x)$.

(31) pg 471, Section 12.3, #20.

(32) Determine which of the following are groups.
   (a) $(\mathbb{Z}, -)$.
   (b) $(\mathbb{R} \setminus \{0\}, \cdot)$
   (c) $(\{r \in \mathbb{Q} \mid r \geq 0\}, +)$.
   (d) $(\{r \in \mathbb{Q} \mid r > 0\}, \cdot)$.

(33) Prove or disprove each of the following:
   (a) $\{f \in S_n \mid f(1) = 1\}$ is a subgroup of S_n.
   (b) $\{e\} \cup \{f \in D_n \mid f \text{ is a rotation}\}$ is a subgroup of D_n.
   (c) $\{e\} \cup \{f \in D_n \mid f \text{ is a reflection}\}$ is a subgroup of D_n.

(34) Prove that isomorphism is an equivalence relation on groups.

(35) Suppose X is set with n elements. Prove that $Bij(X) \cong S_n$.

(36) Let H and K be subgroups of a group G. Prove that $H \cap K$ is a subgroup of G.

(37) Section 8.1, pg 291, #49.

(38) Section 8.1, pg 290, #33-36 (Fill out the multiplication table of D_4, writing each element in the form $y^i x^j$ for $0 \leq i \leq 3$ and $0 \leq j \leq 1$).

(39) Section 8.1, pg 292, #57.

(40) Section 8.1, pg 293, #65.

(41) Section 8.1, pg 293, #67.

(42) Section 8.4, pg 361, #3, 4, 8, 15

(43) Write the permutation f from Problem 42 as a product of transpositions.

(44) Section 8.2, pg 319, #11.

(45) Section 8.2, pg 319, #24.

(46) Section 8.2, pg 320, #30.

(47) Section 8.2, pg 320, #33.

(48) Section 8.2, pg 320, #41.

(49) Section 8.2, pg 320, #43.

(50) Section 8.2, pg 321, #52.

(51) Section 8.2, pg 321, #57.

(52) Section 8.3, pg 344, #35, 36.

(53) Section 8.3, pg 345, #59.

(54) Prove the Second Isomorphism Theorem: Suppose G is a group, $H \leq G$, and $N \trianglelefteq G$. Then
 (a) $HN \leq G$.
 (b) $H \cap N \trianglelefteq H$.
 (c) $HN/N \cong H/(H \cap N)$.

(55) Prove the Third Isomorphism Theorem: Suppose G is a group, $N \trianglelefteq G$, and $N \trianglelefteq K \trianglelefteq G$. Then
 (a) $K/N \trianglelefteq G/N$.
 (b) $(G/N)/(K/N) \cong G/K$.

Let G be a group. An isomorphism $\phi\colon G \to G$ is called an *automorphism* of G. Let $Aut(G)$ denote the set of all automorphisms of G.

(56) Prove that $Aut(G)$ is a subgroup of $Bij(G)$.

(57) Let $g \in G$ and let $\gamma_g\colon G \to G$ be the function defined by $\gamma_g(a) = g^{-1}ag$ for all $a \in G$. Prove that $\gamma_g \in Aut(G)$. γ_g is called an *inner automorphism* of G.

(58) Let $Inn(G)$ denote the set of all inner automorphism of G, that is $Inn(G) = \{\gamma_g \mid g \in G\}$. Prove that $Inn(G)$ is a normal subgroup of $Aut(G)$.

(59) Prove that $Inn(G) \cong G/Z(G)$.

(60) Let $\sigma \in S_n$ be a k-cycle. Prove that every conjugate of σ is a k-cycle.

(61) Let x and y be conjugate elements of a group G. Prove that $x^n = e$ if and only if $y^n = e$, hence x and y have the same order.

(62) Let G be a group acting on a set X. Let $x \in X$, and define $Stab_G(x) = \{g \in G \mid g \cdot x = x\}$. Prove that $Stab_G(x)$ is a subgroup of G.

(63) Let G be a group acting on a set X. Let $x, y \in X$ such that $y = g \cdot x$. Prove that $Stab_G(y) = gStab_G(x)g^{-1}$.

(64) Let $H \leq G$ and let X denote the set of left cosets of H. Define an action of G on X by $g \cdot xH = gxH$. First describe $Stab_G(H)$ and then describe $Stab_G(xH)$ for an arbitrary $x \in G$.

(65) Let G be a finite group acting on a finite set X with exactly n distinct orbits. For $g \in G$, define $Fix(g) = \{x \in X \mid g \cdot x = x\}$. Prove that

$$n = \frac{1}{|G|} \sum_{g \in G} |Fix(g)|$$