HOMEWORK
MATH 445

11/7/14

(1) Let $\mathcal{T}$ be a topology for $\mathbb{R}$ defined as follows: $U \in \mathcal{T}$ iff, for each $p \in U$ there is an open interval I_p such that $p \in I_p$ and $\mathbb{Q} \cap I_p \subseteq U$. Is $(\mathbb{R}, \mathcal{T})$ Hausdorff? is it regular?

(2) Let X be second countable, and $A \subseteq X$ an uncountable subset. Prove that $A \cap A' \neq \emptyset$. Give an example of a space X and an uncountable subset $A \subseteq X$ such that $A \cap A' = \emptyset$.

(3) Prove that every seperable metric space is second countable. Find an example of a seperable space which is not second countable.

(4) Prove that every compact metric space is seperable.

(5) Let $x_1, x_2, ...$ be a squence of points in the product space ΠX_α. Prove that $\lim_{n\to\infty} x_n = x$ if and only if $\lim_{n\to\infty} \pi_\alpha(x_n) = \pi_\alpha(x)$ for all α.

(6) Prove or disprove: $\mathbb{R}^\omega$ is seperable.

(7) Let X be a compact Hausdorff space. Prove that X is metrizable if and only if X is second countable. What happens if "compact" is replaced by "locally compact"?

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(1) Let X be a compact Hausdorff topological space. Let A, B be disjoint closed subsets of X. Prove that there exist disjoint open sets U, V such that $A \subseteq U$ and $B \subseteq V$.

(2) Let $f\colon X \to Y$ and $g\colon X \to Y$ be continuous functions and let Y be Hausdorff. Prove that $W = \{x \in X \mid f(x) = g(x)\}$ is closed.

(3) Let $f\colon X \to Y$ be a continuous map and suppose X is compact. Prove $f(X)$ is compact.

(4) Prove that the following spaces are pairwise not homeomorphic: $[0, 1]$, $(0, 1)$, and $[0, 1)$.

(5) Suppose X and Y are connected, prove $X \times Y$ is connected.

(6) Consider the following subsets of $\mathbb{R}^2$. Prove if they are connected or not.
$A = \{(x, y) \mid x, y \in \mathbb{Q}\}$, $B = \{(x, y) \mid x, y \in \mathbb{R} \setminus \mathbb{Q}\}$, $C = A \cup B$, $D = \mathbb{R}^2 \setminus A$, $E = \mathbb{R}^2 \setminus B$.

(7) Classifly the letters

$$\{A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, P, Q, R, S, T, U, V, W, X, Y, Z\}$$

up to homeomorphism, where each letter is given a topology by considering it as a subset in the plane.

(8) Prove or disprove: $GL_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc \neq 0 \right\}$ is connected.

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(1) Prove that the following collections of spaces are pairwise homeomorphic:
   (a) $(0,1)$, $(0,\infty)$, (a,b), $\mathbb{R}$.

   (b) $\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\leq 1\}$, $\{(x,y)\in\mathbb{R}^2 \mid (x-a)^2+(y-b)^2\leq R^2\}$, $\{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2=1, z\geq 0\}$.

   (c) $\mathbb{R}^2\setminus\{(0,0)\}$, $\{(x,y)\in\mathbb{R}^2 \mid 1<x^2+y^2<4\}$, $\{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2=1\}$.

(2) For each of the following equivalence relations $\sim$ on $\mathbb{R}^2$, identify the the quotient space $\mathbb{R}^2/\sim$ (it is homeomorphic to a familar space).
   (a) $(x_1,y_1)\sim(x_2,y_2)$ if $x_1^2+y_1^2=x_2^2+y_2^2$.

   (b) $(x_1,y_1)\sim(x_2,y_2)$ if $(x_2-x_1,y_2-y_1)\in\mathbb{Z}\times\mathbb{Z}$.

(3) Let $GL_2^+(\mathbb{R})=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix} \mid ad-bc>0\right\}$ with the topology induced by considering $GL_2^+(\mathbb{R})$ and a subspace of $\mathbb{R}^4$. Prove that $GL_2^+(\mathbb{R})\cong S^1\times\mathbb{R}^3$. Hint: show that for each such matrix there exists a unique θ, $p>0$, $r>0$, and q such that $\begin{pmatrix} a & b\\ c & d\end{pmatrix}=\begin{pmatrix} \cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix} p & q\\ 0 & r\end{pmatrix}$

(4) Prove that the following are equivalent for a function $f\colon X \to Y$:
   (a) f is a continuous function.
   (b) For all $A \subseteq X$, $f(\overline{A}) \subseteq \overline{f(A)}$.
   (c) For all $B \subseteq Y$, $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$
   (d) For every closed $B \subseteq Y$, $f^{-1}(B)$ is closed in X.
   (e) For every $x \in X$ and every open set $V \subseteq Y$ such that $f(x) \in V$, there is an open set U such that $x \in U$ and $f(U) \subseteq V$.

(5) (a) Suppose that A and B are closed subsets of X such that $X = A \cup B$. Let $f\colon X \to Y$ be a function such that $f|_A$ and $f|_B$ are both continuous. Prove that f is continuous.
   (b) Suppose $f\colon X \to Y$ and for every $x \in X$, there exists an open set U containing x such that $f|_U$ is continuous. Prove that f is continuous.

(6) Suppose X and Y are Hausdorff. Prove $X \times Y$ is Hausdorff.

(7) Suppose X is T_1 and $A \subseteq X$. Prove that A' is closed.

8/27/14

(1) Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$, and $(Z, \mathcal{T}_Z)$ be topological spaces. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be continuous functions. Prove that $g \circ f\colon X \to Z$ is continuous.

(2) Let X be a set and let $\mathcal{T}_{cofin} = \{A \subseteq X \mid |X \setminus A| < \infty\} \cup \{\emptyset\}$. Prove that $\mathcal{T}_{cofin}$ is a topology on X.

(3) Let $\mathcal{B} = \{(a, b) \mid a, b \in \mathbb{Q},\ a < b\}$. Show that $\mathcal{B}$ is a basis for the standard topology on $\mathbb{R}$.

(4) Let $\mathcal{A}$ be a basis for a topology on X and $\mathcal{B}$ a basis for a topology on Y. Prove that $\{A \times B \mid A \in \mathcal{A},\ B \in \mathcal{B}\}$ is a basis for the product topology on $X \times Y$.

(5) Let X and Y be topological spaces. Prove that the projections $\pi_X\colon X \times Y \to X$ and $\pi_Y\colon X \times Y \to Y$ defined by $\pi_X((x, y)) = x$ and $\pi_Y((x, y)) = y$ are continuous with respect to the product topology on $X \times Y$. Furthermore, prove that π_X and π_Y are *open maps*, that is for every open $U \subset X \times Y$, $\pi_X(U)$ and $\pi_Y(U)$ are open.

(6) Let X be a topological space and let A be a subset of X with the subspace topology. Let $\iota\colon A \to X$ be the inclusion map, that is $\iota(a) = a$.

(a) Prove ι is continuous.

(b) Given another topological space Y and a function $f\colon Y \to A$, prove that f is continuous if and only if $\iota \circ f$ is continuous.

(c) If $g\colon X \to Y$ is continuous, prove that the restricion $g|_A\colon A \to Y$ is continuous.