Hyperbolic groups
Lecture Notes*

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1 Geometric group theory basics

Word metric and Cayley graphs Let G be a group generated by $S \subseteq G$; for convenience, we will always assume that our generating sets are *symmetric*, that is $S = S^{-1}$. A *word* in S is a finite concatenation of elements of S. For such a word W, let $\|W\|$ denote its length. If two words W and U are letter for letter equivalent, we write $W \equiv U$, and if W and U represent the same element of the group G, we write $W =_G U$. For an element $g \in G$, let $|g|_S$ denote the length of the shortest word in S which represents g in the group G. Given $g, h \in G$, let $d_S(g,h) = |g^{-1}h|_S$. d_S is called the *word metric* on G with respect to S.

We let $\Gamma(G,S)$ denote the *Cayley graph of G with respect to S*. This is the graph whose vertex set is G and there is an oriented edge e labeled by $s \in S$ between any two vertices of the form g and gs. We typically identify the edges labeled by s and s^{-1} with the same endpoints and consider these as the same edge with opposite orientations. $\mathbf{Lab}(e)$ denotes the label of the edge e; similarly, for a (combinatorial) path p, $\mathbf{Lab}(p)$ will denote the concatenation of the labels of the edges of p. Also for such a path p, we let p_- and p_+ denote the intial and the terminal vertex of p respectively, and $\ell(p)$ will denote the number of edges of p. The metric obtained on the vertices of $\Gamma(G,S)$ by the shortest path metric is clearly equivalent to the word metric d_S; identifying each edge with the unit interval $[0,1]$ in the natural way allows us to extend this metric to all of $\Gamma(G,S)$.

Metric spaces Throughout these notes, we denote a metric space by X and its metric by d (or d_X if necessary). For $x \in X$ and $n \geq 0$, let $B_n(x) = \{y \in X \mid d(x,y) \leq n\}$, that is the closed ball of radius n centered at x. For a subset $A \subseteq X$, we usually denote the closed n-neighborhood of A by A^{+n}, that is $A^{+n} = \{x \in X \mid d(x,A) \leq n\}$.

A *path* in X is a continuous map $p\colon [a,b] \to X$ for some $[a,b] \subseteq \mathbb{R}$. We will often abuse notation by using p to refer to both the function and its image in X. As above, we let $p_- = p(a)$ and $p_+ = p(b)$. Similarly, a *ray* is a continuous map $p\colon [a,\infty) \to X$, and a *bi-infinite path* is a continuous map $p\colon (-\infty,\infty) \to X$.

*Disclaimer: Nothing in these notes is my own original work. However, most of the material is standard so I will not attempt to provide citations for every result. The interested reader is refered to the standard text [5]. For further resources, see [7, 11, 13, 17, 19, 25].

Given a path p in a metric space X, the *length* of p, denoted, $\ell(p)$, is defined as

$$\ell(p) = \sup_{a \leq t_1 \leq \ldots \leq t_n \leq b} \sum_{i=1}^{n-1} d(p(t_i), p(t_{i+1}))$$

Where the supremum is taken over all $n \geq 1$ and all possible choices of $t_1, ..., t_n$. In general this may be infinite, but we will usually only consider *rectifiable* paths, that is paths p for which $\ell(p) < \infty$. A path p is called a *geodesic* if $\ell(p) = d(p_-, p_+)$. Geodesic rays and bi-infinite geodesics are defined similarly. X is called a *geodesic metric space* if for all $x, y \in X$, there exists a geodesic path p such that $p_- = x$ and $p_+ = y$. Note that geodesic metric spaces are clearly path connected. For x, y in a geodesic metric space X, we let $[x, y]$ denote a geodesic from x to y.

We will usually assume throughout these notes that X is a geodesic metric space. However, most statements and proofs will also work under the weaker assumption that X is a *length space*, that is a path connected space such that for any $x, y \in X$, $d(x, y) = \inf\{\ell(p) \mid p_- = x, p_+ = y\}$.

Let X and Y be metric spaces and $f \colon X \to Y$. If f is onto and for all $x_1, x_2 \in X$, $d_X(x_1, x_2) = d_Y(f(x_1), f(x_2))$, then f is called *isometry.* If f is onto and there is a constant $\lambda \geq 1$ such that for all $x_1, x_2 \in X$,

$$\frac{1}{\lambda} d_X(x_1, x_2) \leq d_Y(f(x_1), f(x_2)) \leq \lambda d_X(x_1, x_2).$$

Then f is called a *bi-lipschitz equivalence.* In this case, we say that X and Y are bi-lipschitz equivalent and write $X \sim_{lip} Y$. Now suppose there are constant $\lambda \geq 1$, $C \geq 0$, and $\varepsilon \geq 0$ such that $f(X)$ is *ε-quasi-dense* in Y, i.e. $f(X)^{+\varepsilon} = Y$, and for all $x_1, x_2 \in X$,

$$\frac{1}{\lambda} d_X(x_1, x_2) - C \leq d_Y(f(x_1), f(x_2)) \leq \lambda d_X(x_1, x_2) + C.$$

Then f is called a *quasi-isometry*, or a $(\lambda, c, \varepsilon)$-quasi-isometry if we need to keep track of the constants.. In this case we say X and Y are *quasi-isometric* and write $X \sim_{qi} Y$. Note that unlike isometries and bi-lipschitz equivalences, quasi-isometries are not required to be continuous.

If the condition that f is onto (or $f(X)$ is quasi-dense) is dropped from the above definitions, then f is called an *isometric embedding*, *bi-lipschitz embedding*, or a *quasi-isometric embedding* respectively.

Exercise 1.1. Show that $\sim_{lip}$ and $\sim_{qi}$ are both equivalence relations on metric spaces.

Exercise 1.2. Let X and Y be bounded metric spaces. Prove that $X \sim_{qi} Y$.

Exercise 1.3. Suppose $S \subseteq G$ and $T \subseteq G$ are two finite generating sets of G. Show that $(G, d_S) \sim_{lip} (G, d_T)$, and hence $\Gamma(G, S) \sim_{qi} \Gamma(G, T)$.

It follows from this exercise that any finitely generated group is canonically associated to a $\sim_{qi}$-equivalence class of metric spaces. We will often abuse notation by considering the group G itself as a metric space, but it should be understood that the metric on G is only well-defined up to quasi-isometry.

Group actions Let G be a group acting on a metric space X. We will always assume that such actions are by isometries, that is for all $x, y \in X$ and $g \in G$,

$$d(x, y) = d(gx, gy).$$

There is a natural correspondence between actions of G on X and homomorphisms $\rho\colon G \to Isom(X)$, where $Isom(X)$ denotes the group of all isometries of X. We say the action is *faithful* if the corresponding homomorphism is injective. This is equivalent to saying for all $g \in G$, there exists $x \in X$ such that $gx \neq x$. The action is called *free* if for all $x \in X$, $Stab_G(x) = \{1\}$, where $Stab_G(x) = \{g \in G \mid gx = x\}$; equivalently, for all $x \in X$ and for all $g \in G$, $gx \neq x$. The action is called *proper*[1] if for any bounded subset $B \subseteq X$, $\{g \in G \mid gB \cap B \neq \emptyset\}$ is finite. The action is called *cobounded* if X/G is bounded, or equivalently there exists a bounded subset $B \subseteq X$ such that

$$X = \bigcup_{g \in G} gB$$

The following lemma is fundamental to geometric group theory. It was first proved by Efremovic.

Lemma 1.4 (Milnor-Svarč Lemma)**.** *Let G be a group acting properly and coboundedly on a geodesic metric space X. Then G has a finite generating set S and*

$$\Gamma(G, S) \sim_{qi} X.$$

Proof. Fix a point $o \in X$. Since the action of G is cobounded, there exists a constant K such that for all $x \in X$, there exists $g \in G$ such that $d(x, go) \leq K$. Let $S = \{g \in G \mid d(o, go) \leq 2K + 1\}$. By properness, the set S is finite. Note that if $s_1, s_2 \in S$, then $d(o, s_1s_2o) \leq d(o, s_1o) + d(s_1o, s_1s_2o) = d(o, s_1o) + d(o, s_2o) \leq 2(2k + 1)$. Similarly, it is easy to show by induction that for all $g \in \langle S \rangle$, $d(o, go) \leq |g|_S(2K + 1)$.

Now fix $g \in G$, and let p be a geodesic from o to go. Choose points $o = x_0, x_1, ..., x_n = go$ on p such that $d(x_i, x_{i+1}) = 1$ for $0 \leq i \leq n - 2$ and $d(x_{n-1}, x_n) \leq 1$. For each $1 \leq i \leq n - 1$, choose $h_i \in G$ such that $d(x_i, h_io) \leq K$, and set $h_0 = 1$ and $h_n = g$. By the triangle inequality, $d(o, h_i^{-1}h_{i+1}o) = d(h_io, h_{i+1}o) \leq 2K + 1$ for all $0 \leq i \leq n - 1$. Hence $h_i^{-1}h_{i+1} \in S$. Furthermore,

$$h_1(h_1^{-1}h_2)(h_2^{-1}h_3)...(h_{n-1}^{-1}h_n) = h_n = g$$

Thus $g \in \langle S \rangle$, and since g is arbitrary we get that S generates G. Furthermore, $|g|_S \leq n$ and by our choice of x_i, $n - 1 < d(o, go) \leq n$. Let $f\colon G \to X$ be the function defined by $f(g) = go$. Then we have shown that

$$|g|_S - 1 \leq d(o, go) \leq (2K + 1)|g|_S.$$

Furthermore, our choice of K implies that $f(G)$ is K-quasi-dense in X. It follows easily that the map f is a quasi-isometry from G with the metric d_S to X. □

Corollary 1.5.

1. *If G is finitely generated and H is a finite index subgroup of G, then H is finitely generated and $G \sim_{qi} H$.*
2. *If $N \trianglelefteq G$ is a finite normal subgroup of G and G/N is finitely generated, then G is finitely generated and $G \sim_{qi} G/N$.*
3. *If M is a closed Riemannian manifold with universal cover $\widetilde{M}$, then $\pi_1(M)$ is finitely generated and $\pi_1(M) \sim_{qi} \widetilde{M}$.*

[1]This is the metric version of properness. There is also a topological version, where bounded is replaced by compact.

4. *If G is a connected Lie group with a left-invariant Riemannian metric and $\Gamma \leq G$ is a uniform lattice in G, then Γ is finitely generated and $\Gamma \sim_{qi} G$.*

Exercise 1.6. Prove parts (1) and (2) of Corollary 1.5.

Group presentations and algorithmic problems Given a set S, we denote the free group on S by $F(S)$. Recall that the elements of this group are equivalence classes of words in S, where words two words are equivalent if you can obtain one from the other by adding or removing subwords of the form ss^{-1} finitely many times. Equivalently, $F(S)$ can be defined as the unique group (up to isomorphism) such that for any group G and any function $f: S \to G$, there is a unique homomorphism $\bar{f}: F(S) \to G$ extending f. If $S = \{s_1, ..., s_n\}$, we typically denote $F(S)$ by F_n.

Given a subset $R \subseteq G$, where G is a a group, the *normal closure* of R, denoted $\langle\langle R \rangle\rangle$, is defined as the intersection of all normal subgroups of G which contain R. Equivalently,

$$\langle\langle R \rangle\rangle = \{f_1^{-1} r_1 f_1 f_2^{-1} r_2 f_2 ... f_k^{-1} r_k f_k \mid k \geq 0, f_i \in G, r_i \in R^{\pm 1}\}.$$

Given a set S and $R \subseteq F(S)$, we say that

$$\langle S \mid R \rangle \tag{1}$$

is a *presentation* of the group G if $G \cong F(S)/\langle\langle R \rangle\rangle$. In this case S is called the set of *generators* and R is called the set of *relations* of the presentation. The presentation is called *finite* if both S and R are finite sets, and G is called *finitely presentable* if G has a finite presentation. For convenience we will always assume that our set of relations is symmetric, that is $r \in R$ implies $r^{-1} \in R$.

The following are classical problems in group theory; the first three were introduced by Max Dehn in 1912.

Word Problem: Given a presentation $\langle S \mid R \rangle$ of a group G, find an algorithm such that for any word W in S, the algorithm determines whether or not $W =_G 1$.

Conjugacy Problem: Given a presentation $\langle S \mid R \rangle$ of a group G, find an algorithm such that for any two words W and U in S, the algorithm determines whether or not W and U represent conjugate elements of the group G.

Isomorphism Problem: Find an algorithm which accepts as input two group presentations and determines whether or not they represent isomorphic groups.

Membership Problem: Given a presentation $\langle S \mid R \rangle$ of a group G and words $U_1, ..., U_n$ in S, find an algorithm such that for any word W in S, the algorithm determines whether or not W belongs to the subgroup of G generated by $U_1, ..., U_n$.

Exercise 1.7. Describe an algorithm which solves the word problem for the standard presentation of $\mathbb{Z}^n$, that is $\langle a_1, ..., a_n \mid [a_i, a_j], 1 \leq i < j \leq n \rangle$.

It is known that there are finite group presentations for which the word probelm is undecidable. Note that the word problem can be viewed as a special case of the conjugacy problem, since $W =_1 G$ if and only if W is conjugate to 1 in G. It follows that any group with undecibable word problem will also have undecidable conjugacy problem. Simlarly, The word problem can also be viewed as a special case of the membership problem corresponding to the trivial subgroup. There do, however, exist group presentations with decidable word problem but undeciable conjugacy and/or membership problems.

Similarly, the isomorphism problem is undecidable in general, though as with the other algorithmic problems it can solved in certain special cases, that is if one only considers presentations which represent groups belonging to a specific class of groups.

Van Kampen Diagrams and Dehn functions Suppose $\langle S \mid R\rangle$ is a presenation for a group G and W is a word in S. Then $W =_G 1$ if and only if there exist $r_1, ..., r_k \in R$ and $f_1, ..., f_k \in F(S)$ such that

$$W =_{F(S)} f_1^{-1} r_1 f_1 ... f_k^{-1} r_k f_k. \tag{2}$$

We now show how this can be encoded geometrically. Let Δ be a finite, connected, simply connected, planar 2-complex in which every edge is oriented and labeled by an element of S. If e is an edge of Δ with label s and $\bar{e}$ is the same edge with the opposite orientation, then $\mathbf{Lab}(\bar{e}) = s^{-1}$. Labels of paths in Δ are defined the same as in Cayley graphs. If Π is a 2-cell of Δ, then $\mathbf{Lab}(\partial\Pi)$ is the word obtained by choosing a base point $v \in \partial\Pi$ and reading the label of the path $\partial\Pi$ starting and ending at v. Note that a different choice of basepoint results in a cyclic permutation of the word $\mathbf{Lab}(\partial\Pi)$, so we consider $\mathbf{Lab}(\partial\Pi)$ as being defined only up to cyclic permutations. $\mathbf{Lab}(\partial\Delta)$ is defined similarly. Δ is called a *van Kampen diagram* over the presentation $\langle S \mid R\rangle$ if for every 2-cell Π of Δ, (a cyclic permutation of) $\mathbf{Lab}(\partial\Pi)$ belongs to R. In this case it can be shown by a reasonably straightforward induction on the number of 2-cells of Δ that $\mathbf{Lab}(\partial\Delta) =_G 1$. It turns out the converse is also true.

Exercise 1.8. Suppose G is a group with presentation $\langle S \mid R\rangle$ and Δ is a van Kampen diagram over $\langle S \mid R\rangle$. Prove that $\mathbf{Lab}(\partial\Delta) =_G 1$.

Lemma 1.9 (van Kampen Lemma). *Suppose $\langle S \mid R\rangle$ is a presentation for a group G and W is a word in S. Then $W =_G 1$ if and only if there exists a van Kampen diagram Δ over the presentation $\langle S \mid R\rangle$ such that $\mathbf{Lab}(\partial\Delta) \equiv W$.*

Proof. If W is the boundary label of a van Kampen diagram, then $W =_G 1$ be the previous exercise. Now suppose that $W =_G 1$. Then there exist $r_1, ..., r_k \in R$ and $f_1, ..., f_k \in F(S)$ such that

$$W =_{F(S)} f_1^{-1} r_1 f_1 ... f_k^{-1} r_k f_k.$$

Each word of the form $f_i^{-1} r_i f_i$ is the label of a van Kampen diagram consisting of a path labeled by f_i connected to a 2-cell with boundary label r_i. glueing the initial points of each of these paths together produces a van Kampen diagram with boundary label $f_1^{-1} r_1 f_1 ... f_k^{-1} r_k f_k$ (sometimes called a "wedge of lollipops"). Now $f_1^{-1} r_1 f_1 ... f_k^{-1} r_k f_k$ can be transformed into the word W by a finite sequence of moves consisting of adding or deleting subwords of the form ss^{-1}. One can check that if one of these moves is applied to a word U produces U' and U is the boundary label of a van Kampen diagram, then there is a natural move on the diagram which produces a new van Kampen diagram with boundary label U'. It follows that the "wedge of lollipops" diagram can be modified by a finite sequence of moves to produce a van Kampen diagram with boundary label W.

□

Exercise 1.10. Suppose Δ is a van Kampen diagram over a presentation $\langle S \mid R\rangle$ for a group G, and p is a closed (combinatorial) path in Δ. Prove that $\mathbf{Lab}(p) =_G 1$.

From this exercise, it follows that if you fix a vertex $v \in \Delta$, there is a well-defined, label preserving map from the 1-skeleton of Δ to $\Gamma(G, S)$ which sends v to 1.

Given a van Kampen diagram Δ, let $Area(\Delta)$ be the number of 2-cells of Δ. For a fixed group presentation $\langle S \mid R\rangle$ and a word W in S, let

$$Area(W) = \min\{Area(\Delta) \mid \Delta \text{ is a van Kampen diagram over } \langle S \mid R\rangle \text{ and } \mathbf{Lab}(\partial\Delta) \equiv W\}.$$

equivalently, $Area(W)$ is equal to the minimal k such that is equal to a product of k conjugates of elements of R (see (2)). The *Dehn function* of a finitely presented group G, denoted δ_G, is the function $\delta_G \colon \mathbb{N} \to \mathbb{N}$ defined by

$$\delta_G(n) = \max_{\|W\| \leq n} Area(W)$$

Of course, this depends not only on G, but also on the chosen presentation of G. In order to make the Dehn function of G independent of the presentation (as is suggested by the notation δ_G), we consider this function as defined only up to the following equivalence relation: functions $f, g \colon \mathbb{N} \to \mathbb{N}$ are equivalent if there exist constant A_1, B_1, C_1 and A_2, B_2, C_2 such that for all $n \in \mathbb{N}$,

$$f(n) \leq A_1 g(B_1 n) + C_1 n \text{ and } g(n) \leq A_2 f(B_2 n) + C_2 n.$$

Note that the linear term in the above equivalence is indeed necessary, since even the trivial group has the presentation $\langle s \mid s = 1\rangle$ and $Area(s^n) = n$.

Exercise 1.11. (a) Show that this is indeed an equivalence relation.

(b) Show that $f_1(n) = 1$, $f_2(n) = \log n$, and $f_3(n) = n$ are all equivalent.

(c) Show that two polynomials p and q are equivalent if and only if they have the same degree.

(d) Show that 2^n and 3^n are equivalent.

Exercise 1.12. Prove that a finite group has at most linear Dehn function.

Exercise 1.13. Prove that a finitely generated abelian group has at most quadratic Dehn function, and that the Dehn function of $\mathbb{Z}^2$ is equivalent to n^2.

Examples 1.14. 1. If G is nilpotent of class c, then $\delta_G(c) \leq n^{c+1}$

2. If G is the fundamental group of a compact, orientable surface of genus $g \geq 2$, then Dehn's algorithm shows that δ_G is linear.

3. The Dehn function of $BS(1, 2) = \langle a, t \mid t^{-1}at = a^2\rangle$ is equivalent to 2^n

4. For $G = \langle a, b, c \mid a^b = c, a^c = a^2\rangle$, δ_G is equivalent to $2^{2^{\cdot^{\cdot^{\cdot^{2^n}}}}}$, where this tower has length $\log_2(n)$.

It can be shown using *Tietze transformations* that up to this equivalence, the choice the Dehn function of a finitely presented group G is independent of the choice of finite presentation. Furthermore, the Dehn function is also invariant under quasi-isometry.

Theorem 1.15. *Suppose G is finitely presented and H is finitely generated. If $G \sim_{qi} H$, then H is finitely presented and δ_G is equivalent to δ_H.*

Proof. Let $\langle S \mid R\rangle$ be a finite presentation for G and let T a finite generating set for H. Let $M = \max\{\|r\| \mid r \in R\}$. Let $f\colon \Gamma(H,T) \to \Gamma(G,S)$ be a $(\lambda, c, \varepsilon)$ quasi-isometry. Let p be a closed (combinatorial) path in $\Gamma(H,T)$, and let $v_1, v_2, ..., v_n, v_{n+1} = v_1$ denote the vertices of p. Let q be the closed path in $\Gamma(G,S)$ formed by connecting each $f(v_i)$ to $f(v_{i+1})$ by a geodesic. Since $d_T(v_i, v_{i+1}) = 1$, $d_S(f(v_i), f(v_{i+1})) \leq \lambda + c$, and hence $\ell(q) \leq (\lambda + c)n$. Since q is a closed path, $\mathbf{Lab}(q) =_G 1$, so there exists a van Kampen diagram Δ with $\mathbf{Lab}(\partial\Delta) \equiv \mathbf{Lab}(q)$. We also choose Δ such that $Area(\Delta) \leq \delta_G((\lambda+c)n)$. We identify the 1-skeleton of Δ with its image in $\Gamma(G,S)$ under the natural map $\Delta^{(1)} \to \Gamma(G,S)$ which sends $\partial\Delta$ to q. Now we build a map $g\colon \Delta^{(1)} \to \Gamma(H,T)$ for each interior vertex $v \in \Delta$, choose a vertex $u \in \Gamma(H,T)$ such that $d_S(v, f(u)) \leq \varepsilon$, and set $g(v) = u$. Each exterior vertex $v \in \Delta$ lies on some geodesic $[f(v_i), f(v_{i+1})]$; if v is closer to $f(v_i)$ we set $g(v) = v_i$, otherwise we set $g(v) = v_{i+1}$. Now if two vertices v and u are adjacent, then we join $g(v)$ and $g(u)$ by geodesics in $\Gamma(H,T)$. Note that for such u and v, $d_S(f(g(u)), f(g(v))) \leq 2\varepsilon + 1$, and hence $d_T(g(u), g(v)) \leq (2\varepsilon + 1 + c)\lambda$. It follows that if Π is a 2-cell of Δ, there is a closed loop in $g(\Delta^{(1)})$ corresponding to the image of $\partial\Pi$ of length at most $(2\varepsilon+1+c)\lambda\ell(\partial\Pi) \leq (2\varepsilon+1+c)\lambda M$.

Let $R' = \{r \in F(T) \mid \|r\| \leq (2\varepsilon + 1 + c)\lambda M \text{ and } r =_H 1\}$. From above, we have that there is a van Kampen diagram Δ' whose 1-skeleton is $g(\Delta^{(1)})$ and each two cell is labeled by an element of R'. Hence Δ' is a van Kampen diagram over $\langle T \mid R'\rangle$ and $W \equiv \mathbf{Lab}(\partial\Delta')$. Thus $\langle T \mid R'\rangle$ is a presentation for H, in particular H is finitely presented. Furthermore,

$$Area(W) \leq Area(\Delta') = Area(\Delta) \leq \delta_G((\lambda + c)n)$$

since W is an arbitrary word of length n, we get that $\delta_H(n) \leq \delta_G((\lambda + c)n)$. Reversing the roles of G and H in the above proof will result in the reverse inquality (with possibly different constants), hence δ_G is equivalent to δ_H.

□

Given a van Kampen diagram Δ, we define the *type* of Δ by the ordered pair of natural numbers $(Area(\Delta), \ell(\partial\Delta))$.

Exercise 1.16. Show that for a finite presentation $\langle S \mid R\rangle$ and a fixed type (k, n), there are only finitely many van Kampen diagrams over $\langle S \mid R\rangle$ of type (k, n).

A function $f\colon \mathbb{N} \to \mathbb{N}$ is called *recursive* is there exists an algorithm which computes $f(n)$ for all $n \in \mathbb{N}$.

Theorem 1.17. *Let G be a finitely presented group. The following are equivalent.*

1. *δ_G is recursive.*

2. *There exists a recursive function $f\colon \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$, $\delta_G(n) \leq f(n)$.*

3. *The word problem in G is solvable.*

Proof. Fix a finite presentation $\langle S \mid R\rangle$ for the group G.

(1) $\implies$ (2)

Trivial.

(2) $\implies$ (3)

Let W be a word in S with $\|W\| = n$. By assumption, there exists a van Kampen diagram Δ with $\mathbf{Lab}(\partial\Delta) \equiv W$ and $Area(\Delta) \leq \delta_G(n) \leq f(n)$. However, by the previous exercise there are only finitely many van Kampen diagrams of type (k, n) with $1 \leq k \leq f(n)$. Hence one can list all of these diagrams; if a some diagram in this list has boundary label W, then $W =_G 1$, otherwise $W \neq_G 1$.

(3) $\implies$ (2).

Fix $n \in \mathbb{N}$, and let R_n be the set of words W in S such that $\|W\| \leq n$ and $W =_G 1$. This set can be explicitly computed by applying the algorithm which solves the word problem in G to each word of length at most n. Now for each $W \in R_n$, we can compute $Area(W)$ by listing all van Kampen diagrams of type $(1, \|W\|)$, then type $(2, \|W\|)$ etc. Since we know $W =_1 G$, there must be some k such that this list produces a van Kampen diagram with boundary label W and area k; if k is the smallest natural number for which such a diagram occurs, then $Area(W) = k$. Hence we can compute the area of each of the the finitely many words in R_n, and by definition $\delta_G(n)$ is the maximum of these areas.

□

2 Geometry of hyperbolic metric spaces

2.1 Definitions

Before we define hyperbolic groups, we need to defined hyperbolic metric spaces and study some basic properties of their geometry. In particular, we need to show that hyperbolicity is invariant under quasi-isometry in order for hyperbolicity to be well-defined in the world of groups.

We will start by listing several equivalent definitions of hyperbolicity for metric spaces.

Let X be a metric space and let $x, y, o \in X$. The *Gromov product* of x and y with respect to o is defined as

$$(x|y)_o = \frac{1}{2}(d(o, x) + d(o, y) - d(x, y)).$$

Loosely speaking, $(x|y)_o$ measures how long geodesics $[o, x]$ and $[o, y]$ would fellow travel before diverging, and hence the Gromov product should be thought of as a type of angle between $[o, x]$ and $[o, y]$. Indeed, in the special case where X is a tree (i.e. a geodesic metric space where there is a unique arc between any two points), then the triangle obtained by connecting o, x and y is actually a *tripod*, that is a wedge of three arcs. In this case $(x|y)_o$ is exactly the distance from o to the center point of the tripod. It is important to note, however, that the definition of the Gromov product does not require X to be geodsic.

Definition 2.1. Let $\delta \geq 0$. We say that a metric space X satisfies $\mathrm{Hyp}_1(\delta)$ if for all $x, y, z, o \in X$,

$$(x|y)_o \geq \min\{(x|z)_o, (y|z)_o\} - \delta.$$

Assume now that X is a geodesic metric space and $x, y, z \in X$. Let T be the triangle with sides $[x, y]$, $[x, z]$ and $[y, z]$.

Exercise 2.2. Show that for any such triangle T is a geodesic metric space, there always exist points $o_x \in [y, z]$, $o_y \in [x, z]$ and $o_z \in [x, y]$ such that

$$\begin{aligned} d(x, o_y) &= d(x, o_z) \\ d(y, o_x) &= d(y, o_z) \\ d(z, o_x) &= d(z, o_y) \end{aligned}$$

In fact, a solution to this exercise will also show that $d(x, o_y) = d(x, o_z) = (y|z)_x$, $d(y, o_x) = d(y, o_z) = (x|z)_y$ and $d(z, o_x) = d(z, o_y) = (x|y)_z$.

Now we can construct a tripod T', called a *comparison tripod* by taking a center point c and attaching arcs $[x', c]$, $[y', c]$, and $[z', c]$, where the length of $[x', c]$ is $d(x, o_y) = d(x, o_z)$, the length of $[y', c]$ is $d(y, o_x) = d(y, o_z)$, and the length of $[z', c]$ is $d(z, o_x) = d(z, o_y)$. There is a natural map $T \to T'$ which sends each side of T isometrically onto the corresponding side of T', that is $[x, y]$ is set isometrically onto $[x', y']$, etc. o_x, o_y, and o_z all map to c, the center point of the tripod T'. Every other point of T' will have a preimage consisting of at most 2 points.

Definition 2.3 (Thin Triangles)**.** Let $\delta \geq 0$. We say that a geodesic metric space X satisfies $\mathrm{Hyp}_2(\delta)$ if for any geodesic triangle T in X with two points $a, b \in T$ which map to the same point of the comparison tripod T', $d(a, b) \leq \delta$.

A triangle T which satisfies the conditions in this definition is called δ*-thin.*

The following is the mostly commonly cited definition of hyperbolicity and is attributed to Rips.

Definition 2.4 (Slim Triangles or the Rips Condition)**.** Let $\delta \geq 0$. We say that a geodesic metric space X satisfies $\mathrm{Hyp}_3(\delta)$ if for any geodesic triangle T in X with sides p, q, r and any point $a \in p$, there exists $b \in q \cup r$ such that $d(a, b) \leq \delta$.

A triangle T which satisfies the conditions in this definition is called δ*-slim.*

Proposition 2.5. *For any geodesic metric space X and any $\delta \geq 0$,*

1. *$Hyp_1(\delta) \implies Hyp_2(4\delta)$.*
2. *$Hyp_2(\delta) \implies Hyp_1(\delta)$.*
3. *$Hyp_2(\delta) \implies Hyp_3(\delta)$.*
4. *$Hyp_3(\delta) \implies Hyp_2(6\delta)$.*

Proof. (1) Suppose X satisfies $\mathrm{Hyp}_1(\delta)$. Let T be a geodesic triangle in X with vertices x, y, z, and let a, b be points of T which have the same image in the comparison tripod. For concreteness, we assume that $a \in [x, z]$, $b \in [x, y]$ such that $d(x, a) = d(x, b) \leq (y|z)_x$. That is, a and b are such that they map to the same point on the arc $[x', c]$ of the comparison tripod T'. Let $N = d(x, a) = d(x, b)$. Then

$$(a|b)_x = \frac{1}{2}(d(x, a) + d(x, b) - d(a, b)) = N - \frac{1}{2}d(a, b).$$

Note that $a \in [x,z]$, $b \in [x,y]$ implies that $(a|z)_x = N = (b|y)_x$. Now applying $\mathrm{Hyp}_1(\delta)$ twice gives

$$(a|b)_x \geq \min\{(a|y)_x, (b|y)_x\} - \delta \geq \min\{(y|z)_x, (a|z)_x, (b|y)_x\} - 2\delta = N - 2\delta$$

Combining this with the above inequality gives that $N - 2\delta \leq N - \frac{1}{2}d(a,b)$, so $d(a,b) \leq 4\delta$. The proof can be easily modified to apply to pairs of points which map to the other two arcs of the comparison tripod, hence X satisfies $\mathrm{Hyp}_2(4\delta)$.

(2) Suppose X satisfies $\mathrm{Hyp}_2(\delta)$ and $x, y, z, o \in X$. Without loss of generality, we assume that $(y|z)_o \leq (x|z)_o$. Let T_1 be a geodesic triangle with vertices o, x, z and T_2 a geodesic triangle with vertices o, y, z. Let b_z, b_y be points on $[o,z]$ and $[o,y]$ respectively which map to the center point of the comparison tripod for T_2. Let $u \in [o,x]$ such that $d(o,u) = d(o,b_z) = (y|z)_o \leq (x|z)_o$. In particular this means that u and b_z map to the same point in the comparison tripod for T_1, so $d(u,b_z) \leq \delta$ by $\mathrm{Hyp}_2(\delta)$. Similarly $d(b_z,b_y) \leq \delta$, so $d(u,b_y) \leq 2\delta$. Applying the triangle inequality and the fact that $u \in [o,x]$ and $b_y \in [o,y]$ we get

$$\begin{aligned} d(x,y) &\leq d(x,u) + 2\delta + d(b_y,y) \\ &= d(o,x) - d(o,u) + 2\delta + d(o,y) - d(o,b_y) \\ &= d(o,x) + d(o,y) - 2(y|z)_o + 2\delta \end{aligned}$$

Plugging this inequality to the definition of $(x|y)_o$ yeilds

$$(x|y)_o \geq (z|y)_o - \delta$$

Hence X satisfies $\mathrm{Hyp}_1(\delta)$

(3) Trivial.

(4) Suppose X satisfies $\mathrm{Hyp}_3(\delta)$. Let T be a geodesic triangle in X with vertices x, y, z, and let $o_x \in [y,z]$, $o_y \in [x,z]$, and $o_z \in [x,y]$ be the points provided by exercise 2.2, that is the points which map to the center of the comparison tripod. We will first show that $\max\{d(o_x,o_y), d(o_x,o_z), d(o_y,o_z)\} \leq 4\delta$. Let u be a point on $[x,y] \cup [x,z]$ such that $d(o_x,u) \leq \delta$. Suppose for concreteness that $u \in [x,z]$; a similar proof works when $u \in [x,y]$. There are two cases to consider.

Case 1: $u \in [o_y, z]$. In this case, we see that $d(z,u) + d(u,o_y) = d(z,o_y) = d(z,o_x) \leq d(z,u) + \delta$, hence $d(u,o_y) \leq \delta$, so $d(o_x,o_y) \leq 2\delta$.

Case 2: $u \in [x, o_y]$. In this case, we get that $d(z,o_y) + d(o_y,u) + d(u,x) = d(z,x) \leq d(z,o_x) + \delta + d(u,x) = d(z,o_y) + d(u,x) + \delta$, hence again $d(u,o_y) \leq \delta$, so $d(o_x,o_y) \leq 2\delta$.

Now a similar proof applied to o_z will show that o_z is within 2δ of either o_x or o_y. Whichever one is within 2δ of o_z, by the triangle inequality o_z will be within 4δ of the other one, hence $\max\{d(o_x,o_y), d(o_x,o_z), d(o_y,o_z)\} \leq 4\delta$.

Let a, b be points of T which have the same image in the comparison tripod. For concreteness, we assume that $a \in [y,z]$, $b \in [x,z]$ such that $d(z,a) = d(z,b) \leq (x|y)_z$. That is, a and b are such that they map to the same point on the arc $[z', c]$ of the comparison tripod T'. Let u be a point of

$[x,y]\cup[x,z]$ such that $d(a,u)\le\delta$. If $u\in[x,z]$, then same proof as above with o_x replaced by a and o_y replaced by b will show that $d(a,b)\le 2\delta$. It only remains to deal with the case where $u\in[x,y]$. The proof is again similar to above. There are two cases, corresponding to $u\in[x,o_z]$ or $u\in[o_z,y]$. We will show the first case and leave the second as an exercise for the reader. If $u\in[x,o_z]$, then $d(z,o_y)+d(o_y,x)=d(z,x)\le d(z,a)+\delta+d(u,x)=d(z,o_x)-d(o_x,a)+\delta+d(o_z,x)-d(o_z,u)=d(z,o_y)-d(o_x,a)+\delta+d(o_y,x)-d(o_z,u)$. This implies that $d(o_x,a)+d(o_z,u)\le\delta$. Finally, we get

$$d(a,b)\le d(a,u)+d(u,o_z)+d(o_z,o_y)+d(o_y,b)=d(a,u)+d(u,o_z)+d(o_z,o_y)+d(o_x,a)\le\delta+\delta+4\delta=6\delta.$$

□

Exercise 2.6 (optional). Let T be a geodesic triangle in X with vertices x,y,z and let $o_x\in[y,z]$, $o_y\in[x,z]$ and $o_z\in[x,y]$ be the points from exercise (2.2). Define $Size(T)=\max\{d(o_x,o_y),d(o_x,o_z),d(o_y,o_z)\}$. Define $MinSize(T)$ as the infimum of $\max\{d(a,b),d(a,c),d(c,b)\}$ over all $a\in[x,y],b\in[x,z],c\in[y,z]$. We say X satisfies $\mathrm{Hyp}_4(\delta)$ if for every geodesic triangle T in X, $Size(T)\le\delta$, and X satisfies $\mathrm{Hyp}_5(\delta)$ if for every geodesic triangle T in X, $MinSize(T)\le\delta$. Prove that these are equivalent to the above hyperbolicity conditions.

Definition 2.7. A metric space X is called δ-hyperbolic if X satisfies $Hyp_1(\delta)$. If X is a geodesic metric space, we will also assume that δ is chosen such that X satisfies the conditions $\mathrm{Hyp}_2(\delta)$ and $\mathrm{Hyp}_3(\delta)$. We say that X is hyperbolic if X is δ-hyperbolic for some $\delta\ge 0$.

Exercise 2.8. Let X be a δ-hyperbolic geodesic metric space and $P=p_1p_2...p_n$ a geodesic n-gon in X for $n\ge 3$. Let a be a point on p_i for some $1\le i\le n$. Prove that there exists $j\ne i$ and $b\in p_j$ such that $d(a,b)\le(n-2)\delta$. (In fact, $n-2$ can be replaced by $\log_2(n)$).

Examples 2.9.

1. If X is a bounded metric space, then X is δ-hyperbolic for $\delta=diam(X)$.
2. $\mathbb{R}$ with the standard metric is 0-hyperbolic.
3. Is X is a simplicial tree, that is a connected graph with no cycles equipped with the combinatorial metric, then X is 0-hyperbolic (equivalently, every triangle is a tripod).
4. Generalizing the previous two examples, a 0-hyperbolic geodesic metric space is called a $\mathbb{R}$-*tree*. Some more examples of $\mathbb{R}$-trees:
   (a) $X=\{(x,y)\mid x\in[0,1],y=0\}\cup\{(x,y)\mid x\in\mathbb{Q},y\in[0,1]\}$ with the metric $d((x_1,y_1),(x_2,y_2))=|y_1|+|x_2-x_1|+|y_2|$ when $x_1\ne x_2$ and $d((x_1,y_1),(x_2,y_2))=|y_2-y_1|$ otherwise.
   (b) $X=\mathbb{R}^2$ with the following metric: If the line containing (x_1,y_1) and (x_2,y_2) passes through the origin, then $d((x_1,y_1),(x_2,y_2))$ is the usual Euclidean distance. Otherwise, $d((x_1,y_1),(x_2,y_2))=\sqrt{x_1^2+y_1^2}+\sqrt{x_2^2+y_2^2}$.
5. $\mathbb{R}^n$ with the Euclidean metric is not hyperbolic for any $n\ge 2$.
6. The classical hyperbolic space $\mathbb{H}^2$ is δ-hyperbolic. Recall that a triangle T in $\mathbb{H}^2$ with angles α, β, and γ has area $=\pi-\alpha-\beta-\gamma$. For a point x on T, consider the largest semi-circle contained in T and centered at x. This semi-circle has area at most the area of T which

is at most π; this provides a bound on the radius of the semi-circle, which can be explicity computed to show that $\mathbb{H}^2$ satisfies $\mathrm{Hyp}_3(\delta)$ for $\delta = 4\log\varphi$, where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio.

7. From the previous example, it follows that $\mathbb{H}^n$ is δ-hyperbolic for all $n \geq 2$.

8. If (X, d) is any metric space, then we can define a new metric $\hat{d}$ on X by $\hat{d}(x, y) = \log(1 + d(x, y))$. Then $(X, \hat{d})$ is $2\log 2$ hyperbolic.

9. $O(n, 1)$, $U(n, 1)$, $SP(n, 1)$ are all hyperbolic when given left-invariant Riemannian metrics.

2.2 Quasi-geodesic stability

Definition 2.10. Suppose X is a metric space and p is a path in X. p is called a (λ, C) *quasi-geodesic* if for any subpath q of p,

$$\ell(q) \leq \lambda d(q_-, q_+) + C$$

Definition 2.11. Suppose X is a metric space and p is a path in X. p is called a *k-local geodesic* if every subpath of p of length $\leq k$ is a geodesic.

Remark 2.12. *Quasi-geodesic rays*, *bi-infinite quasi-geodesics*, *local geodesic rays*, and *bi-infinite local geodesics* are all similarly defined in the obvious ways.

Definition 2.13. Let X be a metric space and A, B closed subsets of X. The *Hausdorff distance* between A and B is the infimum of all ε such that $A \subseteq B^{+\varepsilon}$ and $B \subseteq A^{+\varepsilon}$. We denote this distance by $d_{Hau}(A, B)$.

We will assume for the rest of this section that X is a geodesic and δ-hyperbolic metric space.

Lemma 2.14. *Let p be a (rectifiable) path in X from x to y. Then for any geodesic $[x, y]$ and any point $a \in [x, y]$, there exists $b \in p$ such that*

$$d(a, b) \leq \delta|\log_2(\ell(p))| + 1$$

Proof. We assume that p is paramterized such that $p\colon [0, 1] \to X$ and for all $0 \leq i < j \leq 1$, $\ell(p|_{[i,j]}) = \frac{1}{j-i}\ell(p)$. Choose N such that $2^N \leq \ell(p) \leq 2^{N+1}$. Let $z_1 = p(\frac{1}{2})$. Let T_1 be a triangle with sides $[x, y]$, $[x, z_1]$, and $[z_1, y]$. Since T_1 is δ-slim, there exists a point $b_1 \in [x, z_1] \cup [z_1, y]$ with $d(a, b_1) \leq \delta$. If $b_1 \in [x, z_1]$, let $z_2 = p(\frac{1}{4})$ and $T_2 = [x, z_1][x, z_2][z_1, z_2]$; if $b_1 \in [z_1, y]$, let $z_2 = p(\frac{3}{4})$ and $T_2 = [z_1, y][z_1, z_2][z_2, y]$. We apply slimness to T_2 and b_1 to find a point b_2 on one of the other two sides of T_2 that is δ close to b_1. We then define z_3 as the midpoint of the subpath of p that is "above" the side of T_2 containing b_2 and T_3 as the triangle which contains the side of T_2 that contains b_2 and geodesics connecting the endpoints of this side to z_3. Continue this process inductively until we obtain b_N.

Note that by construction, for each $1 \leq i \leq N - 1$, $d(b_i, b_{i+1}) \leq \delta$, and hence $d(a, b_N) \leq N\delta \leq \delta|\log_2(\ell(p))|$. Furthermore, b_N belongs to a geodesic q with endpoints on p such that $\ell(q) \leq \frac{\ell(p)}{2^N}$. Let $b \in p$ be the closest endpoint of q to b_N, hence $d(b, b_N) \leq \frac{1}{2}\ell(q) \leq 1$. Thefore,

$$d(a, b) \leq d(a, b_N) + d(b_N, b) \leq \delta|\log_2(\ell(p))| + 1.$$

$\square$

Theorem 2.15 (Morse Lemma)**.** *Let X be a δ-hyperbolic metric space. Then for any $\lambda \geq 1$, $C \geq 0$, there exists $K = K(\delta, \lambda, C)$ such that for any geodesic p and any (λ, C)-quasi-geodesic q with $p_- = q_-$ and $p_+ = q_+$, $d_{Hau}(p, q) \leq K$.*

Proof. Let $D = \sup_{x \in p}\{d(x, q)\}$; our first goal will be to bound D in term of δ, λ and C. Since p and q are compact, there is point $x_0 \in p$ which realizes this supremum. In particular, the the interior of $B_D(x_0)$ does not intersect q. Now choose $y \in [p_-, x_0]$ such that $d(x_0, y) = 2D$, or if no such y exists then set $y = p_-$. Choose $z \in [x_0, p_+]$ similarly. By definition of D, there exists some $y', z' \in q$ such that $d(y, y') \leq D$ and $d(z, z') \leq D$. By the triangle inequality,

$$d(y', z') \leq d(y', y) + d(y, z) + d(z, z') \leq 6D$$

If q' is the subpath of q joining y' to z', then since q is a (λ, C)-quasi-geodesic, $\ell(q') \leq 6\lambda D + C$. Let $c = [y, y']q'[z', z]$, and note that $l(c) \leq 6\lambda D + C + 2D$ and $d(x_0, c) = D$. By Lemma 2.14, $d(x_0, c) \leq \delta|\log_2(\ell(c))| + 1$, and combinging this with the previous estimates gives

$$D \leq \delta|\log_2(6\lambda D + 2D + C)| + 1.$$

This equation implies that D must be bounded in terms of δ, λ and C.

It remains to show that q is contained in a bounded neighborhood of p. Suppose $q = q_1q_2q_3$ such that q_2 is a maximal subpath of q which lies outside p^{+D}. Now every point of p is within D of some point on either q_1 or q_3; by connectedness of p, there must exist some $x \in p$ and $y \in q_1$, $z \in q_3$ such that $d(x, y) \leq D$ and $d(x, z) \leq D$. In particular, this means that $\ell(q_2) \leq \lambda(2D) + C$. It follows that q is contained in the $2\lambda D + D + C$ neighborhood of p. $\square$

Corollary 2.16. *Let X be a δ-hyperbolic metric space. Then for any $\lambda \geq 1$, $C \geq 0$, there exists $\kappa = \kappa(\delta, \lambda, C)$ such that for any (λ, C)-quasi-geodesics p and q with $p_- = q_-$ and $p_+ = q_+$, $d_{Hau}(p, q) \leq \kappa$.*

Exercise 2.17. Prove that there exist $\lambda \geq 1$ and $C \geq 0$ such that for any $K \geq 0$, there exists a (λ, C)-quasi-geodesic q in $\mathbb{R}^2$ such that $d_{Hau}(q, [q_-, q_+]) \geq K$.

Exercise 2.18. Let X be a geodesic metric space. Prove X is hyperbolic if and only if for all $\lambda \geq 1$, $C \geq 0$ that there exists δ' such that for any triangle T in X whose sides are (λ, C)-quasi-geodesics is δ'-slim.

Proposition 2.19. *Suppose Y is a geodesic metric space and $X \sim_{qi} Y$. Then Y is hyperbolic.*

Proof. Let $f\colon Y \to X$ be a $(\lambda, c, \varepsilon)$ quasi-isometry and let $T = pqr$ be any geodesic triangle in Y. Choose points $p_- = a_0, a_1, ..., a_{n_1} = p_+ \in p$ such that $d_Y(a_i, a_{i+1}) = 1$ for $0 \leq i \leq n_1 - 1$ and $d_Y(a_{n_1-1}, a_{n_1}) \leq 1$. For each a_i, let $a_i' = f(a_i)$, and let p' be the path in X obtained by connected each a_i to a_{i+1} by a geodesic. We will first show that p' is a quasi-geodesic. Let q be a subpath of p', and let a_i' and a_j' be the closest points to q_- and q_+ respectively. Then q is composed of at most $|j - i| + 2$ geodesic segments, each of length at most $(\lambda + c)$. Hence $\ell(q) \leq (|j - i| + 2)(\lambda + c)$. On the other hand, $|j - i| - 1 \leq d_Y(a_i, a_j) \leq |j - i|$, so $d_X(a_i', a_j') \geq \frac{1}{\lambda}(|j - i| - 1) - c$. It follows that $d_Y(q_-, q_+) \geq \frac{1}{\lambda}(|j - i| - 1) - c - \lambda - c$, which we rewrite as $|j - i| \leq \lambda d_Y(q_-, q_+) + \lambda^2 + 2c\lambda + 1$. Comparing these estimates gives that

$$\ell(q) \leq (\lambda + c)|j - i| + 2(\lambda + c) \leq (\lambda + c)(\lambda d_Y(q_-, q_+) + \lambda^2 + 2c\lambda + 1) + 2(\lambda + c).$$

This shows that p is a (λ', c') quasi-geodesic for λ' and c' depending only on λ and c.

Choose points $q_- = b_0, b_1, ..., b_{n_2} = q_+ \in q$ and $r_- = c_0, c_1, ..., c_{n_3} = r_+ \in r$ similarly, and define q' and r' as before. Then $T' = p'q'r'$ is a (λ', c')-quasi-geodesic triangle, and hence is δ'-slim for some $\delta' = \delta'(\delta, \lambda', c')$ by Exercise (2.18).

Now, let $x \in p$. Then there exists some $a_i \in p$ such that $d_Y(x, a_i) \leq 1$. Since T' is δ'-slim, there exists some $z \in q' \cup r'$ such that $d_X(a_i', z) \leq \delta'$. For concreteness we assume $z \in q'$ and choose b_j' such that $d_X(z, b_j') \leq \lambda + c$, and hence $d_X(a_i', b_j') \leq \delta' + \lambda + c$. It follows that

$$d_Y(x, b_j) \leq 1 + d_Y(a_i, b_j) \leq 1 + \lambda(\delta' + \lambda + 2c).$$

In particular, we have shown that Y satisfies $\mathrm{Hyp}_3(1 + \lambda(\delta' + \lambda + 2c))$. □

Remark 2.20. By the same proof as above, if X and Y are geodesic metric spaces such that Y is hyperbolic and $f\colon X \to Y$ is a quasi-isometric embedding, then X is hyperbolic.

Definition 2.21. A finitely generated group G is *hyperbolic* if for some (equivalently, any) finite generating set S, $\Gamma(G, S)$ is a hyperbolic metric space.

Remark 2.22. By the Milnor-Svarc Lemma, a group G is hyperbolic if and only if G admits a proper, cobounded action on a geodesic hyperbolic metric space.

Examples 2.23.

1. Finite groups are hyperbolic.
2. $\mathbb{Z}$ is hyperbolic. More generally, any group which is virtually $\mathbb{Z}$, such as $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$.
3. F_n is hyperbolic for any $n \geq 1$.
4. If M is a closed hyperbolic manifold, then $\pi_1(M)$ is hyperbolic. In particular, if S is an orientable surface of genus g, then $\pi_1(S)$ is hyperbolic if and only if $g \geq 2$.
5. $\mathbb{Z}^n$ is hyperbolic if and only if $n = 1$.

Lemma 2.24. *Suppose p is a k-local geodesic in X from x to y for $k > 8\delta$. Then*

1. $p \subseteq [x, y]^{+2\delta}$.
2. $[x, y] \subseteq p^{+3\delta}$.
3. *p is a (λ, c)-quasi-geodsic for $\lambda = \frac{k+4\delta}{k-4\delta}$ and $c = 2\delta$.*

Proof. (1) Choose a point $a \in p$ which maximizes the distance to $[x, y]$. Choose $b, c \in p$ such that a is the midpoint of the subpath of p from b to c and $8\delta < d(b, c) \leq k$. (if such points do not exist, we use the endpoints of p instead and an obvious modification of the following arguement will work). Choose b', c' as the points on $[x, y]$ closest to b and c respectively, and consider the quadrilateral (b', b, c, c'). a must be 2δ from one of the other sides of this quadrilateral by hyperbolicity. If a is within 2δ of a point on $[b', b]$ or $[c, c']$, it would contridict our choice of a as the point which maximizes the distance to $[x, y]$. Hence a is within 2δ of a point on $[b', c'] \subseteq [x, y]$.

(2) Now let $a \in [x, y]$. Since p is connected, there exists some $b \in p$ such that $d(b, [x, a]) \leq 2\delta$ and $d(b, [a, y]) \leq 2\delta$. Applying hyperbolicity to the triangle spanned by b and the two points which realize these inequalities produces the desired result.

(3) We subdivide p into subpaths $p = p_1 p_2 ... p_{n+1}$ such that $\ell(p_i) = k' = \frac{k}{2} + 2\delta$ for $1 \leq i \leq n$ and $0 \leq \ell(p_{n+1}) = \eta < k'$. Note that

$$\ell(p) = nk' + \eta$$

Now let $a_i = (p_i)_-$, and let a'_i be a point on $[x, y]$ with $d(a_i, a'_i) \leq 2\delta$. We first need to show that each a'_i is "between" a'_{i-1} and a'_{i+1} on $[x, y]$, which will imply that $x = a'_1, a'_2 ... a'_{n+1}, y$ forms a monotone sequence along $[x, y]$.

Let $x_0 \in p_{i-1}$ with $d(a_{i-1}, x_0) = 2\delta$ and $y_0 \in p_i$ with $d(a_{i+1}, y) = 2\delta$. Note that $d(x_0, y_0) = 2k' - 4\delta = k$, hence a geodesic $[x_0, y_0]$ can be chosen as a subpath of p. Consider the triangle T with endpoints a_{i-1}, a'_{i-1}, and x_0. By hyperbolicity, $T \subseteq B_{3\delta}(a_{i-1})$. Since $d(a_{i-1}, a_i) = k' > 6\delta$, T does not intersect $B_{3\delta}(a_i)$. Similarly, a triangle with endpoints a_{i+1}, a'_{i+1}, and y_0 will not intersect $B_{3\delta}(a_i)$. Now we apply hyperbolicity to the quadrilateral with vertices $a'_{i-1}, x_0, y_0, a'_{i+1}$ and the point a_i, we get a point $a''_i \in [a'_{i-1}, a'_{i+1}]$ with $d(a_i, a''_i) \leq 2\delta$. By hyperbolicity of the triangle (a_i, a'_i, a''_i), $d(a_i, z) \leq 3\delta$ for any point z which is between a'_i and a''_i. In particular, neither a'_{i-1} nor a'_{i+1} are between a'_i and a''_i, and since $a''_i \in [a'_{i-1}, a'_{i+1}]$, we must also have $a'_i \in [a'_{i-1}, a'_{i+1}]$.

Since $x = a'_1, a'_2 ... a'_{n+1}, y$ forms a monotone sequence along $[x, y]$, we get that

$$d(x, y) = \sum_{i=1}^{n} d(a'_i, a'_{i+1}) + d(a'_{n+1}, y)$$

Now for each $1 \leq i \leq n$, $d(a'_i, a'_{i+1}) \geq k' - 4\delta$, and $d(a'_{n+1}, y) \geq \eta - 2\delta$. Hence,

$$d(x, y) \geq nk' - 4\delta n + \eta - 2\delta = \ell(p) - 4\delta n - 2\delta$$

Finally, since $n \leq \frac{\ell(p)}{k'}$,

$$d(x, y) \geq (\frac{k' - 4\delta}{k'})\ell(p) - 2\delta.$$

Finally, it only remains to note that every subpath of p is again a k-local geodesic to which the above proof applies. $\square$

Corollary 2.25. *Suppose p is a k-local geodesic in X for $k > 8\delta$. Then either p is constant or $p_- \neq p_+$.*

3 Algorithmic and isoperimetric characterizations of hyperbolic groups

Given a group presentation $\langle S \mid R \rangle$ and a word W in S, *Dehn's algorithm* is the following procedure: First freely reduce W; if this produces the empty word, the algorithm stops. Now if W is freely reduced and non-empty, search W for subwords U such that U is also a subword of relation (or a cyclic shift of a relation) $r \in R$ and $\|U\| > \frac{1}{2}\|r\|$. If no such subword exists, the algorithm stops.

If such a U exists, then there is a (possibly empty) word V (the complement of U in r) such that $UV^{-1} =_G 1$ and $\|V\| < \|U\|$. In this case, the algorithm replaces U with V and repeats.

If the presentation $\langle S \mid R \rangle$ is finite, then Dehn's algorithm terminates after finitely many steps for any word W.

Definition 3.1. Let $\langle S \mid R \rangle$ be a finite presentation for a group G. Dehn's algorithm solves the word problem for $\langle S \mid R \rangle$ if for any non-empty word W for which Dehn's algorithm stops, $W \neq_G 1$.

Exercise 3.2. Find a group presentation for which Dehn's algorithm does not solve the word problem.

Exercise 3.3. Suppose $\langle S \mid R \rangle$ is a finite presentation for a group G for which Dehn's algorithm solves the word problem. Prove that G has linear Dehn function.

Theorem 3.4. *For any finitely generated group G, the following are equivalent.*

1. *G is hyperbolic.*
2. *G has a finite presentation $\langle S \mid R \rangle$ for which Dehn's algorithm solves the word problem.*
3. *G is finitely presented and has linear Dehn function.*
4. *G is finitely presented and has subquadratic Dehn function.*

Exercise 3.5. Suppose G and H are hyperbolic. Prove that $G * H$ is hyperbolic.

Proof. (1) $\implies$ (2)

Let S be a finite, symmetric generating set of G, and let δ be the hyperbolicity constant of $\Gamma(G, S)$. Let $R = \{U \mid U \text{ is a word in } S\ , \|U\| \leq 16\delta, U =_G 1\}$. We will show that $\langle S \mid R \rangle$ is a presentation for G for which Dehn's algorithm solves the word problem.

Let W be a non-empty word in S such that $W =_G 1$. Let p be the path in $\Gamma(G, S)$ with $p_- = 1$ and $\mathbf{Lab}(p) \equiv W$. By assumption, p is not constant and $p_- = p_+$, so by Corollary 2.25, p is not a k-local geodesic for any $k > 8\delta$. This means that p contains a subpath q with $\ell(q) \leq 8\delta$ such that q is not a geodsic. Let r be a geodesic from q_- to q_+. Then $\ell(r) < \ell(q) \leq 8\delta$, so qr^{-1} is a closed loop with $\ell(qr^{-1}) \leq 16\delta$. This means that $\mathbf{Lab}(qr^{-1}) \in R$, and since $\ell(r) < \ell(q)$, Dehn's algorithm will not stop on W.

Thus we have shown that every word in S which is equal to 1 in G by be reduced to the empty word via Dehn's algorithm using only relations from R. Therefore, $\langle S \mid R \rangle$ is a finite presentation for G and Dehn's algorithm solves the word problem for $\langle S \mid R \rangle$.

(2) $\implies$ (3) by Exercise 3.3.

(3) $\implies$ (4) is trivial.

$\square$

We will need a few auxillary results before proving the final implication. First, however, I would like to highlight the following consequence of the above theorem, which is a purely algebraic consequence of the geometric assumption of hyperbolicity.

Corollary 3.6. *If G is a hyperbolic group, then G is finitely presented.*

In fact, this corollary is a special case of a more general finiteness phenomenon for hyperbolic groups which we will see later when we introduce the *Rips complex*.

Now, we return to the proof of Theorem 3.4. It will be convienent to use the following characterization of hyperbolic metric spaces.

Definition 3.7. Let X be a geodesic metric space. Given a triangle $T = pqr$ in X, a point $o \in p$ is called a *bisector* if $d(o,q) = d(o,r)$. We say that X satisfies $\mathrm{Hyp}_6(\delta)$ if for every geodesic triangle $T = pqr$ in X and ever bisector point $o \in p$, $d(o,q) = d(o,r) \leq \delta$.

Exercise 3.8. Show that $\mathrm{Hyp}_6(\delta)$ is equivalent to previous characterizations of hyperbolicity.

Given a polygon $P = p_1 p_2 ... p_n$ in X, we say P is *t-slim* if for any point $a \in p_i$, there exists $j \neq i$ and a point $b \in p_j$ such that $d(a,b) \leq t$. We define the *thickness* of P, denoted $t(P)$, as the minimal constant t such that P is t-slim. Clearly, if X is non-hyperbolic it will have triangles of arbitrarily large thickness. We will show that in this case there are polygons or arbitrarily large thickness t whose perimeter length is linear in t. Next we will show that the area of a polygon P is bounded below by a quadratic function of the thickness of P. These results together will finish the proof of Theorem 3.4.

For the next two lemmas I am following the proofs from [21].

Lemma 3.9 (Thick polygons with linear perimeter)**.** *Suppose a geodesic metric space X is not hyperbolic. Then for all $t_0 \geq 0$, there exists $t \geq t_0$ such that X contains a polygon of thickness t whose perimeter length is at most $46t$.*

Proof. Let $T = pqr$ be a geodesic triangle in X with $x = p_- = q_-$, $y = q_+ = r_-$, and $z = p_+ = r_+$. Let $a \in p$ a point such that $d(a,q) = d(a,r) = t \geq t_0$, and $b \in q, c \in r$ such that $d(a,b) = t = d(a,c)$. Let $e \in [b,y]$ such that $d(d,e) = 7t$ (or $e = y$ if no such point exists). Let f be the point of $[y,c]$ which is closest to e.

Case 1: $d(e,f) \geq 4t$. In this case we analyze the triangle with vertices b, c, and y. Choose a point $o \in [b,y]$ which maximizes $d(o,[y,c])$. Note that by our assumption, $d(o,[y,c]) \geq 4t \geq \frac{1}{2}d(b,c)$ (hence this is an example of a *wide* triangle). Let $D = d(o,[y,c])$, and let $g \in [o,y]$ with $d(o,g) = \frac{3d}{2}$ and $i \in [o,b]$ with $d(o,i) = \frac{3D}{2}$ (as usual, choose g and i to be the endpoints if needed). By definition of o, there exist $h, j \in [y,c]$ with $d(g,h) \leq D$ and $d(i,j) \leq D$. In case $i = b$, we set $j = c$, and if $g = y$, then $h = g = y$.

Now we show the quadrilateral Q with vertices $[g,h,i,j]$ is $\frac{D}{2}$-thick. Indeed, $d(o,[h,j]) \leq d(o,[y,c]) = 2D$. Since $d(o,g) = \frac{3D}{2}$ and $d(g,h) \leq D$, $d(o,[g,h]) \geq \frac{D}{2}$. Similarly, if $i \neq b$, $d(o,[i,j]) \geq \frac{D}{2}$. For the case $i = b$, observe that $d(b,c) \leq \frac{D}{2}$, and since $d(o,c) \geq D$ we must have $d(o,[b,c]) \geq \frac{D}{2}$. Hence, $\frac{t}{2} \leq \frac{D}{2} \leq t(Q)$.

Finally, $d(g,h)$ and $d(i,j)$ are both bounded by D, $d(g,i) \leq 3D$, and hence the triangle inequality gives $d(h,j) \leq 5D$. Therefore the length of the perimeter of Q is at most $10D \leq 20t(Q)$.

Case 2: $d(e,f) \leq 4t$. First, we are going to show that for any $k \in [e,f]$, $d(a,k) \geq t$. First note that $d(f,c) \geq d(b,e) - d(b,c) - d(e,f) \geq 7t - 2t - 4t = t$.

Now, the following two inequalities can be extracted via applying the triangle inequality to the relevent sequences of points, which can be easily traced out if the right picture is drawn.

$$d(x,z) \leq d(x,b) + d(b,c) + d(c,z) \leq d(x,b) + d(c,z) + 2t. \tag{3}$$

$$d(x,e) + d(z,f) \leq d(x,a) + d(a,k) + d(k,e) + d(z,a) + d(a,k) + d(k,f)$$

substituting $d(b,e) + d(c,f) \geq 7t + t$, $d(e,f) \leq 4t$, and $d(x,z) = d(x,a) + d(a,z)$ into the above equation gives

$$d(x,b) + d(z,c) + 8t \leq d(x,z) + 2d(a,k) + 4t$$

Summing this with 3 produces $d(a,k) \geq t$, as desired.

We now continue constructing the desired polygon. Let $g \in [x,a]$ and $i \in [a,z]$ with $d(g,a) = d(a,i) = 3t$ (as usual, we may need to choose the end points, and the proof is easily modified to work in this case). Furthermore, we can assume that there are points $h \in [x,b]$ and $j \in [z,c]$ such that $d(g,h) \leq 2t$ and $d(i,j) \leq 2t$. If these points do not exists, then we will get a wide triangle, and we can then procede as in Case 1.

Now the inequalites $d(a,g) = 3t$ and $d(g,h) \leq 2t$ implie that $d(a,[g,h]) \geq t$. Similarly, $d(a,[i,j]) \geq t$. It follows that the hexigon H with vertices $[g,h,e,f,j,i]$ is at least t-thick, since the distance from a to any other side of H is at least t.

It only remains to esitmate the perimeter of H. I will leave this as an exercise, but using known lengths and estimating the rest with the triangle inequality will produce a bound on the perimeter of $46t$. ☐

Exercise 3.10. Show the hexegon H constructed in the above proof as perimeter $\leq 46t$.

The following lemma can be proved in the context of general geodesic metric spaces. However, we will restrict our attention to the case of Cayley graphs of finitely presented groups. This restriction is purely for convenience of notation, there are no essential differences in the following proofs for general geodesic metric space once a suitable notion of area is defined.

Given a polygon $P = p_1 ... p_n$ in a Cayley graph $\Gamma(G,S)$, let $\mathbf{Lab}(P) \equiv \mathbf{Lab}(p_1)...\mathbf{Lab}(p_n)$. Also, we slightly modify our notion of thickness for such polygons by only measuring distance between points which are vertices of the Cayley graph. This change decreases the thickness of a polygon by at most 1, so it clearly does not affect our previous result.

Lemma 3.11 (Thick polygons have quadratic area)**.** *Let G be a group given by a finite presentation $\langle S \mid R \rangle$, and let $M = \max_{r \in R}\{\|r\|\}$. Let P be a polygon in $\Gamma(G,S)$ of thickness t with $W \equiv \mathbf{Lab}(P)$. Then Area$(W) \geq \frac{4}{M^3}t^2$.*

Proof. By definition of thickness, there exists some side p of P and a vertex $a \in p$ such that $d(a, P \setminus p) \geq t$. Let q be the remaining sides of P, so $P = pq$.

We now fill the closed loop pq with a van Kampen diagram Δ. We now set $x_0 = y_0 = a$ and inductively define a sequence of simple closed paths, $z_i = x_i y_i$ for $0 \leq i \leq \frac{2t}{M} + 1$ which satisfy the following properties:

1. x_i is a subpath of p containing x_{i-1}.

2. For every vertex $b \in y_i$, $d(a,b) \leq \frac{Mi}{2}$.

3. The subdiagram Δ_i bound by $z_i = x_i y_i$ contains the maximal area over all simple closed paths which satisfy the first two properties.

Increasing x_i is necessary, we can assume that each y_i has no edges in common with p. Furthermore, if $i \leq \frac{2t}{M}$ then y_i does not intersect q, since if $b \in y_i$, $d(a,b) \leq \frac{Mi}{2} < t$. Suppose b is a vertex of both y_{i-1} and y_i. Then b is a vertex of $\partial\Delta_i$, and since b does not belong to the boundary of Δ, there must exist some 2-cell Π such that $b \in \partial\Pi$ but Π does not belong to Δ_i. But since $b \in y_{i-1}$, $d(a,b) \leq \frac{M(i-1)}{2}$, and the definition of M gives that for any vertex $c \in \partial\Pi$, $d(b,c) \leq \frac{M}{2}$. Hence Δ_i could be enlarged by adding Π without violating the first two conditions, which contradicts the third condition of the definiton of Δ_i. Thus the vertices of y_i and y_{i-1} are disjoint.

It follows that every edge of y_i belongs to the boundary of a 2-cell which is contained in Δ_i but not in Δ_{i-1}. Let m_i be the number of such faces , and note that m_i is at least $\frac{\ell(y_i)}{M}$. Since y_i and y_{i-1} are disjoint, x_i must contain at least 2 more vertices then x_{i-1}, one on each end. Thus, $\ell(x_i) \geq 2i$, and since each x_i is a subpath of a geodesic, $d((x_i)_-, (x_i)_+) \geq 2i$. y_i has the same endpoints as x_i, so $\ell(y_i) \geq 2i$, which implies that $m_i \geq \frac{2i}{M}$. Finally, we get

$$Area(\Delta) \geq \sum_{i=1}^{\frac{2t}{M}+1} m_i \geq \sum_{i=1}^{\frac{2t}{M}+1} \frac{2i}{M} \geq \frac{4t^2}{M^3}.$$

□

Combining the previous two lemmas gives the following corollary, which finishes the proof of Theorem 3.4 (in particular, it shows that (4) $\Longrightarrow$ (1) part the proof of Theorem 3.4).

Corollary 3.12. *Let G be a finitely presented group which is not hyperbolic. Then the Dehn function of G is at least quadratic.*

Exercise 3.13. Suppose G has a presentation $\langle S \mid R \rangle$ with sublinear Dehn function. Prove that $R = \emptyset$, so in fact G is the free group on S.

Theorem 3.4 shows that hyperbolic groups have solvable word problem. In fact, they also have solvable conjugacy problem.

Lemma 3.14. *Suppose G is group generated by a finite set S and $\Gamma(G,S)$ is δ-hyperbolic. Let W, U, and V be words in S such that no shorter words in S represents the same elements of G as U and V, $W^{-1}UW =_G V$, and W is the shortest word which conjugates a cyclic shift of U to a cyclic shift of V. Then either:*

1. $\|W\| \leq \|U\| + \|V\| + 4\delta + 2$

   or

2. There exists words Y and Z in S with $\|Z\| \leq 4\delta$ and $\|Y\| < \|W\|$ such that $Y^{-1}UY =_G Z$.

Proof. Suppose $\|W\| \geq \|U\|+\|V\|+4\delta+2$. Let $Q = s_1 q (s_2)^{-1} p^{-1}$ be a quadrilateral in $\Gamma(G,S)$ such that $\mathbf{Lab}(p) \equiv U$, $\mathbf{Lab}(q) \equiv V$, and $\mathbf{Lab}(s_1) \equiv \mathbf{Lab}(s_2) \equiv W$. Also s_1 is the shortest path which connects p to q, otherwise there would be a word shorter then W which conjugates a cyclic shift of U to a cyclic shift of V. In particular, s_1 and s_2 are geodesic, so Q is a geodesic quadrilateral.

Let $x \in s_1$ such that $d(x,(s_1)_-) = \|U\| + 2\delta + 1$. Then $d(x,p) > 2\delta$, and similarly $d(x,q) > 2\delta$. By hyperbolicity, there must exist a point $y \in s_2$ such that $d(x,y) \leq 2\delta$. Let $z \in s_2$ be the point "parallel" to x, that is $d(z,(s_2)_-) = d(x,(s_1)_-)$. We will show that y must be close to z.

Let $s_1 = \sigma_1\sigma_2$ where $(\sigma_1)_+ = (\sigma_2)_- = x$, and let $s_2 = \tau_1\tau_2$ where $(\tau_1)_+ = (\tau_2)_- = y$. If $d(y,z) > 2\delta$, then one of the two paths $\sigma_1[x,y]\tau_2$ or $\tau_1[y,x]\sigma_2$ will be a path from p to q which is shorter then s_1. Since we assumed that no such path exists, we get that $d(y,z) \leq 2\delta$, and hence $d(x,z) \leq 4\delta$.

Let $Z \equiv \mathbf{Lab}([x,z])$ and $Y \equiv \mathbf{Lab}(\sigma_1)$. Note that Y is also the label of the initial segment of s_2 which ends at z, which give the desired equality $Y^{-1}UY =_G Z$. Finally we note that $\|Y\| < \|W\|$, since σ_1 is a proper subpath of s_1. □

Exercise 3.15. Let G be a hyperbolic group. Describe an algorithm which solves the conjugacy problem in G.

Theorem 3.16. *If G is a hyperbolic group, then the conjugacy problem is solvable in G.*

4 The Rips complex and finiteness properties

Let X be a metric space and $d > 0$. The *Rips complex* is the simplicial complex $P_d(X)$ which has a vertex for every point in X and an n-simplex corresponding to every $n+1$ tuple of vertices $\{x_0, x_1, ..., x_n\}$ which satisfies $d(x_i, x_j) \leq d$ for all $0 \leq i < j \leq n$.

Proposition 4.1. *Let Y be a geodesic and δ-hyperbolic metric space and X and r-dense subset of Y. Then for all $d \geq 4\delta + 2r$, $P_d(X)$ is contractible.*

Proof. The space $P_d(X)$ is contractible if and only if $\pi_n(P_d(X)$ is trivial for all $n \geq 1$. Since any continuous map $S^n \to P_d(X)$ has compact image, it suffices to show that every finite subcomplex of $P_d(X)$ is contractible.

Let x_0 be a fixed vertex of $P_d(X)$, and let K be a finite subcomplex of $P_d(X)$. Let v be the vertex of K of maximal distance (using the metric from Y) to x_0.

Case 1: $d(x_0, v) \leq \frac{d}{2}$. In this case, K is contained in a single simplex of $P_d(X)$, hence K is contractible.

Case 2: $d(x_0, v) > \frac{d}{2}$. In this case, let $[x_0, v]$ be a geodesic in Y from x_0 to v, and let y be a point on this geodesic such that $d(y,v) = \frac{d}{2}$. Let v' be a point in X such that $d(y,v') \leq r$. We are going to show that any simplex σ of K containing v is a face of a larger simplex obtained by joining v' to σ.

Suppose u is a vertex in K and $d(u,v) \leq d$. Considering the triangle in Y with vertices at x_0, v and u, there is a point $w \in [x_0,u] \cup [v,u]$ such that $d(y,w) \leq \delta$. Suppose first that $w \in [x_0,u]$. In this case, the fact that $d(x_0,u) \leq d(x_o,v)$ by our choice of v and the triangle inequality yields:

$$d(x_0, v) \leq d(x_0, w) + d(w, y) + d(y, v) \leq d(x_0, u) - d(u, w) + \delta + \frac{d}{2} \leq d(x_0, v) - d(u, w) + \delta + \frac{d}{2}.$$

Subtracting $d(x_0, v) - d(u, w)$ from both sides of this equation gives $d(u, w) \leq \delta + \frac{d}{2}$. Hence

$$d(v', u) \leq d(v', y) + d(y, w) + d(w, u) \leq r + \delta + \delta + \frac{d}{2} \leq d.$$

Now suppose $w \in [v, u]$. In this case, $d(v, w) \geq \frac{d}{2} - \delta$, otherwise there would be a shortcut from y to v. Then

$$d(u, w) = d(u, v) - d(v, w) \leq d - \frac{d}{2} + \delta = \frac{d}{2} + \delta.$$

Which implies

$$d(v', u) \leq d(v', y) + d(y, w) + d(w, u) \leq r + \delta + \frac{d}{2} + \delta \leq d.$$

Hence, if σ is a simplex of $P_d(X)$ containing v, then $\sigma \cup \{v'\}$ is also a simplex of $P_d(X)$. Let $\sigma' = (\sigma \setminus \{v\}) \cup \{v'\}$, that is the simplex where v is replaced by v'. There is a natural affine homotopy which takes σ to σ' through the simplex $\sigma \cup \{v'\}$. Hence we get a homotopy $K \to K'$, where K' is the finite subcomplex obtained by replacing v with v' in every simplex of K which contains v. Note $d(x_0, v') < d(x_0, v)$, so applying the above procedure finitely many times (since K is a fintie complex) will show that K is homotopic to a subcomplex that is contained in a single simplex of $P_d(X)$ as in Case 1.

□

In particular, if G is a hyperbolic group generated by a finite set S, we apply this theorem where X is the set of vertices of the Cayley graph $Y = \Gamma(G, S)$. The corresponding Rips complex $P_d(X)$ will be locally finite and finite dimensional because there are only finitely many vertices in any ball of radius d in $\Gamma(G, S)$. Furthermore, the natural action of G on X extends to a simplicial action of G on $P_d(X)$ with compact quotient. This action is free and transitive on the set of vertices of $P_d(X)$, and it follows that the stabilizer of any simplex is finite; in particular, if G is torsion-free then the action of G on $P_d(X)$ is free, hence the quotient is a $K(G, 1)$ whenever $P_d(X)$ is contractible. We collect these properties in the following theorem.

Theorem 4.2. *Let G be a hyperbolic group. Then there exists a simplicial complex P such that*

1. *P is locally finite, finite dimensional, and contractible.*
2. *G acts simplicially and cocompactly on P.*
3. *G acts freely and transitively on the vertices of P, hence the stabilizer of any simplex of P is finite.*
4. *If G is torsion-free then the quotient P/G is a finite $K(G, 1)$.*

We will now mention some applications of Theorem 4.2 to topological finiteness properties of hyperbolic groups. For more background on finiteness properties of groups, see [10].

Recall that a CW-complex Y is called a $K(G,1)$ if $\pi_1(Y) = G$ and Y is *aspherical*, that is the universal cover $\tilde{Y}$ is contractible (equivalently, $\pi_n(Y)$ is trivial for all $n \geq 2$). Every group G admits a $K(G,1)$, and such a space is unique up to homotopy equivalence. If Y is a $K(G,1)$, then the homology (respectively, cohomology) of G can be defined by $H_n(G) := H_n(Y)$ (respectively, $H^n(G) := H^n(Y)$). For our purposes, we will only consider homology with coefficients in $\mathbb{Z}$.

Definition 4.3. A group G is said to be of type F_m for $m \in \mathbb{N} \cup \{\infty\}$ if G admits a $K(G,1)$ with finitely many cells in each dimension $\leq m$. G is said to be of type F if G admits a finite $K(G,1)$.

Note that G satisfies F_1 if and only if G is finitely generated, and G satisfies F_2 if and only if G is finitely presented. Also, if G is type F_n then $H_n(G)$ is finitely generated.

Clearly there are natural implications:

$$F_1 \impliedby F_2 \impliedby F_3 \impliedby ... \impliedby F_\infty \impliedby F.$$

The *geometric dimension* of G, denoted $gd(G)$, as the minimal d such that G has a d-dimensional $K(G,1)$, or ∞ if G has no finite-dimensional $K(G,1)$. Clearly $gd(G) = d$ implies that $H_n(G) = \{0\}$ for all $n \geq d$. It is well-known that this can only happen for groups that are torsion-free, hence we get the following:

$$F \implies \text{finite geometric dimension} \implies \text{torsion-free}.$$

If G is a torsion-free hyperbolic group, then the quotient of the Rips complex $P_d(G)/G$ is a finite $K(G,1)$ for sufficiently large d, hence we get the following.

Corollary 4.4. *If G is a torsion-free hyperbolic group, then G is type F. Hence G has finite geometric dimension, and $H_n(G) = \{0\}$ for all $n > gd(G)$.*

If G is a hyperbolic group with torsion, then it cannot be type F. Howeve the action on the Rips complex is still sufficient to prove it is type F_∞, see [10, Remark 7.3.2].

Corollary 4.5. *If G is any hyperbolic group, then G is type F_∞, hence $H_n(G)$ is finitely generated for all $n \geq 1$.*

4.1 Hyperbolicity of subgroups

Exercise 4.6. Find the mistake in the following proof.

Claim: if G is hyperbolic and $H \leq G$, then H is hyperbolic.

Proof. Let S be a finite generating set, and consider H with the metric d_S. Equivalently, you can consider H with the metric induced by considering H as a subspace of $\Gamma(G,S)$. Since $\Gamma(G,S)$ is hyperbolic, every subspace with the induced metric is hyperbolic; this follows immediately from $\mathrm{Hyp}_1(\delta)$. Furthermore, it is easy to check that the natural action of H on (H, d_S) is proper and cobounded. Hence by the Milnor-Svarc lemma, H is finitely generated and a Cayley graph of H is quasi-isometric to (H, d_S), therefore this Cayley graph is hyperbolic. □

It is natural to ask whether all subgroups of hyperbolic groups are hyperbolic. The answer is no, simply because even the free group F_2 contains infinitely generated subgroups, for example $[F_2, F_2]$. A better question is whether all finitely generated subgroups of hyperbolic groups are hyperbolic. In this case the answer is yes for free groups, as all subgroups of free group are free. It is also yes for fundamental groups of compact orientable surfaces of genus ≥ 2, as all subgroups are either free or are fundamental groups of compact orientable surfaces of higher genus.

However in general the answer to this question is no, as there are finitely generated subgroups of hyperbolic groups which are not finitely presented. We will see how to constuct such examples in the next section using small cancellation theory.

This raises another natural question: are all finitely presented subgroups of hyperbolic groups hyperbolic? The answer again is no, as there are examples of subgroups of hyperbolic groups which are finitely presented but not type F_3. The first such examples were constructed by Brady [4], and more examples were constructed recently by Lodha [16].

Again, this leads to another natural question which is currently open.

Question 4.7. *If G is a hyperbolic group and $H \leq G$ is type F_3, is H hyperbolic?*

This question is also open if 3 is replaced by any $n \geq 3$. One can also ask whether any finiteness property is sufficient to guarantee hyperbolicity of subgroups, which leads to another open question.

Question 4.8. *If G is a hyperbolic group and $H \leq G$ is type F, is H hyperbolic?*

5 Small cancellation theory

5.1 Small cancellation conditions, hyperbolicity and asphericity

Throughout this section, for all group presentations $\langle S \mid R \rangle$ the set of relations R is assumed to be *cyclically reduced* that is, every $r \in R$ is cyclically reduced. Also, we denote by $\overline{R}$ the *symmetrized closure* of R, that is the set of all cyclic shifts of elements of R and their inverses.

Definition 5.1. Given a group presentation $\langle S \mid R \rangle$, a word U is called a *piece* if there exist $r, r' \in \overline{R}$ and (possibly empty) words V, W, V', W' such that

1. $r \equiv VUW$.
2. $r' \equiv V'UW'$.
3. $WV^{-1} \not\equiv W'(V')^{-1}$.

Exercise 5.2. Describe which words are pieces for the following presentations.

1. $\langle a, b \mid aba^{-1}b^{-1} \rangle$.
2. $\langle a, b \mid (ab)^4 \rangle$.
3. $\langle a, b \mid abab^{10}ab^{100} \rangle$.

4. $\langle a_1, a_2, b_1, b_2 \mid [a_1, b_1][a_2, b_2]\rangle$.

The main idea of small cancellation theory is to put restrictions on the ways that pieces can occur in a group presentation. These restrictions then imply a number of group theoretic properties, and it is usually straightforward to construct presentations which satisfy these restrictions. Hence, small cancellation theory provides a set of tools for building a large variety of examples (and counterexamples) in group theory.

The following is the most commonly used small cancellation condition, sometimes called the *metric small cancellation condition.*

Definition 5.3. Let $0 \leq \lambda \leq 1$. A presentation $\langle S \mid R\rangle$ satisfies the $C'(\lambda)$-condition if any piece U which is a subword of a relation $r \in \overline{R}$ satisfies $\|U\| < \lambda\|r\|$.

Exercise 5.4. For each presentation in the previous exercise, find the optimal λ such that the presentation satisfies $C'(\lambda)$. Also find the optimal λ for the standard presentation of the fundamental group of a compact, orientable, genus g surface.

Exercise 5.5. Let r be a cyclically reduced word in S which is not a proper power in $F(S)$. Then $\langle S \mid r^n\rangle$ satisfies $C'(\frac{1}{n})$.

Let Δ be a van Kampen diagram. Suppose Π_1 and Π_2 are cells of Δ whose boudaries intersect in a path q. Let $\partial\Pi_1 = qp_1$ and $\partial\Pi_2 = qp_2$. If $\mathbf{Lab}(p_1) \equiv \mathbf{Lab}(p_2)$, then (Π_1, Π_2) are called a *cancellable pair* of cells. Note that if this happens, then $\partial(\Pi_1 \cup \Pi_2) = p_1 p_2^{-1}$ which is labeled by a freely trivial word. Hence the cells Π_1 and Π_2 can be removed and the resulting hole can be "sewed" together by attaching each edge in p_1 to the edge in p_2 which freely cancels its label. Note that this process does not affect any part of Δ outside of $\Pi_1 \cup \Pi_2$.

More generally, (Π_1, Π_2) is a cancellable pair if there is a path t in Δ from Π_1 to Π_2 such that $\mathbf{Lab}(t) =_{F(S)} 1$ and the label of $\partial\Pi_1$ read counterclockwise starting at t_- is the same as the label of $\partial\Pi_2$ read counterclockwise starting at t_+. This means that the label of $\partial\Pi_1 t \partial\Pi_2 t^{-1}$ is freely trivial, and the same procudure as above can be applied to remove such pairs of cells.

A van Kampen diagram is called *reduced* if it contains no cancellable pair. Since we have seen that a cancellable pair of cells can be removed to produce a van Kampen diagram with fewer cells and the same boundary label, it follows that if Δ has the minimal area among all diagrams with a given boundary label, then Δ must be reduced. Hence the get the following strengthening of the van Kampen lemma.

Lemma 5.6. *If $G = \langle S \mid R\rangle$ and $W =_G 1$, then there exists a reduced van Kampen diagram Δ with $\mathbf{Lab}(\partial\Delta) \equiv W$.*

The reason we want to consider reduced diagrams is the following: If Π_1 and Π_2 are cells of Δ and q is a maximal connected component of $\Pi_1 \cap \Pi_2$, then $\mathbf{Lab}(q)$ is a piece if and only if (Π_1, Π_2) is not a cancellable pair. Hence we obtain the following geometric interpretation of the $C'(\lambda)$-condition.

Lemma 5.7. *Suppose Δ is a reduced van Kampen diagram over a presentation $\langle S \mid R\rangle$ which satisfies $C'(\lambda)$. If Π_2 and Π_2 are 2-cells of Δ and q is a subpath of $\partial\Pi_1 \cap \partial\Pi_2$, then*

$$\ell(q) < \lambda \min\{\ell(\partial\Pi_1), \ell(\partial\Pi_2)\}.$$

Exercise 5.8. Let Φ be a finite, connected, planar graph and let v, e, and f denote the number of vertices, edges, and faces of Φ. Show that if Φ has no 1-gons, 2-gons, or vertices of degree 1, then

$$e < 3(v-1). \tag{4}$$

The following is a fundamental result in small cancellation theory.

Theorem 5.9 (Greendlinger Lemma)**.** *Let G be a group defined by a presentation $\langle S \mid R\rangle$ which satisfies $C'(\lambda)$ for some $0 \le \lambda \le \frac{1}{6}$. Then for any cyclically reduced non-empty word W with $W =_G 1$ and no proper subword of W trivial in G, W contains a subword V such that V is also a subword of some relation $r \in \overline{R}$ and*

$$\|V\| > (1-3\lambda)\|r\|.$$

Proof. Without loss of generality, we assume that $\lambda = \frac{1}{n}$ for some $n \in \mathbb{N}$, $n \ge 6$. Let W be a word in S with $W =_G 1$, and let Δ be reduced a van Kampen diagram over $\langle S \mid R\rangle$ with $\mathbf{Lab}(\partial\Delta) \equiv W$.

We will first prove the result under the following assumption, then we will show how to remove this assumption:

$(*)$ For every cell Π of Δ, $\partial\Pi$ is a simple path.

We build a planar graph Φ duel to Δ in the following way: we put one vertex inside each 2-cell of Δ and one vertex O in the exterior of Δ. We put an edge between the vertex inside Π_1 and Π_2 for each maximal conneced component of $\partial\Pi_1 \cap \partial\Pi_2$ which contains at least one edge. Similarly, we put an edge between the vertex inside Π and O for each maximal connected compenent of $\partial\Pi\cap\partial\Delta$. Φ is clearly a finite planar graph, since it can be naturally realized as a graph duel to the planar graph obtained from the 1-skeleton of Δ by replacing degree 2 vertex by a singe edge.

$(*)$ implies that Φ has no 1-gons and no vertices of degree 1. If Φ contained a 2-gon, then the arcs corresponding to these edges would not be maximal, which contridicts our construction of Φ. Therefore the formula (4) applies to Φ.

Now, we suppose towards a contridiction that for every 2-cell Π every maximal connected component $q \subset \partial\Pi \cap \partial\Delta$ satisfes $\ell(q) \le (1-\frac{3}{n})\ell(\partial\Pi)$. We are going to show that this assumption contradicts (4).

For a vertex $y \ne O$ of Φ, define the *exterior degree* of y, $e(y)$, as the number of edges connecting y to O, and define the *interior degree* as the number of edges joining y to vertices inside other 2-cells of Δ. Define the *weight* of a vertex as $w(y) = e(y) + \frac{1}{2}i(y)$. The point of this weight is that $\sum_{y\ne O} w(y) = e$.

If $e(y) = 0$, then the $C'(\frac{1}{n})$-condition implies the $i(y) \ge n+1$, since the arc corresponding to any edge of Φ adjacent to e is labeled by a piece. Hence $w(y) \ge (\frac{n+1}{2}) \ge 3$ since $n \ge 6$.

If $e(y) = 1$, then $i(y) \ge 4$ by the $C'(\frac{1}{n})$-condition and the bound on the length of the exterior arc of the corresponding 2-cell. Hence $w(y) \ge 1 + \frac{4}{2} = 3$.

If $e(y) = 2$, then $i(y) \ge 2$, since there must be at least two adjacent 2-cells in order to separate these arcs on the boundary. Hence $w(y) \ge 3$.

If $e(y) \ge 3$, then clearly $w(y) \ge 3$.

Thus we have shown that for every $y \neq O$, $w(y) \geq 3$. It follows that

$$e = \sum_{y \neq O} w(y) \geq 3(v-1)$$

which clearly contradicts (4). Thus we have shown that the desired result holds whenever Δ satisfies $(*)$.

We show now that every such graph Φ satisfies $(*)$. Suppose not; then Δ contains a 2-cell Π such that some subpath $q \subset \partial\Pi$ is a simple closed path. It follows that q is the boundary of a subdiagram Δ' of Δ. If Δ' is chosen as a minimal such subdiagram, then Δ' satisfies $(*)$. Hence the proof above shows that some 2-cell Π' of Δ' contains a subpath $q \subseteq \partial\Pi' \cap \partial\Delta' \subseteq \partial\Pi' \cap \partial\Pi$ with $\ell(q) > \frac{1}{2}\ell(\partial\Pi')$, which clearly violates the C'-condition. $\square$

Corollary 5.10. *Let G be a group given by a presentation $\langle S \mid R\rangle$ which satisfies $C'(1/6)$. Then G is hyperbolic.*

Proof. If W is a word in S with $W =_G 1$, then the words V and r provided by the Greendlinger Lemma satisfy

$$\|V\| > (1 - \frac{3}{6})\|r\| = \frac{1}{2}\|r\|.$$

This is precisely what it means for Dehn's algorithm to solve the word problem for the presentation $\langle S \mid R\rangle$, hence G is hyperbolic by Theorem 3.4. $\square$

Corollary 5.11. *Let G be a group given by a fniite $C'(\frac{1}{n})$ presentation $\langle S \mid R\rangle$ for some $n \geq 6$, and let $\rho = \min\{\|r\| \mid r \in R\}$. If $f\colon F(S) \twoheadrightarrow G$ is the natural quotient map and $B = \{g \in F(S) \mid |g|_S \leq \frac{n-3}{2n}\rho\}$, then $f|_B$ is injective.*

Proof. Suppose W and U are words in S with $W \neq_{F(S)} U$ but $W =_G U$. Let V be the word obtained by freely and cyclicaly reducing WU^{-1}. Then $V =_G 1$ so by Theorem 8.3 V contains a subword of length $\frac{n-3}{n}\rho$, hence $\|WU^{-1}\| \geq \|V\| \geq \frac{n-3}{n}\rho$. It follows that at least one of W or U has length $\geq \frac{n-3}{2n}\rho$. $\square$

Remark 5.12. While there are some results for $C'(\lambda)$ groups when $\frac{1}{6} < \lambda \leq \frac{1}{5}$ (including solvability of the word problem), the C' condition becomes useless for larger λ. Indeed, if $\lambda > \frac{1}{5}$, then every finitely presented group G can be given a presentation which satisfies $C'(\lambda)$ (see [21]).

A *spherical diagram* is a van Kampen diagram which is embedded on the sphere instead of on the plane. Equivalently, this can be viewed as an ordinary van Kampen diagram Δ with the extra requirement that $\mathbf{Lab}(\partial\Delta) \in \overline{R}$. Hence spherical diagrams encode "relations among relations."

The notion of a cancellable pair and a reduced diagram are the same for spherical diagrams as for ordinary van Kampen diagrams. Applying the same proof as in Theorem 8.3 together with the fact that the constructed graph Φ will have no "exterior" vertex will produce a contradiction with the existence of a reduced spherical diagram.

Proposition 5.13. *If $\langle S \mid R\rangle$ satisfies $C'(\frac{1}{6})$, then there are no reduced spherical diagrams over $\langle S \mid R\rangle$.*

Given a group presentation $\langle S \mid R \rangle$ for a group G, there is an associated CW-complex Y with $\pi_1(Y) \cong G$, called the *presentation complex.* This Y contains a single vertex v, one edge (labeled by s) with both ends glued to v for each $s \in S$, and one 2-cell Π for each $r \in R$, glued to the 1-skeleton of Y such that $\partial\Pi$ is labeled by r.

The universal cover $\tilde{Y}$ is called the *Cayley complex* associated to $\langle S \mid R \rangle$. Note that the 1-skeleton of the Cayley complex can be naturally identified with $\Gamma(G, S)$, and the 2-skeleton of $\tilde{Y}$ is obtained by gluing, for each $g \in G$ and $r \in R$, a 2-cell with boundary a loop based at g and labeled by r.

The presentation $\langle S \mid R \rangle$ is called *asperical* if the associated presentation complex Y is aspherical, or equivalently if the Cayley complex $\tilde{Y}$ is contractible[2]. Since Y is a 2-dimensional CW complex, Y is aspherical if and only if $\pi_2(Y) = \pi_2(\tilde{Y}) = \{1\}$.

Now any map $S^2 \to \tilde{Y}$ can be encoded as a spherical diagram. If $\langle S \mid R \rangle$ is a $C'(\frac{1}{6})$ presentation, this diagram must be reducible (as long as it is non-empty), and "cancelling" a cancellable pair of 2-cells corresponds to a homotopy in the Cayley complex unless the diagram contains exactly 2 2-cells. In this case, the boundary of these 2 cells, which we picture as the equator of the spherical diagram, is a loop in $\Gamma(G, S)$ and the upper and lower hemispheres of the sphere are mapping to discs corresponding to relations whose boundary in this loop. If these are both the same disc, then this map is homotopically trivial. However, $\tilde{Y}$ may have two different discs with the same boundary label, which will produce a non-homotopically trivial sphere in $\tilde{Y}$.

If two discs can be glued to the same boundary in $\tilde{Y}$, there must be some $r, r' \in R$ are proper cyclic shifts of each other. There are essentially two ways that this can happen. The first is that r and r' are distinct as elements of R. To rule this out we assume that R is *concise*, that for any $r \in R$, no cyclic shift of r or r^{-1} belongs to R (equivalently, for all $R_0 \subsetneq R$, $\overline{R_0} \neq \overline{R}$). Clearly from any presentation we can obtain a concise presentation by removing unnecessary elements of R.

The second way for two discs to be glued to the same boundary label is if some relation r is equal to a proper cyclic shift of itself, which will imply that that r is a proper power, that is $r \equiv r_0^n$ for some word r_0 and some $n \geq 2$. Such a relation will produce an essential sphere in Caley complex $\tilde{Y}$, hence it is necessary to rule out these relations if we want $\langle S \mid R \rangle$ to be aspherical.

Theorem 5.14. *Suppose $\langle S \mid R \rangle$ is a concise, $C'(\frac{1}{6})$ presentation and no $r \in R$ is a proper power. Then $\langle S \mid R \rangle$ is an apherical presentation.*

In particular, this theorem implies that the corresponding presentation complex Y is a $K(G, 1)$. Since Y is clearly 2-dimensional, this implies that $gd(G) \leq 2$, and hence G is torsion-free.

Corollary 5.15. *If G has a $C'(\frac{1}{6})$ presentation in which no relation is a proper power, then $gd(G) \leq 2$.*

Corollary 5.16. *If G has a $C'(\frac{1}{6})$ presentation $\langle S \mid R \rangle$, then G is torsion-free if and only if no relation $r \in R$ is a proper power.*

If G is the fundamental group of a closed hyperbolic n-dimensional manifold M, then M is a $K(G, 1)$, hence $gd(G) = n$.

[2]There are actually a few different defintions of an aspherical presentation, the relationship between then is examined in [6].

Corollary 5.17. *There are hyperbolic groups which do not admit a $C'(\frac{1}{6})$ presentation.*

5.2 Rips construction

The Rips construction uses small cancellation theory to transfer pathological subgroups of finitely presented groups to create pathological subgroups of hyperbolic groups.

Theorem 5.18 (The Rips Construction)**.** *Let Q be a finitely presented group. There there exists a short exact sequence*

$$1 \to N \to G \to Q \to 1$$

such that N is 2-generated and G is given by a $C'(\frac{1}{6})$ presentation.

Remark 5.19. It will be obvious from the proof that $C'(\frac{1}{6})$ can be replaced by $C'(\lambda)$ for any $\lambda > 0$ and the theorem will still hold.

Proof. Let $\langle s_1, ..., s_n \mid r_1, ..., r_m \rangle$ be a finite presentation for Q, and let $M = \max\{\|r_i\|\}$. Let G be the group given by the following presentation:

$$\langle s_1, ..., s_n, a, b \mid r_1 w_1, ..., r_m w_m, s_1^{-1} a s_1 u_1, ..., s_n^{-1} a s_n u_n, s_1^{-1} b s_1 v_1, s_n^{-1} b s_n v_n \rangle \tag{5}$$

where each w_i, u_i, and v_i is chosen as a word in $\{a, b\}$ such that each of these words has length much larger then M, and the whole set $\{w_1, ..., w_m, u_1, ..., u_n, v_1, ..., v_n\}$ satisfies $C'(\lambda)$ for sufficiently small λ. It follows in the presentation 5, if a piece is a subword of some r_i, then is is short compared to any relation it occurs in, and the pieces which do not occur as subwords of r_i are short compared to the relation they occur in by construction of the words w_i, u_i, and v_i. There may be a piece which is partially a subword of some r_i and partially a subword of some w_i, u_i, or v_i, but then this piece is composed of 2 short pieces. Hence can choose sufficiently long words w_i, u_i, and v_i and sufficiently small λ such that the presentation 5 satisfies $C'(\frac{1}{6})$.

Let $N = \langle a, b \rangle \leq G$. Note that if we conjugate a or b by any generator of G we obtain an element of N, hence N is normal in G. If we add the relations $a = 1$ and $b = 1$ to the presentation 5, then we obtain a presentation of the group Q. Thus, $G/N \cong Q$. ☐

Exercise 5.20. Show how to explicitly construct the words $w_1, ..., w_m$, $u_1, ..., u_n$, and $v_1, ..., v_n$ used in the previous proof.

Exercise 5.21. Let Q be a finitely presented group, and let G be the group provided by the Rips construction and $\varphi\colon G \twoheadrightarrow Q$ the natural quotient map. Then for any finitely generated subgroup $H \leq Q$, prove that $\varphi^{-1}(H)$ is a finitely generated subgroup of G.

In order to apply our strategy of transferring pathological subgroups from finitely presented groups to hyperbolic groups, we first need a way to construct pathological subgroup of finitely presented groups. It is much easier to construct infinitely presented groups with desired properties, and the following allows us to embed these groups into finitely presented groups.

A presentation $\langle S \mid R \rangle$ is called *recursive* if S is finite or countable and indexed by $\mathbb{N}$ and there exists an algorithm which lists the elements of R.

Theorem 5.22 (Higman embedding theorem)**.** *Every recursively presented group embeds into a finitely presented group.*

We now apply the Rips construction to build various examples.

Corollary 5.23. *1. There exists a hyperbolic group G and finitely generated subgroups H_1, H_2 such that $H_1 \cap H_2$ is not finitely generated.*

2. *There exists a hyperbolic group G and a finitely generated subgroup H which is not finitely presented.*

3. *There exists a hyperbolic group G and a finitely generated subgroup N such that the membership problem for N is not solvable.*

4. *There is no algorithm which can accept as input a finite presentation of a hyperbolic group G and determines the rank of G.*

Proof. (1) Consider the amalgamated product of two copies of F_2 over an infinitely generated subgroup, for example

$$P = \langle a, b, c, d \mid b^{-i}ab^{i} = d^{-i}cd^{i}, i \in \mathbb{N} \rangle$$

This group is clearly recursively presented, and $H_1 \cap H_2$ is not finitely generated for $H_1 = \langle a, b \rangle$, $H_2 = \langle c, d \rangle$. Let Q be a finitely presented group which contains P as a subgroup, and let G be the group provided by the Rips constuction. Then $\varphi^{-1}(H_1)$ and $\varphi^{-1}(H_2)$ is finitely generated, but $\varphi^{-1}(H_1) \cap \varphi^{-1}(H_2)$ cannot be finitely generated since it surjects onto $H_1 \cap H_2$.

(2) Let H be any finitely generatd, recursively presented group which is not finitely presented. For example, we can choose H to be the *lamplighter group* $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}$. Apply the Higman embedding theorem to obtain a finitely presented group Q which contains H, and the Rips construction to obtain G. Then $\varphi^{-1}(H)$ is finitely generated, and if this group was finitely presented then we could add the relation $a = 1, b = 1$ to obtain a finite presentation of H.

(3) Let Q be a finitely presented group such that the word problem in Q is not solvable. Applying the Rips construction to get G and N, and it follows immediatly that the membership problem for N is not solvable.

(4) Let P be a group given by a finite presentation $\langle S \mid R \rangle$, and let Q be the free-product of k copies of P. By Grushko's theorem, $rank(Q) \geq k$ if and only if $P \neq \{1\}$. Applying the Rips construction to Q, we get a group G where $rank(G) \geq k$ if $P \neq \{1\}$ and $rank(G) \leq 2$ if $P = \{1\}$. Since there is no algorithm to determine if a given presentation represents the trivial group, the rank problem is undecidable for hyperbolic groups.

□

Remark 5.24. In fact, it can be shown that the finitely generated normal subgoup N provided by the Rips construction will not be finitely presented as long as Q is infinite.

5.3 Random groups

It is commonly said that "almost every (finitely presented) group is hyperbolic." The "almost every" here is meant in a statistical sense, and in order to make this precise one needs a suitble notion of a "random group." There are a few different models of random groups, we will mention here two of the most commonely used models, the *few relators model* and the *density model* both of which were introduced by Gromov. For more information about random groups as well as proofs of the following theorems, see the survey by Ollivier [18] and references therein.

For each model of random groups, we fix a finite set S with $|S| \geq 2$, and we probabilistically choose some set of freely reduced words in S of length $\leq l$ as a set of relations. If P is some property of groups, we say that a *random group has P* (in this model) if the probability that a presentation chosen in this way defines a group with P goes to 1 as $l \to \infty$.

In the *few relators model*, we fix some constant k and choose $r_1, ..., r_k$ uniformly at random among all reduced words in S of length $\leq l$. It is not hard to see that as l gets large, the pieces which occur in these relations will typically be very small.

Theorem 5.25. *A random group in the few relators model satisfies $C'(\lambda)$ for any $\lambda > 0$. In particular, a random group in this model is hyperbolic, torsion-free, and has geometric dimension 2.*

The few relator model is quite restrictive, it is usually more interesting to let the number of relations grow (in a controlled way) with l. This leads to the *density model*, which is actually a continuous family of models of randoms groups, depending on some constant $0 \leq d \leq 1$. Let $f(l)$ be the number of reduced words in S of length l. Now for a fixed density d, we choose $f(l)^d$ relations of length l uniformly at random.

In this model a random group will only satisfy small cancellation conditions at sufficiently low densities. Specifially,

Proposition 5.26. *[18, Proposition 10] A random group in the density model with density d satisfies $C'(\lambda)$ if $d < \frac{\lambda}{2}$ and does not satisfy $C'(\lambda)$ if $d > \frac{\lambda}{2}$.*

Hence we get the same conclusion as in Thereorm 5.25 in the case where the density $d < \frac{1}{12}$. Despite not satisfying small cancellation conditions, it turns out that random groups in the density model are still almost always hyperbolic.

Theorem 5.27.
1. *If $d < \frac{1}{2}$, a random group in the density model with density d is hyperbolic, torsion-free, and has geometric dimension 2.*
2. *If $d > \frac{1}{2}$, a random group in the density model with density d is trivial or $\mathbb{Z}/2\mathbb{Z}$.*

The "trivial or $\mathbb{Z}/2\mathbb{Z}$" part of the conclusion is a parity issue, depending on whether l is even or odd.

6 Boundaries and isometries of hyperbolic spaces

6.1 Definitions and properties of ∂X

Throughout this sections, we will assume that any path or ray γ is *parameterized by arc length*, that is for any $a < b$ in the domain of γ, $|b - a| = \ell(\gamma|_{[a,b]})$. Also, we will assume that X is a geodesic,

δ-hyperbolic metric space wihich is *proper*, that is all closed balls are compact. The reason we assume this is that we will need to use the Arzelà-Ascoli theorem.

Recall that a sequence of functions $f_i \colon Y \to X$ converges *uniformly on compact sets* to f if for every compact $K \subset Y$ and for all $\varepsilon > 0$, there exists N such that for all $i \geq N$ and $y \in K$, $d_X(f_i(x), f(x)) \leq \varepsilon$.

Theorem 6.1 (Arzelà-Ascoli)**.** *If Y is separable and X is compact, every equicontinuous sequence of maps $Y \to X$ has a subsequence which converges uniformly on compact sets to a continuous map $Y \to X$.*

Corollary 6.2. *Suppose $o \in X$ and $\gamma_i \colon [0, \infty) \to X$ is a sequence of geodesic rays with $\gamma_i(0) = o$ for all i. Then there exists a subsequence of (γ_i) which converges uniformly on compact sets to a geodesic ray γ.*

Proof. For all i and for all n, $\gamma_i([o, n]) \subseteq B_n(o)$, whcih is compact since X is proper. Hence a subsequence of $\gamma_i|_{[0,1]}$ converges to a continuous function by Arzelà-Ascoli. Now we pass to furhter subsequence which converges when restricted to $[0, 2]$, and repeat for all $n \geq 1$. Finally we take the diagonal sequence corresponding to this process, which is converge uniformly on compact sets to a continuous function γ. Showing that γ is a geodesic is an exercise. □

Exercise 6.3. Suppose $\gamma_i \colon [0, \infty) \to X$ is a sequence of geodesics which converges uniformly on compact sets to a continuous function $\gamma \colon [0, \infty) \to X$. Prove that γ is a geodesic.

Let γ_1, γ_2 be rays in X. We say that γ_1 and γ_2 are *equivalent* and write $\gamma_1 \sim \gamma_2$ if $d_{Hau}(\gamma_1, \gamma_2) < \infty$. Clearly this defines an equivalence relation on geodesic rays.

Exercise 6.4. Suppose $\gamma_1, \gamma_2 \colon [0, \infty) \to X$ are geodesic rays parameterized by arc length with $\gamma_1(0) = \gamma_2(0)$. Prove that $d_{Hau}(\gamma_1, \gamma_2) < \infty$ if and only if $d(\gamma_1(t), \gamma_2(t)) \leq 2\delta$.

Remark 6.5. Modifying the above proof, you can also show that for any $\lambda \geq 1$ and $c \geq 0$, there exists a constant M such that if $\gamma_1, \gamma_2 \colon [0, \infty) \to X$ are (λ, c)-quasi-geodesic rays parameterized by arc length, then $d_{Hau}(\gamma_1, \gamma_2) < \infty$ if and only if $d(\gamma_1(t), \gamma_2(t)) \leq M$.

Let $o \in X$ be a fixed base point. Let $\partial_{g,o} X$ denote the set of equivalence classes of geodesic rays $\gamma \colon [0, \infty) \to X$ with $\gamma(0) = o$.

Let $\partial_g X$ denote the set of all equivalence classes of geodesic rays in X.

Let $\partial_q(X)$ denote the set of equivalence classes of quasi-geodesic rays in X. By a quasi-geodesic ray, we mean that γ is a (λ, c)-quasi-geodesic ray for *some* $\lambda \geq 1$ and $c \geq 0$.

Exercise 6.6. Let $o, o' \in X$. Prove that for any $x, y \in X$,

$$(x|y)_o \leq (x|y)_{o'} + d(o, o')$$

Given a sequence (x_i) of points in X, we say that (x_i) *converges to infinity* if for some $o \in X$ (equivalently, any $o \in X$) $(x_i|x_j)_o \to \infty$ as $i, j \to \infty$. Two sequences (x_i) and (y_i) are *equivalent* if $(x_i|y_j)_o \to \infty$ as $i, j \to \infty$, in which case we write $(x_i) \sim (y_i)$. (Note that transitivity of this relation follows from $\mathrm{Hyp}_1(\delta)$). Let $\partial_s X$ denote the set of all equivalence classes of sequences of points in X.

Clearly there are natural inclusion maps $\partial_{g,o} X \hookrightarrow \partial_g X \hookrightarrow \partial_q X$. Furthermore there is map $\partial_q X \to \partial_s$ which sends a quasi-geodesic ray γ to the sequence $(\gamma(i))$.

Lemma 6.7. *The map $\partial_q X \to \partial_s X$ is well-defined.*

Proof. Let γ_1 and γ_2 be equivalent (λ, c) quasi-geodesic rays. Let M be the constant from the Remark 6.5. Let $x_i = \gamma_1(i)$ and $y_i = \gamma_2(i)$. Then $d(x_i, y_i) \leq M$. Let $K = K(\lambda, c, \delta)$ be the constant provided by the Morse Lemma 2.15. Then for all $j \geq i$, there exists a point $a_{i,j} \in [o, y_j]$ such that $d(y_i, a_{i,j}) \leq K$, and hence $d(x_i, a_{i,j}) \leq M + K$. Now $d(x_i, y_j) \leq d(x_i, a_{i,j}) + d(a_{i,j}, y_j) = d(x_i, a_{i,j}) + d(o, y_j) - d(o, a_{i,j})$, hence

$$(x_i|y_j)_o \geq d(x_i, o) + d(y_j, o) - (d(x_i, a_{i,j}) + d(o, y_j) - d(o, a_{i,j})) \geq d(o, x_i) + d(o, y_i) - M - 2K$$

Therefore, $(x_i|y_j)_o \to \infty$ as $i, j \to \infty$. □

Proposition 6.8. *The maps $\partial_{g,o} X \hookrightarrow \partial_g X \hookrightarrow \partial_q X \to \partial_s X$ are all bijections.*

Proof. Let f denote map $\partial_q X \to \partial_s X$. It suffices to show that f is injective and $f|_{\partial_{g,o} X}$ is surjective.

Suppose that γ_1 and γ_2 are (λ, c) quasi-geodesic rays and $f(\gamma_1) = f(\gamma_2)$, that is $(\gamma_1(i)|\gamma_2(j))_o \to \infty$ as $i, j \to \infty$. After possibly increasing the constant c, without loss of generality, we can assume that γ_1 and γ_2 both originate at o. Let δ' be the constant where all (λ, c)-quasi-geodesic triangles are δ'-thin. Let $p_i \in \gamma_1$ and $q_i \in \gamma_2$ be the points such that $d(o, p_i) = d(o, q_i) = (\gamma_1(i)|\gamma_2(i))_o$. Then thinness of the triangle $[o, \gamma_1(i), \gamma_2(i)]$ implies that $d(p_i, q_i) \leq \delta'$. Since p_i and q_i are both unbounded sequences, it follows that $d_{Hau}(\gamma_1, \gamma_2) \leq \infty$.

Now suppose (x_i) is a sequence converging to infinity in X. Let $\gamma_i \colon [0, d(o, x_i)] \to X$ be a geodesic from o to x_i. By the Arzelà-Ascoli-theorem, after passing to a subsequence γ_i will converge uniformly on compact sets to a geodesic ray γ. It is straightforward to check that $(x_i|(\gamma(j))_o \to \infty$, hence $f(\gamma) = (x_i)$. □

$\partial_{g,o} X$, $\partial_g X$, $\partial_q X$, and $\partial_s X$ are all models for the *boundary* of X. From now on we will use these model interchangable and denote the boundary simply by ∂X. Given a (quasi-)geodesic ray γ or a sequence (x_i), we denote the corresponding element of ∂X by $\gamma(\infty)$ or x_∞ respectively, and we say that γ converges to $\gamma(\infty)$ and (x_i) converges to x_∞.

Given any two non-equivalent $\gamma_1, \gamma_2 \in \partial_{g,o} X$, we can construct a sequence of geodesics $\sigma_i \colon [a_i, b_i] \to X$ which connect $\gamma_1(i)$ to $\gamma_2(i)$. We parameterize each σ_i such that for $a_i \leq t \leq 0$, $d(\sigma_i(t), \gamma_1) \leq \delta$ and for $0 \leq t \leq b_i$, $d(\sigma_i(t), \gamma_2(t)) \leq \delta$. Since $\gamma_1 \not\sim \gamma_2$, there exists t_0 such that for all $t > t_0$, $d(\gamma_1(t), \gamma_2(t)) > 2\delta$. In particular, this gives a bound on $d(\sigma_1(0), o)$ which is independent of i and hence for all n and all i, $c_i([-n, n])$ maps to bounded subset of X. So we can apply Arzelà-Ascoli to get a subsequence of σ_i converging to a bi-infinte geodesic σ such that $\sigma(-\infty) = \gamma_1(\infty)$ and $\sigma(\infty) = \gamma_2(\infty)$.

Lemma 6.9. *For any $\xi_1, \xi_2 \in \partial X$, there exists a bi-infinite geodesic $\gamma \colon (-\infty, \infty) \to X$ such that $\gamma(-\infty) = \xi_1$ and $\gamma(\infty) = \xi_2$.*

Our next goal is to describe the topology on ∂X. The simpliest description is to give $\partial_g X$ the topology induced by the compact-open topology on the set of all continuous maps $[0, \infty) \to X$. Equivalently, given a geodesic ray γ and a fixed constant $k > 2\delta$, we can define a basis for the neighborhood system of $\gamma(\infty)$ to be $\{V(\gamma(\infty), r) \mid r \geq 0\}$, where

$$V(\gamma(\infty), r) = \{\gamma'(\infty); \mid \gamma(0) = \gamma'(0), d(\gamma(r), \gamma'(r)) < k\}.$$

Examples 6.10.

1. If X is a bounded metric space, $\partial X = \emptyset$.
2. $\partial\mathbb{R} = \{-\infty, \infty\}$, that is a discrete space with 2 points.
3. If T is a regular tree where each vertex has degree ≥ 3, then ∂T is a Cantor set.
4. $\partial\mathbb{H}^n$ is homeomorphic to S^{n-1}.

Furthermore, there is a natural topology on $\overline{X} = X \cup \partial X$. In this space, a basis for the neighborhood system of $\gamma(\infty) \in \partial X$ can be defined as $\{V'(\gamma(\infty), r) \mid r \geq 0\}$, where

$$V'(\gamma(\infty), r) = \{\gamma'(t); \mid \gamma(0) = \gamma'(0), d(\gamma(r), \gamma'(r)) < k, r < t \leq \infty\}.$$

Equivalently, we can identify X with the set of geodesic paths with a fixed base points, where two such paths are equivalent if they have the same endpoint. The following is just a restatement of Corollary 6.2, it shows that $\overline{X}$ is a *compactification* of X.

Lemma 6.11. *$\overline{X}$ and ∂X are both compact.*

In the classical hyperbolic space $\mathbb{H}^n$, there is a unique geodesic $\gamma \in \partial_{g,o}\mathbb{H}^n$ representing each point of the boundary. We can use the angles between any two such geodesics to define a metric on $\partial\mathbb{H}^n$ which is isometric to the standard metric on S^{n-1}.

For general (proper) δ-hyperbolic metric spaces, the boundary is still metrizable in a similar way. In order to state this result, we extend the Gromov product to points on the boundary in order to have a notion of "angle" between two points $\xi_1, \xi_2 \in \partial X$.

Given $\xi_1, \xi_2 \in \partial X$, we can define

$$(\xi_1|\xi_2)_o = \sup \liminf_{i,j\to\infty} (x_i|y_j)_o$$

Where the supremum is taken over all sequences (x_i), (y_i) with $x_\infty = \xi_1$ and $y_\infty = \xi_2$.

Proposition 6.12. *∂X admits a metric (called a* visual metric*) d such that for some constants k_1, k_2 and parameter a,*

$$k_1 a^{-(\xi_1|\xi_2)_o} \leq d(\xi_1, \xi_2) \leq k_2 a^{-(\xi_1|\xi_2)_o}.$$

For all $\xi_1, \xi_2 \in \partial X$.

We refer to [5] for the proof.

One particularly useful aspect of the boundary of a hyperbolic space is that it provides a quasi-isometry invariant.

Theorem 6.13. *Suppose X and Y are proper, geodesic, hyperbolic metric spaces and $f\colon X \to Y$ is a quasi-isometry. Then f induces a homeomorphism $f_\partial\colon \partial X \to \partial Y$.*

Proof. Let $f\colon X \to Y$ be a $(\lambda, c, \varepsilon)$ quasi-isometry, and let δ be a constant such that X and Y are both δ-hyperbolic. For any geodesic ray γ in X, $f \circ \gamma$ will be a (λ, c)-quasi-geodesic ray in Y[3]. It

[3]Formally, we defined quasi-geodesics as continuous maps, and $f \circ \gamma$ may not be continuous. In this case we think of $f \circ \gamma$ as the quasi-geodesic constructed by connecting $f(\gamma(i))$ to $f(\gamma(i+1))$ by a geodesic for each $i \in \mathbb{N}$.

is not hard to see that $\gamma_1 \sim \gamma_2$ if and only if $(f \circ \gamma_1) \sim (f \circ \gamma_2)$, hence f induces a well-defined, injective map $f_\partial \colon \partial_g X \to \partial_q Y$.

For $i = 1, 2$, let $\gamma_i \in \partial_{g,o} X$ and γ_i' a geodesic ray such that $\gamma_i' \sim f \circ \gamma_i$ and $\gamma_i'(0) = f(o)$. There exists a constant M such that for all t, $d(\gamma_i'(t), f(\gamma_i(t))) \leq M$ (see Remark 6.5). Suppose $d(\gamma_1(r), \gamma_2(r)) < k$. Then $d(\gamma_1'(r), \gamma_2'(r)) \leq 2(\lambda k + c + M)$. Let $s = (\gamma_1'(r)|\gamma_2'(r))_{f(o)}$; using thin triangles, $d(\gamma_1'(s), \gamma_2'(s)) \leq \delta < k$. Furthermore,

$$s = (\gamma_1'(r)|\gamma_2'(r))_{f(o)} = r - \frac{1}{2} d(\gamma_1'(r), \gamma_2'(r)) \geq r - (\lambda k + c + M)$$

In particular, we have shown that $f_\partial(V(\xi, r)) \subseteq V(f(\xi), r - \lambda k - c - M))$ for any $\xi \in \partial X$ and $r > 0$, which implies that f is continuous.

Now, if we let g be a quas-inverse of f, then $g_\partial = (f_\partial)^{-1}$. Therefore f is a homeomorphism. $\square$

Remark 6.14. If $f \colon X \to Y$ is a quasi-isometric embedding, then the induced map $\partial f \colon \partial X \to \partial Y$ is a topological embedding.

Corollary 6.15. *$\mathbb{H}^n \sim_{qi} \mathbb{H}^m$ if and only if $n = m$.*

Corollary 6.16. *If T is a regular tree where each vertex has degree ≥ 3, then $T \not\sim_{qi} \mathbb{H}^n$ for any $n \geq 1$.*

As a consequence, we get that hyperbolic groups have boundaries that are well-defined (up to homeomorphism). That is for any hyperbolic group G, we define ∂G to be $\partial \Gamma(G, S)$ for some (equivalently, any) finite generating set $S \subseteq G$.

Examples 6.17.

1. $\partial \mathbb{Z} = \{-\infty, \infty\}$.
2. ∂F_n is a cantor set for any $n \geq 2$.
3. $\partial \pi_1(S) = S^1$ for any closed, orientable surface S of genus ≥ 2.
4. $\partial G \cong S^{n-1}$ for any group that acts properly and cocompactly on $\mathbb{H}^n$.

We will mention a few further results about boundaries without proofs. For more information, see the survey [15].

Exercise 6.18. Let A and B be hyperbolic groups, and let $G = A * B$. Prove that ∂G is not connected.

Recall that $Ends(X)$ is defined as the set of proper[4] rays $\gamma \colon [0, \infty)$, where two rays γ_1 and γ_2 represent the same end if for any compact set C, there exists $N \geq 0$ such that $\gamma_1([N, \infty))$ and $\gamma_2([N, \infty))$ lie in the same path component of $X \setminus C$.

Lemma 6.19. *Two geodesic rays represent the same end of X if and only if they are contained in a connected component of ∂X.*

[4] a function f is proper if $f^{-1}(C)$ is compact whenever C is compact

A group G *splits over a subgroup* C if either there are subgroups A and B which both contain C and $G \cong A *_C B$ or G has a subgroup A containing C and $G = A*_C$. A famous theorem of Stallings is that a finitely generated group splits over a finite subgroup if and only if it has more then one end. Rephrasing this in terms of boundaries, we get the follwing:

Theorem 6.20. *A hyperbolic group G splits over a finite subgroup if and only if ∂G is not connected.*

There are more connections between connectedness properties of boundaries and splittings of hyperbolic groups, see [15].

For any hyperbolic group G, $|\partial G| \in \{0, 2, \infty\}$, where $|\partial G| = 0$ if and only if G is finite and $|\partial G| = 2$ if and only if G is virtually $\mathbb{Z}$. When $|\partial G| = \infty$, G is called *non-elementary*; in this case ∂G will be an infinite, perfect, compact, metrizable space. Since G acts by isometries on $\Gamma(G, S)$, Theorem 6.13 implies that each element of G induces a homeomorphism on ∂G.

Proposition 6.21. *If G is a non-elementary hyperbolic group, then G acts by homeomorphism on a infinite, perfect, compact, metrizable space, namely ∂G.*

In fact, Bowditch showed that this action can be used to give a characterization of hyperbolic groups purely in terms of topological dynamics. See [15] for the definition of a uniform convergence action.

Theorem 6.22. *Suppose a group G acts by homeomorphisms on an infinite, perfect, compact, metrizable space X. Then this is a uniform convergence action if and only if G is a non-elementary hyperbolic group and X is G-equivariantly homeomorphic to ∂G.*

We have seen that $\partial\pi_1(S) = S^1$ for any closed, orientable surface S of genus ≥ 2. It turns out that the converse of this statement is also (virtually) true, which is a very deep result which combines the work of several people (see [15])

Theorem 6.23. *Suppose G is a hyperbolic group and $\partial G \cong S^1$. Then G is virtually the fundamental of a closed hyperbolic surface.*

The analogue of this result for S^2 is a famous open problem known as the Cannon conjecture:

Conjecture 6.24. *Suppose G is a hyperbolic group and $\partial G \cong S^2$. Then G acts properly and cocompactly on $\mathbb{H}^3$.*

In particular, this conjecture implies that modulo the (finite) kernel of the action, G is the fundamental group of a compact hyperbolic 3-orbifold, and if G is torsion-free then G is the fundamental group of a compact hyperbolic 3-manifold. This conjecture was one of the major steps in Cannon's approach to Thurston's Hyperbolization conjecture, now proved by Perelman.

Finally, we mention that Bestvina-Mess showed that ∂G can be viewed as a boundary of the Rips complex, which allows one to connect homological properties of G and topological properties of ∂G.

Theorem 6.25. *If G is hyperbolic and torsion-free, $dim(\partial G) = cd(G) - 1$.*

Combining this with a theorem of Stallings gives:

Corollary 6.26. *$dim(\partial G) = 0$ if and only if G is virtualy free.*

Corollary 6.27. *If G is a random group, $dim(\partial G) = 1$.*

6.2 Isometries of $\mathbb{H}^2$

Consider the Poincaré upper-half plane model for $\mathbb{H}^2$, that is identify $\mathbb{H}^2$ with $\{z \in \mathbb{C} \mid im(z) > 0\}$ with the metric $ds^2 = \frac{dx^2+dy^2}{y^2}$. Recall that $PSL(2,\mathbb{C})$ acts on the Riemann sphere $\mathbb{C} \cup \{\infty\}$ by Möbius transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}.$$

Furthermore, the subgroup which fixes the upper-half plane is $PSL(2,\mathbb{R})$. This action also preserves the metric on $\mathbb{H}^2$ and furthermore every orientation preserving isometry of $\mathbb{H}^2$ can be given by a Möbius transformation. Hence we can identify $PSL(2,\mathbb{R})$ with $Isom^+(\mathbb{H}^2)$, the group of orientation preserving isometries of $\mathbb{H}^2$.

We consider a few specific isometries of $\mathbb{H}^2$ and the corresponding Möbius transformations:

Translation: $z \to z + t$ for some $t \in \mathbb{R}$, $\begin{pmatrix} 1 & t \\ 01 & \end{pmatrix}$.

Dilation: $z \to \lambda^2 z$ for some $\lambda \in \mathbb{R}$, $\begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}$.

Exercise 6.28. For any $a, b \in \mathbb{H}^2$, there exists a translation T and a dilation D such that $b = T(D(a))$.

Rotation: about i of angle θ: $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is a rotation about i of angle θ. Conjugating a rotation by translations and dilations produces a rotation about any point.

It is an exercise to show that $PSL(2,\mathbb{R})$ is generated by translations, reflections, and dilations. There is one more type of isometry, which is not orientation preserving:

Refection about a line: for example, $z \to \frac{1}{\bar{z}}$ is a reflection about unit circle, $x + yi \to -x + yi$ is a reflection about y-axis.

Proposition 6.29. *$Isom(\mathbb{H}^2)$ is generated by translations, dilations, reflections, and rotations, and $Isom^+$ is generated by translations, dilations, and rotations.*

sketch. If g fixes 3 non-colinear points, g is the identity.

If g fixes at least 2 points, then g fixes the geodesic between them, and hence g is either the identity or the reflection about this line. Hence if g fixes two points and preserves orientation, g is the identity.

If g fixes a unique point, then g must be a rotation about this point.

If g has no fixed points, then there is a translation T and a dilation D such that $T \circ D \circ g$ has a fix point, and hence belongs to one of the above cases. $\square$

Now given any element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, R)$, we need to determine where the corresponding Möbius transformation has any fixed points.

$$\frac{az+b}{cz+d} = z \iff cz^2 + (d-a)z - b = 0$$

If this quadratic has a non-real root, then will be two conjugate complex roots and exactly one of them will have positive imaginary part. If this happens, then $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ must represent a rotation. This will happen if and only if the discriminant is negative, that is

$$(d-a)^2 + 4bc = d^2 - 2ad + a^a + 4bc = (d+a)^2 - 4 < 0$$

or in other words, $|trace(A)| = |a+d| < 2$. We now consider the case $Trace(A) \geq 2$. For this, we consider the characteristic polynomial

$$(\lambda - a)(\lambda - d) - bc = \lambda^2 - (a+d)\lambda + 1$$

For A to have a real eigenvalue, $(a+d)^2 - 4 \geq 0$, or in other words $|trace(A)| \geq 2$. Suppose $trace(A) > 2$. Then A has 2 distinct real eigenvalues; since the product of these is $det(A) = 1$, the eigenvalues must be λ and $\frac{1}{\lambda}$. So A is similar to a diagonal matrix, that is for some S, $S^{-1}AS = \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}$. Furthermore, after scaling S we can assume $S \in PSL(2, \mathbb{R})$. So the matrix A acts as a dilation "along" the image of the y-axis under S. points on this line are translated long it by a distance of $\ln \lambda^2$.

The final case is when $trace(A) = 2$, that is A has a single real eigenvalue which hence must be 1. In this case A will be similar to $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$, that is a translation, and A itself will translate perpindicular to the image of the y-axis under the conjugating matrix S.

Based on this anaylsis, we have the following terminology: Let $A \in Isom^+(\mathbb{H}^2) = PSL(2, \mathbb{R})$. We now discuss each of these cases in more detail. Let $\tau_0(A) = \inf_{z \in \mathbb{H}^2} d(z, A \cdot z)$, called the *translation length* of A.

A is called *elliptic* if A has a fixed point. In this case the the fixed point is unique, and A acts as a rotation around this point. This happens if and only if $|trace(A)| < 2$. If we consider orientation reversing isometries, this case also includes reflections. Elliptic isometries clearly have translation length 0.

A is called *loxodromic* (or *hyperbolic*) if $S^{-1}AS = \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}$ for some $S \in PSL(2, \mathbb{R})$ and some $\lambda \in \mathbb{R} \setminus \{1\}$. This case happens if and only if $|trace(A)| > 2$. In this case, $l_A = S(y - axis)$ will be a bi-infinite geodesic which is preserved by A, and points on l_A are shifted along l_A by a distance of $\ln(\lambda^2)$. Furthermore, l_A is exactly the subset of $\mathbb{H}^2$ which minimizes the translation length of A, so $\tau_0(A) = \ln(\lambda^2)$. l_A defines two points $\xi_- \xi_+ \in \partial\mathbb{H}^2$ which are fixed by A. All other points on $\partial\mathbb{H}^2$ are moved away from ξ_- and towards ξ_+. In other words, the induced action of A on $\partial\mathbb{H}^2$ exhibits *north-south dynamics.*

A is called *parabolic* if $S^{-1}AS = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ for some $S \in PSL(2, \mathbb{R})$ and $t \in \mathbb{R} \setminus \{0\}$. This case happens if and only if $|trace(A)| = 2$. In this case $\tau_0(A) = 0$; indeed, consider $d(ki, S^{-1}AS \cdot ki)$, which goes to 0 as $k \to \infty$. In this case, A fixes a unique point $\xi \in \partial\mathbb{H}^2$; if ∞ is the point on the boundary defined by the y-axis, then $\xi = S(\infty)$. A preserves horoballs based at ξ, so in this case A can be viewed as a "rotation" about a point at infinity.

Note that for every isometry A, $\tau_0(A^n) = \tau_0(A)$ and $\tau_0(A) = \tau_0(S^{-1}AS)$.

6.3 Isometries of δ-hyperbolic metric spaces

Our next goal is to understand the analogue of the above classification for a δ-hyperbolic metric space X instead of $\mathbb{H}^2$. Here we now longer have tools from linear algebra such as trace, so we replace this with translation length. In fact since we are not interested in small distances, we work instead with *stable translation length*:

Definition 6.30. Let $f\colon X \to X$ be an isometry. The *(stable) translation length* of f, denoted $\tau(f)$, is defined as

$$\lim_{n\to\infty} \frac{1}{n} d(x, f^n(x))$$

for some (equivalently, any) $x \in X$

Note that for any isometry of $\mathbb{H}^2$, $\tau = \tau_0$. This will not be true in arbitrary hyperbolic metric spaces, but it will be true up to an additive error depending only on δ. Also note that unlike translation length, stable translation length is independent of the base-point.

Exercise 6.31. Prove that for any isometry $f\colon X \to X$ and any $x, y \in X$,

$$\lim_{n\to\infty} \frac{1}{n} d(x, f^n(x)) = \lim_{n\to\infty} \frac{1}{n} d(y, f^n(y))$$

Exercise 6.32. Let X be a δ-hyperbolic metric space, and let $f, g \in Isom(X)$. Prove that $\tau(f) = \tau(g^{-1}fg)$ and $\tau(f^n) = n\tau(f)$.

We now categorize the isometries of a δ-hyperbolic metric space X. Let $f \in Isom(X)$.

f is called *elliptic* if f has a bounded orbit. In this case, $\tau(f) = 0$. Unlike $\mathbb{H}^2$, f may have no fixed points. However, it does have a point whose orbit is bounded in terms of δ.

Lemma 6.33. *f is elliptic if and only if there exists $x \in X$ such that $rad(\langle f \rangle \cdot x) \leq 4\delta + 1$.*

f is called *loxodromc* (or *hyperbolic*) if $\tau(f) > 0$. In this case, f will preserve a bi-infinite quasi-geodesic axis, and hence have exactly 2 fixed points on ∂X. equivalently, the map $\mathbb{Z} \to X$ defined by $n \to f^n(x)$ is a quasi-isometry for some (equivalently, any) $x \in X$.

f is called *parabolic* if $\tau(f) = 0$ but f has no bounded orbit. In this case, f preserves a unique point $\xi \in \partial X$.

7 Elements and subgroups of hyperbolic groups

7.1 Elliptic elements

Throughout this section, let G denote a group generated by a finite set S such that $\Gamma(G,S)$ is δ-hyperbolic.

Next we study the elements of a hyperbolic group G under the action of G on $\Gamma(G,S)$, or equivalently under any proper, cobounded action of G. This equivalence follows from the exercise.

Exercise 7.1. Suppose G acts on X and Y, and $f\colon X \to Y$ is a G equivariant isometry. Then for any $g \in G$, g is elliptic (resp. parabolic, loxodromic) with respect to the action on X if and only if g is elliptic (resp.parabolic, loxodromic) with respect to the action on Y.

In particular, the type of an element of a hyperbolic group is well-defined. We first consider the case of elliptic elements:

Exercise 7.2. Let G be a hyperbolic group. Prove that $g \in G$ is elliptic if and only if g has finite order.

Given a hyperbolic space X and a bounded subset $A \subset X$, let $rad(A) = \inf\{\rho \mid A \subseteq B_\rho(x) \text{ for some } x \in X\}$. A point $y \in X$ is called an ε- *quasi-center* of A if $A \subseteq B_{rad(A)+\varepsilon}(y)$.

Lemma 7.3. *Let X be a δ-hyperbolic metric space, and let A be a bounded subset of X. Then for any $\varepsilon > 0$ and any two ε-quasi-centers of A x and y, $d(x,y) \leq 4\delta + 2\varepsilon$.*

Proof. Let x, y be quasi-centers of a bounded set A. Let $r_A = rad(A)$, Fix a geodesic $[x,y]$, and let m be the midpoint of this geodesic. Let $a \in A$, and consider the geodesic triangle $[x,y,a]$. Then there exists some $b \in [a,x] \cup [a,y]$ such that $d(a,b) \leq \delta$. Without loss of generality, suppose $b \in [a,x]$. Then $d(b,x) \geq d(x,m) - \delta = \frac{1}{2}d(x,y) - \delta$, hence

$$d(a,b) = d(a,x) - d(x,b) \leq d(a,x) - \frac{1}{2}d(x,y) + \delta \leq r_A + \varepsilon + \delta - \frac{1}{2}d(x,y)$$

It follows that for any $a \in A$,

$$d(a,m) \leq d(a,b) + \delta \leq r_A + \varepsilon + 2\delta - \frac{1}{2}d(x,y)$$

However for some $a \in A$, $d(a,m) \geq r_A$ by definition of r_A. Therefore $d(x,y) \leq 4\delta + 2\varepsilon$. $\square$

Theorem 7.4. *If $g \in G$ is an element of finite order, then g is conjugate to an element $h \in G$ such that $|h|_S \leq 4\delta + 1$. In particular, the set of conjugacy classes of torsion elements in G is finite.*

Proof. Choose k such that k is a 1-quasi-center of the bounded set $\langle g \rangle \subseteq \Gamma(G,S)$. Then gk is also a 1-quasi-center, hence $d(1, k^{-1}gk) = d(k, gk) \leq 4\delta + 1$.

$\square$

7.2 Quasi-convex subgroups

Definition 7.5. Let X be a metric space and $\sigma \geq 0$. A subset $Y \subseteq X$ is called σ-quasi-convex if for every $x, y \in Y$ and every geodesic $[x, y]$, $[x, y] \subseteq A^{+\sigma}$.

Exercise 7.6. Let X be a δ-hyperbolic metric space. Prove that a subset Y is quasi-convex if and only if there exists (λ, c) such that for every $x, y \in Y$ there exists a (possibly non-continuous) (λ, c) quasi-geodesic γ from x to y such that $\gamma \subseteq Y$.

Theorem 7.7. *Let G be a hyperbolic group and $H \leq G$. Then H is quasi-convex in $\Gamma(G, S)$ if and only if H is generated by a finite set T and the inclusion map $(H, d_T) \hookrightarrow (G, d_S)$ is a quasi-isometric embedding.*

Proof. First, suppose H is σ-quasi-convex in G. Let $T = \{h \in H \mid |h|_S \leq 2\sigma + 1\}$. Now fix $h \in H$, and let p be a geodesic from 1 to h with vertices $v_0 = 1, v_1, ..., v_n = h$. For each $1 \leq i \leq n - 1$, choose $h_i \in H$ such that $d(h_i, v_i) \leq \sigma$. Then $|h_i^{-1} h_{i+1}|_S \leq 2\sigma + 1$, and

$$h = h_1(h_1^{-1}h_2)...(h_{n-1}^{-1}h)$$

Hence $h \in \langle T \rangle$, and $|h|_T \leq n = |h|_S$. Now for all $h \in H$, $|h|_S \leq (2\sigma + 1)|h|_T$. Thus H is quasi-isometrically embedded in G.

Conversely, suppose H is generated by a finite set T and H is quasi-isometrically embedded in $\Gamma(G, S)$. Then any two points in H can be joined by a quasi-geodesic in H, and by the Morse Lemma any geodesic in G must stay a bounded distance from this quasi-geodesic. □

In particular, quasi-convexity is independent of the choice of generating set of G. Furthermore,

Corollary 7.8. *Every quasi-convex subgroup of G is hyperbolic*

Corollary 7.9. *If H is a quasi-convex subgroup of G, then there is a topological embedding $\partial H \hookrightarrow \partial G$.*

Exercise 7.10. Suppose $H_1 \leq H_2 \leq G$ such that $[H_2 : H_1] < \infty$. Prove that H_1 is quasi-convex if and only if H_2 is quasi-convex.

Proposition 7.11. *Let H_1 and H_2 be quasi-convex subgroups of G. Then $H_1 \cap H_2$ is quasi-convex.*

Proof. Let H_1 and H_2 be σ-quasi-convex in $\Gamma(G, S)$. Let q be a geodesic in $\Gamma(G, S)$ from 1 to some $h \in H_1 \cap H_2$. Let v be a vertex on q, and let p be the shortest path from v to $H_1 \cap H_2$ with the property that for every vertex v' on p, $\max\{d(v', H_1), d(v', H_2)\} \leq \sigma$. Note that the subpath of q starting at v has this property, so such a path does exist.

Let r and r' be two initial seqments of p with $\ell(r) < \ell(r')$. Let u and u' be the elements of G such that $u = r_-$ and $u' = r'_-$. respectively. By assumption, for each $i = 1, 2$, there are elements $u_i, u_i' \in H_i$ such that $d(u, u_i) \leq \sigma$ and $d(u', u_i') \leq \sigma$. Suppose that $u^{-1}u_i = (u')^{-1}u_i$ for $i = 1, 2$. Then for any $g \in G$,

$$d(ug, H_j) = d(g, u^{-1}u_i H_i) = d(g, (u')^{-1}u_i' H_i) = d(u'g, H_i)$$

In particular, if $p = r's'$, then the path $p' = rs'$ is stricly shorter then p, and by previous calculation this contradicts our choice of p. Hence, for every vertex on p we can associate a ordered pair of elements of length at most σ, corresponding to the shortest paths from this vertex to H_1 and H_2. In order for all of these pairs to be unique, we must have $\ell(p) \leq |B_\sigma(1)|^2$.

□

Lemma 7.12. *Let $[x, y, z, w]$ be a geodesic quadrilateral in a δ hyperbolic metric space. Let $a \in [x, y]$ and $b \in [w, z]$ such that $d(x, a) = d(w, b)$. Then*

$$d(a, b) \leq 2 \max\{d(x, w), d(y, z)\} + 4\delta$$

Proof. By δ-hyperbolicity, there exists a point $c \in [y, z] \cup [w, z] \cup [x, w]$ with $d(a, c) \leq 2\delta$. First we suppose that $c \in [w, z]$, and that c occurs between w and b (the case where c is between b and z is similar). Then

$$d(x, a) \leq d(x, w) + d(w, c) + 2\delta = d(x, w) + d(w, b) - d(b, c) + 2\delta$$

Hence, $d(b, c) \leq d(x, w) + 2\delta$, so $d(a, b) \leq d(x, w) + 4\delta$.

Suppose now that $c \in [x, w]$. Then $d(w, b) = d(x, a) \leq d(x, w) + 2\delta$. Hence $d(a, b) \leq 2\delta + d(x, w) + d(w, b) \leq 2d(x, w) + 4\delta$. By symmetry, the case $c \in [y, z]$ is the same. □

The following was implicit in our solution to the conjugacy problem in hyperbolic groups.

Lemma 7.13. *For any g and h in G, there exists a constant $K = K(|g|_S, |h|_S, \delta)$ such that g and h are conjugate if and only if there is some $x \in G$ with $|x|_S \leq K$ and $x^{-1}gx = h$.*

Recall that for an element $g \in G$, $C(g)$ denotes the centralizer of g, that is $\{h \in G \mid gh = hg\}$.

Proposition 7.14. *For every $g \in G$, $C(g)$ is quasi-convex in G.*

Proof. Let p be a geodsic from 1 to $h \in C(g)$. Let q be a geodesic from 1 to g. Consider the geodesic quadrilateral with sides q, gp, hq, and p. Let v be a vertex on p. By Lemma 7.12, $d(v, gv) \leq 2|g|_S + 4\delta$. Hence, by Lemma 7.13, there $x \in G$ such that $xgx^{-1} = v^{-1}gv$ and $|x|_S \leq K$, where $K = K(|g|_S, \delta)$. Futhermore,

$$(vx)g = vxgx^{-1}x = g(vx)$$

Hence $vx \in C(g)$, and $d(v, vx) \leq K$. Thus $C(g)$ is K-quasi-convex.

□

Theorem 7.15. *For any $g \in G$, $\langle g \rangle$ is quasi-convex. In particular, if g has infinite order, then the map $\mathbb{Z} \to \Gamma(G, S)$ given by $n \to g^n$ is a quasi-isometric embedding.*

Corollary 7.16. *$g \in G$ is loxodromic if and only if g has infinite order.*

Proof. If g has finite order, then $\langle g\rangle$ is bounded and hence quasi-convex. Suppose now that g has infinite order. Since $C(g)$ is quasi-convex, it is generated by a finite set T. By Proposition 7.11, $Z(C(g)) = \cap_{h\in T} C(h)$ is quasi-convex and hence hyperbolic. However, $Z(C(g))$ is abelian and it is easy to see that every abelian hyperbolic group is isomorphic to $\mathbb{Z}\times A$ where A is a finite abelian group. Therefore, $\langle g\rangle$ is a finite index subgroup of $Z(C(g))$, and since $Z(C(g))$ is quasi-convex so is $\langle g\rangle$. □

Corollary 7.17. *If $g \in G$ has infinite order and $x^{-1}g^kx = g^l$, then $k = \pm l$.*

Proof.

$$|k|\tau(g) = \tau(g^k) = \tau(x^{-1}g^lx) = \tau(g^l) = |l|\tau(g)$$

□

Corollary 7.18. *If $g \in G$ has infinite order, $[C(g) : \langle g\rangle] < \infty$.*

Proof. Passing to a sufficiently high power of g, g is not conjugate to any element of length $\leq 4\delta$. If $h \in C(g)$ v is a vertex on $[1, h]$ such that $d(1, v) = d(v, h) > d(1, g) + 2\delta$, then $d(v, gv) \leq 4\delta$, contradicting our assumption about g. □

Recall that the *Baumslag-Solitar* groups are groups given by the presentations

$$BS(n, m) = \langle a, t \mid t^{-1}a^nt = a^m\rangle$$

For some $n, m \in \mathbb{Z}$.

Theorem 7.19. *Let G be a hyperbolic group. Then for any $n, m \in \mathbb{Z}$, G has no subgroups isomorphic to $BS(n, m)$. In particular, G has no subgroup isomorphic to $\mathbb{Z}^2$.*

Proof. First, if g is infinite order, then $[C(g) : \langle g\rangle] < \infty$, so G contains no subgroup isomorphic to $\mathbb{Z}^2$. Now if $n = \pm m$, then $\langle a^n, t^2\rangle \cong \mathbb{Z}^2$, so G does not contain a subgroup isomorphic to $BS(n, \pm n)$. If $n \neq \pm n$, then G does not contain $BS(n, m)$ by Lemma 7.17. □

The following is a well-known open question which generalizes Question 4.8.

Question 7.20. *Is every group of type F which contains no Baumslag-Solitar subgroups hyperbolic?*

7.3 Elementary (sub)groups

A group G is called *elementary* if G is virtually cyclic. Every such group is a finite extension of either $\mathbb{Z}$ or the infinite dihedral group $D_\infty \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} \cong Isom(\mathbb{Z}) \cong \mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$.

Lemma 7.21. *Let E be a torsion-free elementary group. Then E is cyclic.*

Proof. Let $g \in E$ such that $\langle g\rangle$ is an infinite cyclic normal subgroup of minimal index. In this case g is central, since if $x^{-1}gx = g^{-1}$ and $x^n = g^k$, then $x^{2n} = 1$. Let $F = E/\langle g\rangle$. If F is cyclic, then E is abelian since it is generated by the central element g and the pre-image of the generator of F. Clearly every torsion-free abelian virtually cyclic group is cyclic.

If F contains a non-trivial proper normal cyclic subgroup $H/\langle g\rangle$, then H is a normal subgroup of E which is cyclic previous argument. But this contradicts the minimality of the index of $\langle g\rangle$.

If F contains a non-trivial proper normal abelian subgroup $H/\langle g\rangle$, then every cyclic subgroup of $H/\langle g\rangle$ is normal in $H/\langle g\rangle$. Hence by the previous argument $H/\langle g\rangle$ is cyclic so H is cyclic, so again we contradict the minimality of the index of $\langle g\rangle$.

Now if F is a non-abelian finite group, then it contains a non-abelian subgroup L such that L contains a normal abelian subgroup M. But then the previous arguements show that the inverse image of L in E is cyclic, which contradicts the fact that L is non-abelian. $\square$

Exercise 7.22. Show that every non-abelian finite group F contains a non-abelian subgroup L such that L contains a normal abelian subgroup M.

Theorem 7.23. *Let E be an elementary group. Then E contains normal subgroups $T \leq E^+ \leq E$ such that $[E : E^+] \leq 2$, T is finite, and $E^+/T \cong \mathbb{Z}$. Furthermore, if $E \neq E^+$, then $E/T \cong D_\infty$.*

Proof. Let $g \in E$ such that $\langle g\rangle$ is an infinite cyclic normal subgroup. By normality, for all $x \in E$, $x^{-1}gx = g^{\pm 1}$. Let
$$E^+ = \{x \in E \mid x^{-1}gx = g\}.$$
Clearly E^+ is a subgroup of index at most 2, hence it is normal in E. Let T be the set of elements of E^+ of finite order. Note that $|T| \leq [E : \langle g\rangle]$, so T is finite. Since the center of E^+ has finite index in E^+, it follows that T is a subgroup of E^+. T is clearly normal, so E^+/T is torsion-free and elementary, hence cyclic by Lemma 7.21.

T is also normal in E, since $x^{-1}Tx \leq E^+$ for all $x \in E$ and conjugation preserves orders of elements. Now if $E^+ \neq E$, then E/T contains E^+/T as a cyclic subgroup of index 2. This implies that $E/T \cong \mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z} \cong D_\infty$.

$\square$

Exercise 7.24. Suppose G is a group such that $[G : Z(G)] < \infty$. Prove that the set of torsion elements of G is a subgroup of G.

Next we show that every infinite order element of g is contained in a unique, maximal, elementary subgroup.

Definition 7.25. Let G be a hyperbolic group and $g \in G$ be an element of infinite order. Then
$$E(g) = \{x \in G \mid x^{-1}g^n x = g^{\pm n} \text{ for some } n = n(x) \in \mathbb{Z}\}.$$

The following observations are elementary, the proof is left as an exercise.

Lemma 7.26. *Let $g \in G$ be an element of infinite order.*

1. *$E(g)$ is a subgroup of G.*
2. *$E^+(g) := \{x \in G \mid x^{-1}g^n x = g^n$ for some $n = n(x) \in \mathbb{Z}\}$ is a subgroup of $E(g)$ of index ≤ 2.*
3. *$E(g) = E(g^k)$ for any $k \in \mathbb{Z} \setminus \{0\}$.*

4. $E(y^{-1}gy) = y^{-1}E(g)y$ *for any* $y \in G$.

5. $\langle g \rangle \le C(g) \le E^+(g) \le E(g)$.

Theorem 7.27. $[E(g) : \langle g \rangle] < \infty$. *Furthermore,* $E(g)$ *is the unique, maximal, virtually cyclic subgroup of* G *containing* g.

Proof. It suffices to show that $[E^+(g) : \langle g \rangle] < \infty$. Let $x \in G$ such that $x^{-1}g^n x = g^n$. Let W be a geodesic word representing g and X a geodesic word representing x. Choosing m as a sufficiently large multiple of $2n$, there is a (λ, c) quasi-geodesic quadrilateral with the top and bottom sides labeled by W^m, and the sides labeled by X. Furthermore, if u and v are the middle vertices on the top and bottom respectively, there is a and a path from u to v labeled by X. Let $K = K(\delta, \lambda, c)$ be the constant such that every (λ, c)-quasi-geodesic triangle in a δ hyperbolic space is K-slim. Choosing m large enough we can ensure that $d(x, u) \ge 2K + \|X\|$, so by hyperbolicity there exists a vertex w on the bottom path such that $d(u, w) \le 2K$. Then there is a vertex z on the bottom path such $z = g^l$ for some l and $d(z, w) \le \frac{\|W\|}{2} = \frac{|g|_S}{2}$. Then $z^{-1}u = g^{-l}x$ and $d(z, u) \le 2K + \frac{|g|_S}{2}$. Therefore $[E^+(g) : \langle g \rangle] \le |B_{2K + \frac{|g|_S}{2}}(1)| < \infty$.

Finally, if E is any elementary group containing g, then for some $n \in \mathbb{N}$ $\langle g^n \rangle$ is normal in E. This implies that $E \le E(g)$. □

Let g be an element of infintie order in G, and let γ_g be the bi-infinite quasi-geodesic obtained by connecting g^i to g^{i+1} by a translate of a fixed geodesic $[1, g]$ for all $i \in \mathbb{Z}$. Then $E(g) = \{x \in G \mid d_{Hau}(\gamma_g, x\gamma_g) < \infty\}$.

We let $g^{\pm\infty}$ denote the points $\gamma(\pm\infty) \in \partial X$. Hence, $E(g) = Stab_G(\{g^{+\infty}, g^{-\infty}\})$. $E^+(g)$ is the pointwise stabilizer of $\{g^{+\infty}, g^{-\infty}\}$.

7.4 Torsion subgroups are finite

The following proposition says that a concatenation of geodesics will be a global quasi-geodesic as long as the angles at the concatenation points are sufficiently small. This is similar to the fact that local geodesic in a hyperbolic space are global quasi-geodesics, i.e. Lemma 2.24. Given points $x_1, ..., x_n \in X$, let $[x_1, ..., x_n]$ denote the path p obtained by concatenating geodesics from x_i to x_{i+1}. Such a path p is called a *broken line*.

Proposition 7.28. *[21] Let* $p = [x_0, x_1, ..., x_n]$ *be a broken line such that* $d(x_i, x_{i+1}) \ge c_1$ *and* $(x_{i-1}|x_{i+1})_{x_i} \le c_0$ *where* $c_1 > 2c_0 + 2\delta$. *Then* $d(x_0, x_n) \ge n(c_1 - 2c_0 - 2\delta)$.

Proof. We induct on n. If $n = 1$ the statement is obvious. For $n = 2$,

$$c_0 \ge (x_0|x_2)_{x_1} = \frac{1}{2}(d(x_0, x_1) + d(x_1, x_2) - d(x_0, x_2)) \ge c_1 - \frac{1}{2}d(x_0, x_2)$$

Which implies that $d(x_0, x_2) \ge 2(c_1 - c_0)$.

Suppose now $n \ge 3$ and that the statement holds for all $k \le n-1$. By the inductive hypothosis, $d(x_0, x_{n-2}) < d(x_0, x_{n-1})$. It follows that

$$(x_0|x_{n-2})_{x_{n-1}} = \frac{1}{2}(d(x_{n-1}, x_0) + d(x_{n-1}, x_{n-2}) - d(x_0, x_{n-2})) > \frac{1}{2}c_1 > c_0 + \delta.$$

Applying $\mathrm{Hyp}_1(\delta)$, $c_0 \geq (x_n|x_{n-2})_{x_{n-1}} \geq \min\{(x_0|x_n)_{x_{n-1}}, (x_{n-2}|x_0)_{x_{n-1}}\} - \delta$. By the previous calculation, this minimum cannot be realized by $(x_0|x_{n-2})_{x_n-1}$, and hence $(x_0|x_n)_{x_n-1} \leq c_0 + \delta$.

Using this and the inductive hypothesis,

$$d(x_0, x_n) = d(x_0, x_{n-1}) + d(x_{n-1}, x_n) - 2(x_0|x_n)_{x_{n-1}} \geq d(x_0, x_{n-1}) + c_1 - 2(c_0 + \delta).$$

□

For $g, h \in G$, let $(g|h) = (g|h)_1$.

Lemma 7.29. *Suppose $g \in G$ such that $(g^{-1}|g) < \frac{1}{2}|g| - \delta$. Then g has infinite order.*

Proof. By Proposition 7.28 applied to $[1, g, g^2, ..., g^n]$, $d(1, g^n) > n(|g| - 2(g^{-1}|g) - 2\delta) > 0$, so $g^n \neq 1$ for all $n \in \mathbb{N}$. □

Lemma 7.30. *Let g, $h \in G$ be elements of finite order such that $(h^{-1}|g) \leq \frac{1}{2}\min\{|g|, |h|\} - 3\delta$. Then gh has infinite order in G.*

First, applying $\mathrm{Hyp}_1(\delta)$ twice gives

$$(h^{-1}|g) \geq \min\{(g^{-1}|g), (h^{-1}|h), (g^{-1}|h)\} - 2\delta$$

By the previous lemma and the fact that g and h have finite order, this minimum must be realized by $(g^{-1}|h)$. Hence $(g^{-1}|h) \leq (h^{-1}|g) + 2\delta \leq \frac{1}{2}\min\{|g|, |h|\} - \delta$. This means that we can apply Proposition 7.28 to $[1, g, gh, ghg, ...(gh)^n]$ to get that $d(1, (gh)^n) > 0$ for all $n \in \mathbb{N}$.

Proposition 7.31. *Let M be a subset of a hyperbolic group G such that $M = M^{-1}$ and M and MM both consist of torsion elements. Then there exists $u \in G$ such that $u^{-1}Mu \subseteq B_{16\delta}(1)$. In particular, $|M| \leq |B_{16\delta}(1)|$.*

Proof. It suffices to assume that M if a finite set. Let $L_M = \max\{|g| \mid g \in M\}$; after conjugating M, we assume that $L_M \leq L_{M'}$ for any M' which is conjugate to M. Let $M_1 = \{g \in G \mid |g| > 6\delta\}$. If M_1 is empty then we are done. By the previous two lemmas, for all $g, h \in M_1$,

$$(h^{-1}|g) \geq \frac{1}{2}\min\{|g|_S, |h|_S\} - 3\delta > 3\delta$$

Otherwise $gh \in MM$ would have infinite order.

Now fix $g_0 \in M_1$, and let u be a vertex on $[1, g_1]$ such that $d(1, u) = 3\delta$. By the previous calculation, for any $h \in M_1$, there exists $o_1 \in [1, h]$ with $d(o_1, u) \leq \delta$. Similarly, there exists $o_2 \in [1, h]$ such that $d(o_2, hu) \leq \delta$.

Now $d(1, o_1) \geq d(1, u) - d(u, o_1) \geq 2\delta$ and $d(o_2, h) \geq 2\delta$. Thus $d(o_1, o_2) \leq |h| - 4\delta$, therefore

$$d(u, hu) \leq 2\delta + |h| - 4\delta = |h| - 2\delta$$

Also, for all $g \in M \setminus M_1$, $|u^{-1}gu| \leq |g| + 6\delta \leq 12\delta$. Therefore, if M contains an element of length $\geq 12\delta$, then $L_{u^{-1}Mu} < L_M$, a contradiction.

□

Corollary 7.32. *G contains only finitely many conjugacy classes of finitie subgroups.*

Theorem 7.33. *Let H be an infinite subgroup of a hyperbolic group. Then H contains an element of infinite order.*

Exercise 7.34. Let G be a hyperbolic group, and let $G_\infty = \{g \in G \mid g \text{ has infinite order}\}$

1. Suppose $g, h \in G_\infty$ such that $E(g) \neq E(h)$. Prove that $E(g) \cap E(h)$ is finite.
2. Let $g \in G_\infty$ and let N be a finite normal subgroup of G. Prove that $N \leq E(g)$.
3. Let
$$K(G) = \bigcap_{g \in G_\infty} E(g).$$
Prove that $K(G)$ is the unique, maximal finite normal subgroup of G.

7.5 Free subgroups, ping-pong, and the Tits alternative

Lemma 7.35. *Suppose $g, h \in G$ are elements of infinite order such that $g^{+\infty} = h^{+\infty}$. Then $E(g) = E(h)$; in particular, $g^{-\infty} = h^{-\infty}$.*

Proof. Let M be a constant such that for all $t > 0$, $d(\gamma_g(t), \gamma_h(t)) \leq M$. Note that $g\gamma(t) = \gamma(t+|g|)$, and similarly for h and γ_h. Now for any $r > 0$,

$$d(g^{-r}hg^r, h) = d(hg^r, g^r h) = d(h\gamma_g(r|g|), g^r\gamma_h(|h|))$$
$$\leq d(h\gamma_h(r|g|), g^r\gamma_g(|h|)) + 2M = d(\gamma_h(|h| + r|g|), \gamma_g(|h| + r|g|)) + 2M \leq 3M.$$

Since $B_{3m}(h)$ is finite, there exists some $r > s > 0$ such that $g^{-r}hg^r = g^{-s}hg^s$, and hence $hg^{r-s}h^{-1} = g^{r-s}$, that is $h \in E(g)$. Therefore $E(h) = E(g)$ by maximality.

□

Proposition 7.36. *Suppose $a \in \partial G$ such that $Stab_G(a)$ is infinite. Then there exists $b \in \partial G$ and an element of infinite order $g \in G$ such that $a = g^{+\infty}$ and $b = g^{-\infty}$. In particular, $Stab_G(a) = Stab_G(b) = E^+(g)$.*

Loxodromic elements g, h are called *independent* if $g^{\pm\infty}$ and $h^{\pm\infty}$ are all distinct; equivalently, $E(g) \neq E(h)$.

Exercise 7.37. Suppose $g \in G$ has infinite order and $x^{-1}gx \in E(g)$. Then $x \in E(g)$. Therefore, if $x \in G \setminus E(g)$, then g and $x^{-1}gx$ are independent.

The following is one of the fundamental tools of geometric group theory. The proof is straightforward, it is left as an exercise.

Lemma 7.38 (Ping-Pong Lemma)**.** *Let G be a group acting on a set X. Suppose X has disjoint subsets A_+, A_-, B_+, B_- such that for some $g, h \in G$, $g \cdot (X \setminus A_-) \subseteq A_+$, $g^{-1} \cdot (X \setminus A_+) \subseteq A_-$, $h \cdot (X \setminus B_-) \subseteq B_+$, and $h \cdot (X \setminus A_+) \subseteq A_-$. Then $\langle g, h \rangle \cong F_2$.*

Given a closed subset $Y \subseteq X$, let $proj_Y : X \to Y$ be a function such that for each $x \in X$, $d(x, Y) = d(x, proj_Y(x))$.

Proposition 7.39. *Let G be a group acting on a geodesic hyperbolic metic space X, and suppose g and h are independent loxodromic isometries. Then there exists $N \in \mathbb{N}$ such that $\langle g^N, h^N \rangle \cong F_2$.*

Proof. Fix a basepoint $o \in X$, and let γ_g and γ_h be the bi-infinite quasi-geodesics formed by connecting adjacent points in $O_g = \langle g \rangle \cdot o$ and $O_h = \langle h \rangle \cdot o$ by geodesics respectively. Let K be the constant such that all (λ, c) quasi-geodesic triangles are K-slim where λ and c are the quasi-geodesic constants for γ_g and γ_h. For all $x \in X$, let $x_g = proj_{O_g}(x)$ and $x_h = proj_{O_h}(x)$.

Let $M > 0$ be a large constant. Let

$$A_\pm = \{x \in G \mid d(o, x_g) = g^{\pm n} o \text{ for some } n \geq M\}$$

$$B_\pm = \{x \in G \mid x_h = h^{\pm n} o \text{ for some } n \geq M\}$$

If $x_g = g^n o$, then $(gx)_g = g^{n+1} o$. Hence there exists some $N = N(M)$ such that $g^{\pm N} \cdot X \setminus A_\mp \subseteq A^\pm$ and $h^{\pm N} \cdot X \setminus B_\mp \subseteq B^\pm$. In order to apply the ping-pong lemma, it only remains to show that (for sufficiently large M these sets are disjoint.

First we show that there exists R such that for any $y \in \gamma_g, z \in \gamma_h$, $[y, z] \cap B_R(o) \neq \emptyset$. Since $\gamma_g(\infty) \neq \gamma_h(\infty)$, there exists $y_0 \in \gamma_g$ such that $d(y_0, \gamma_h) \geq K$. Then if $d(o, y) \geq d(o, y_0)$, then y_0 must be within K of a point on the geodesic triangle with sides γ_g, γ_h, and $[y, z]$. So we can choose $R = d(o, y_0) + K$.

Now let $x \in X$, and let $u \in B_R(o) \cap [x_g, x_h]$. Then in the geodesic triangle $[x, x_g, x_h]$, there exists $v \in [x, x_g] \cup [x, x_h]$ such that $d(u, v) \leq \delta$. Suppose $v \in [x, x_g]$. Then $d(o, v) \leq R + \delta$. By definition of the projection, $d(v, x_g) \leq d(v, o)$, so $d(o, x_g) \leq 2d(o, v) \leq 2(R + \delta)$. Similary, if $v \in [x, x_h]$, then $d(o, x_h) \leq 2R + 2\delta$.

We have shown that for all $x \in X$, either x_h or x_g belongs to $B_{2R+2\delta}(o)$. Hence it suffices to choose M such that for all $n \in \mathbb{Z}$ with $|n| \geq M$, $d(o, g^n o) \geq 2R + 2\delta$ and $d(o, h^n o) \geq 2R + 2\delta$.

□

Theorem 7.40 (Strong Tits alternative)**.** *Let G be a hyperbolic group and let $H \leq G$. Then either H is elementary or H contains F_2.*

Proof. If H is non-elementary, then H contains an element of infinite order g and some $x \in G \setminus E(g)$. Let $h = x^{-1} g x$. Since $h \notin E(g)$, $E(h) \neq E(g)$, and hence g and h are independent. Therefore, by the previosu proposotion there exists $N \in \mathbb{N}$ such that $\langle g^N, h^N \rangle \cong F_2$. □

Theorem 7.41. *Let G be a hyperbolic group. The following are equivalent:*

1. $|\partial G| \geq 3$
2. *G is non-elementary.*
3. *G contains F_2.*

4. *G contains infinitely many pairwise independent loxodromic elements.*

5. *$|\partial G| = \infty$.*

Proof. 1 $\implies$ 2 If G is elementary, then $|\partial G| \in \{0, 2\}$.

2 $\implies$ 3 follows from Theorem 7.40.

3 $\implies$ 4: If $E(g) = E(h)$, then for some n, k, $g^n = h^k$. Thus any $g, h \in F_2 \leq G$ which are independent in F_2 are also independent in G. Clearly F_2 has infintiely many pairwise independent loxodromic elements, hence so does G.

4 $\implies$ 3 If $\{g_1, ...\}$ is an infinte sequence of pairwise independent loxodromic elements in G, then $\{g_1^{+\infty}, ...\}$ is an infinite sequence of distinct points in ∂G.

5 $\implies$ 1 Obvious. ☐

Corollary 7.42. *Suppose G is a hyperbolic group. Then exactly one of the following occurs:*

1. *$|\partial G| = 0$, equivalently G is finite.*

2. *$|\partial G| = 2$, equivalently G is virtually infinite cyclic.*

3. *$|\partial G| = \infty$, equivalently G is non-elementary.*

Moreover, one can show that $\{g^{\pm\infty} \mid g \in G\}$ is a dense subsetset of ∂G. Furthermore, from the proof one can derive that in this case ∂G is *perfect*, that is every point is a limit point.

Corollary 7.42 should be compared to the following general classification of group acting on hyperbolic spaces:

Given G acting on a hyperbolic metric space X, $\Lambda G = \overline{G \cdot o} \cap \partial X$, where $o \in X$ is any fixed base point and the closure is taken in the compacitification $\overline{X} = X \cup \partial X$ of X. If G is acting on $\Gamma(G, S)$, then $\Lambda G = \partial G$, and if H is a quasi-convex subgroup of G, then $\Lambda H = \partial H$ under the natural inclusion $H \hookrightarrow G$.

Theorem 7.43. *Suppose G acts on a hyperbolic metric space X. Then exactly one of the following occurs:*

1. *$|\Lambda G| = 0$, equivalently G is* elliptic *(i.e., G has bounded orbits).*

2. *$|\Lambda G| = 1$, equivalently G has unbounded orbits but contains no loxodromic elements. In this case the action is called* parabolic.

3. *$|\Lambda G| = 2$, equivalently G contains loxodromic elements and any two loxodromic elements have the same limit points on ∂X.*

4. *$|\Lambda G| = \infty$. Here there are two subcases:*

   (a) *There exists $\xi \in \partial X$ which is fixed under the action of G. Equivalently, for any loxodromic element $g \in G$, $\xi \in \{g^{+\infty}, g^{-\infty}\}$. In this case the action is called* quasi-parabolic.

   (b) *G contains infiniitely many pairwise independent loxodromic elements.*

Actions of type 1-3 are called *elementary*.

Finally, conclude this section by highlighting a few more immediate consequences of the Tits alternative.

Given a fintie subset $A \subset \Gamma(G, S)$, let ∂A be the set of vertices v in $\Gamma(G, S)$ with $d(v, A) = 1$.

Definition 7.44 (Folner condition)**.** A finitely generated group G is *amenable* if there exists a sequence of finite subsets $A_n \subseteq G$ such that

$$\lim_{n\to\infty} \frac{|\partial A|}{|A|} = 0$$

Some examples: $\mathbb{Z}$ is amenable; more generally, all solvable groups are amenable. F_2 and any group which contains F_2 is not amenable.

Corollary 7.45. *Every amenable subgroup of a hyperbolic group is elementary.*

Corollary 7.46. *If G is a non-elementary hyperbolic group, then every infinite normal subgroup of G is non-elementary. In particular, G has no infinite amenable normal subgroups.*

8 Quotients of hyperbolic groups

8.1 Small cancellation over hyperbolic groups

In this section, we show how the generalize classical small cancellation theory to study quotients of hyperbolic groups. This generalization has roots in the work of Gromov [11], but the presentation we give here is due to Olshanskii [20]. Our goal is to give a brief overview of the main tools and ideas of this area, so we will omit most of the proofs. Proofs for all of these facts can be found in [20] (see also [24]).

Recall that a set of words $\mathcal{R}$ in an alphabet S, is *symmetrized* if for any $R \in \mathcal{R}$, $\mathcal{R}$ contains all cyclic shifts of $R^{\pm 1}$. Further, if G is a group generated by a set S, we say that a word R is (λ, c)–*quasi–geodesic* in G if any path in the Cayley graph $\Gamma(G, S)$ labeled by R is (λ, c)-quasi-geodesic.

Definition 8.1. Let G be a group generated by a set S, $\mathcal{R}$ a symmetrized set of words in S. For $\varepsilon > 0$, a subword U of a word $R \in \mathcal{R}$ is called an *ε-piece* if there exists a word $R' \in \mathcal{R}$ such that:

(1) $R \equiv UV$, $R' \equiv U'V'$, for some V, U', V'.

(2) $U' =_G YUZ$ for some words Y, Z in S such that $\max\{\|Y\|, \|Z\|\} \le \varepsilon$.

(3) $YRY^{-1} \neq_G R'$.

Similarly, a subword U of $R \in \mathcal{R}$ is called an *ε'–piece* if:

(1$'$) $R \equiv UVU'V'$ for some V, U', V'.

(2$'$) $U' =_G YU^{\pm 1}Z$ for some Y, Z satisfying $\max\{\|Y\|, \|Z\|\} \le \varepsilon$.

Definition 8.2. We say that the set $\mathcal{R}$ satisfies the $C(\varepsilon, \mu, \lambda, c, \rho)$*–condition* for some $\varepsilon \geq 0$, $\mu > 0$, $\lambda > 0$, $c \geq 0$, $\rho > 0$, if

(1) $\|R\| \geq \rho$ for any $R \in \mathcal{R}$.

(2) Each $R \in \mathcal{R}$ is (λ, c)–quasi–geodesic.

(3) For any ε-piece U of any word $R \in \mathcal{R}$, the inequality $\max\{\|U\|, \|U'\|\} < \mu\|R\|$ holds (using the notation of Definition 8.1).

Further the set $\mathcal{R}$ satisfies the $C_1(\varepsilon, \mu, \lambda, c, \rho)$*–condition* if in addition the condition (3) holds for any ε'-piece of any word $R \in \mathcal{R}$.

Suppose that G is a group defined by

$$G = \langle \mathcal{S} \mid \mathcal{O} \rangle. \tag{6}$$

Given a set of words $\mathcal{R}$, we consider the quotient group of G represented by

$$\overline{G} = \langle \mathcal{S} \mid \mathcal{O} \cup \mathcal{R} \rangle. \tag{7}$$

A cell in a van Kampen diagram over (7) is called an $\mathcal{R}$*-cell* if its boundary label is a word from $\mathcal{R}$. Let Δ be a van Kampen diagram over (7) and Π an $\mathcal{R}$-cell of Δ. Suppose that there is a simple closed path

$$p = s_1 q_1 s_2 q_2 \tag{8}$$

in Δ, where q_1 is a subpath of $\partial\Pi$, q_2 is a subpath of $\partial\Delta$, and

$$\max\{\ell(s_1),\, \ell(s_2)\} \leq \varepsilon \tag{9}$$

for some constant $\varepsilon > 0$. By Γ we denote the subdiagram of Δ bounded by p. If Γ contains no $\mathcal{R}$-cells, we say that Γ is an ε*–contiguity subdiagram* (or simply a contiguity subdiagram if ε is fixed) of Π to $\partial\Delta$ and q_1 is the *contiguity arc* of Π to q. The ratio $\ell(q_1)/l(\partial\Pi)$ is called the *contiguity degree* of Π to $\partial\Delta$ and is denoted by $(\Pi, \Gamma, \partial\Delta)$. Since Γ contains no $\mathcal{R}$-cells, it can be considered a diagram over (6).

A van Kampen diagram Δ over (7) is said to be *reduced* if Δ has minimal number of $\mathcal{R}$-cells among all diagrams over (7) having the same boundary label. When dealing with a diagram Δ over (7), it is convenient to consider the following transformations. Let Σ be a subdiagram of Δ which contains no $\mathcal{R}$-cells, Σ' another diagram over (6) with $\mathbf{Lab}(\partial\Sigma) \equiv \mathbf{Lab}(\partial\Sigma')$. Then we can remove Σ and fill the obtained hole with Σ'. Note that this transformation does not affect $\mathbf{Lab}(\partial\Delta)$ and the number of $\mathcal{R}$-cells in Δ. If two diagrams over (7) can be obtained from each other by a sequence of such transformations, we call them $\mathcal{O}$*-equivalent*. The following is an analogue of the well-known Greendlinger Lemma from classical small cancellation theory.

Lemma 8.3 (Greendlinger-Olshanskii Lemma)**.** *Let G be a group with presentation (6). Suppose that the Cayley graph $\Gamma(G, S)$ of G is hyperbolic. Then for any $\lambda \in (0, 1]$, $c \geq 0$, and $\mu \in (0, 1/16]$, there exists $\varepsilon \geq 0$ and $\rho > 0$ with the following property. Let $\mathcal{R}$ be a symmetrized set of words in S satisfying the $C(\varepsilon, \mu, \lambda, c, \rho)$-condition, Δ a reduced van Kampen diagram over the presentation (7)* *such that $\partial\Delta$ is (λ, c)-quasi-geodesic. Assume that Δ has at least one $\mathcal{R}$-cell. Then up to passing to an $\mathcal{O}$-equivalent diagram, then there is an $\mathcal{R}$-cell Π of Δ and an ε-contiguity subdiagram Γ of Π to $\partial\Delta$ such that*

$$(\Pi, \Gamma, \partial\Delta) > 1 - 13\mu.$$

It is worth noting that the proof of this Lemma does not require S to be finite (see [24]), which allows for generalizations of small cancellation theory to the relatively hyperbolic setting and beyond.

Proposition 8.4. *Suppose $G = \langle \mathcal{S} \mid \mathcal{O} \rangle$ is a hyperbolic group. Then for all λ, c, there exists ε, μ and ρ such that if $\mathcal{R}$ is a symmetrized set of words satisftying the $C(\varepsilon, \mu, \lambda, c, \rho)$ condidtion, then the group $\overline{G} = \langle \mathcal{S} \mid \mathcal{O} \cup \mathcal{R} \rangle$ satisfies:*

1. *$\overline{G}$ is hyperbolic.*
2. *If W is a word in S with $\|W\| \leq \frac{1}{2}\lambda\rho - c - 2\varepsilon$, then $W =_{\overline{G}} 1$ if and only if $W =_G 1$.*
3. *If in addition R satisfies $C_1(\varepsilon, \mu, \lambda, c, \rho)$, then every element of $\overline{G}$ of finite order is the image of an element of G of finite order.*

sketch. (1) Suppose Δ is a minimal van Kampen diagram over (7). If Δ has no $\mathcal{R}$-cells, then Δ is also a diagram over (6) and hence the area of δ is linear in the length of $\partial\Delta$ by the hyperbolicity of G. If Δ contains an $\mathcal{R}$-cell, the Greendlinger-Olshanskii Lemma can be used (for sufficiently small ε and μ and sufficiently large ρ) to find an $\mathcal{R}$-cell Π with most of the boundary of Π close to the boundary of Δ. Then we can "cut out" Π to produce a new diagram Δ' such that Δ' has fewer $\mathcal{R}$-cells then Δ and $\ell(\partial\Delta') < \ell(\partial\Delta)$. It follows that $\overline{G}$ will have linear Dehn function and hence be hyperbolic.

(2) If $W =_{\overline{G}} 1$ and $W \neq_G 1$, then every van Kampen diagram over (7) must contain an $\mathcal{R}$-cell. By the Greendlinger-Olshanskii Lemma, such a diagram Δ has a 2-cell Π with most of the boundary of Π close to the boundary of Δ. Since $\ell(\partial\Pi) \geq \rho$, we can derive a lower bound on $\ell(\partial\Delta)$.

(3) If Δ is a van Kampen diagram over (7) which contains an $\mathcal{R}$-cell and whose boundary label is a proper power, then the 2-cell Π provided by the Greendlinger-Oshanskii Lemma can be used to violate the C_1-condition. For details of this proof, see [24, Lemma 6.3]. $\square$

For classical small cancellation theory, checking whether a set of words satisfies a given small cancellation condition is fairly straightforward. For small cancellation over hyperbolic groups, this is much harder. In fact, even showing that there exists a set of words which satisfies these small cancellation conditions is non-trivial

Lemma 8.5. *[20, Lemma 4.1] Let G be a non-elementary hyperbolic group, let $g \in G$ be an infinite order element, and let W be a geodesic word in representing g. Then there exists λ, c such that for any ε, μ, ρ, there exists N such that the set of cyclic shifts of W^N satisfies $C(\varepsilon, \mu, \lambda, c, \rho)$.*

We now show how to find words which satisfy the stronger C_1-condition.

Given a subgroup $H \leq G$, let $K_G(H) := \cap\{E_G(g) \mid g \in H \text{ is an element of infinite order }\} = \{1\}$. Equivalently, $K_G(H)$ is the maximal finite subgroup of G normalized by H.

Definition 8.6. A subgroup $H \leq G$ of a hyperbolic group is called *suitable* if H is non-elementary and $K_G(H) = \{1\}$.

Lemma 8.7. *[24] Let H be a subgroup of a hyperbolic group G. Then H is suitable if and only if H contains independent loxodromic elements g and h such that $E(g) = \langle g \rangle$ and $E(h) = \langle h \rangle$.*

Proposition 8.8. *Let g and h be independent loxodromic elements of a hyperbolic group $G = \langle S \rangle$ such that $E(g) = \langle g \rangle$ and $E(h) = \langle h \rangle$. Let W be a geodesic word representing g and U a geodesic word representing h. Then there exists λ and c such that for all ε, μ, ρ and all $s \in S$, there exists N and m such that for all $N < n_1 < n_2 < ... < n_m$, the symmetrized closure of*

$$\{s^{-1}W^{n_1}U^{n_2}...W^{n_{m-1}}U^{n_m}\}$$

satisfies $C_1(\varepsilon, \mu, \lambda, c, \rho)$.

Theorem 8.9. *Let G be a non-elementary hyperbolic group, and let H be a suitable subgroup. Then for any finite set $F \subset G$, there exists a quotient $f\colon G \twoheadrightarrow \overline{G}$ such that*

1. *$\overline{G}$ is non-elementary hyperbolic.*
2. *$f|_F$ is injective.*
3. *$f|_H$ is surjective.*
4. *Every element of $\overline{G}$ of finite order is the image of an element of G of finite order. In particular, if G is torsion-free then so is $\overline{G}$.*

Proof. Let S be a finite generating set for G and let $s \in S$. By Lemma 8.7 and Proposition **??**, there exists λ and c such that for all ε, μ, and ρ there a word W representing an element of H such that the symmetrized closure of $\{s^{-1}W^{n_1}U^{n_2}...W^{n_{m-1}}U^{n_m}\}$, which we denote by $\mathcal{R}$, satisfies $C_1(\varepsilon, \mu, \lambda, c, \rho)$. In particular after for sufficently small μ and sufficiently large ε and ρ we can apply Proposition 8.4 to get that the quotient $G_1 = G/\langle\langle \mathcal{R} \rangle\rangle$ satisfes conditions (1), (2) and (4). Furthermore, if $f_1\colon G \twoheadrightarrow G_1$ is the natural quotient map, then $f_1(s) \in f_1(H)$. In addition, $f_1(H)$ will again be suitable in G_1, so we can inductively repeat the construction for all elements of S to form a sequence

$$G \twoheadrightarrow G_1 \twoheadrightarrow ... \twoheadrightarrow G_{|S|}.$$

Such that each G_i satisfies conditions (1), (2) and (4) and if $f\colon G \twoheadrightarrow \overline{G} := G_{|S|}$, then $f(s) \in f(H)$ for all $s \in S$. This implies that $f|_H$ is surjective, thus all the conditions hold for f and $\overline{G}$.

□

8.2 Exotic quotients and the von Neumann-Day problem

Theorem 8.10. *[20] Every non-elementary hyperbolic group has an infinite torsion quotient.*

Proof. Let $G = \{1 = g_0, g_1, ...\}$. We proceed by inductively constructing a sequence of quotients of G. Suppose we have constructed a sequence

$$G \twoheadrightarrow G_1 \twoheadrightarrow ... \twoheadrightarrow G_n$$

Such that each G_i is a non-elementary hyperbolic group and the image of g_i has finite order in G_i. Let $\alpha_i\colon G \twoheadrightarrow G_i$ denote the natural quotient map. If $\alpha_n(g_{n+1})$ has finite order, we let $G_{n+1} = G_n$ otherwise, by Lemma 8.5 and Proposition 8.4, there exists $N \in \mathbb{N}$ such that $G_{n+1} := G_n/\langle\langle \alpha_n(g_{n+1})^N \rangle\rangle$ is a non-elementary hyperbolic group. Hence we can continue the sequence.

Now let Q be the direct limit of the sequence $G \twoheadrightarrow G_1 \twoheadrightarrow ...$ constructed above, that is

$$Q = G/\cup_{i=1}^{\infty} \ker(\alpha_i)$$

Let $\alpha\colon G \to Q$ be the natural quotient map. Now for each $g \in G$, there exists i and N such that $g^N \in \ker(\alpha_i)$, so $\alpha(g)$ has finite order. Thus Q is a torsion group, it only remains to prove that Q is infinite.

Suppose Q is finitely presented. Then there exists a finite set $R \subseteq \ker(\alpha)$ such that $\langle\langle R\rangle\rangle = \ker(\alpha)$. Since R is finite, there exists some i such that $R \subseteq \ker(\alpha_i)$. But then $\ker(\alpha_i) \subseteq \ker(\alpha) = \langle\langle R\rangle\rangle \subseteq \ker(\alpha_i)$, so $Q = G_i$. But G_i is non-elementary hyperbolic and hence cannot be a torsion group, a contradiction. Therefore Q is not finitely presented, in particular Q must be infinite.

$\square$

Theorem 8.11. *Let G be a non-elementary torsion-free hyperbolic group. Then G has an infinite, non-abelian quotient Q such that every proper subgroup of Q is infinite cyclic.*

Proof. Enumerate $G \times G = \{(g_1, h_1), ...\}$. Fix non-commuting elements $a, b \in G$ and let $F = \{1, a, b, [a, b]\}$ Let $G_0 = G$. If $E(g_1) = E(h_1)$, then let $G_1 = G_0$. Otherwise, $H = \langle g_1, h_1\rangle$ is non-elementary and hence suitable since G is torsion-free. So we can apply Theorem 8.9 and set $G_1 = \overline{G_0}$.

Continute this process inductively, we get a sequence

$$G = G_0 \twoheadrightarrow G_1 \twoheadrightarrow ...$$

such that each G_i is a non-elementary hyperbolic group, F maps injectively to G_i, and the image of $\{g_i, h_i\}$ in G_i either generates all of G_i or a cyclic subgroup of G_i.

Let Q be the direct limit of this sequence; that is if $\alpha_i\colon G \twoheadrightarrow G_i$ is the natrually induced map, $Q = G/\cup \ker(\alpha_i)$. Now for any two elements g, h in Q, these elements have some preimages g_i, h_i in G. Now consider the images of g_i, h_i in G_i. If $E(g_i) = E(h_i)$ then $\langle g_i, h_i\langle$ is cyclic, and hence $\langle g, h\rangle$ (which is a quotient of $\langle g_i, h_i\langle)$ is also cyclic.

Otherwise, $\langle \alpha_i(g_i), \alpha_i(h_i)\rangle = G_i$, and hence g, h generate Q. Therefore every proper subgroup of Q is infinite cyclic. Finally, since $[a, b] \notin \ker(\alpha_i)$ for all i, the image of a and b do not commute in Q, hence Q is non-abelian. $\square$

The groups constructed in Theorems 8.11 and 8.10 can be used to answer a famous question from the 1950's:

Question 8.12 (von Neumann-Day problem)**.** *Does every non-amenable group contain F_2?*

A negative answer to this question was given by Olshanskii in 1980 using complicated combinatorial small cancellation techniques. Gromov observed that the proof could be simplified using Theorem 8.9 and the notiton of *property (T)*.

Let G be generated by a finite set S, let $\mathcal{H}$ be a Hilbert space, and let $\pi\colon G \to \mathcal{U}(\mathcal{H})$ a unitary representation of G. π is said to have *almost invariant vectors* if for all $\varepsilon > 0$, there exists $\xi \in \mathcal{H}$ with $\|\xi\| = 1$ and $\|\pi(s)\xi - \xi\| \leq \varepsilon$ for all $s \in S$.

Definition 8.13. G has *property* (T) if every unitary representation of G which has almost invariant vectors has a non-zero invariant vector.

We will only need to following elementary facts about Property (T):

1. If G has (T), then every quotient of G has (T).
2. If G is amenable and G has (T), then G is finite.
3. There exist non-elementary hyperbolic groups with (T).

Exercise 8.14. Let G be non-elementary hyperbolic, let $Q = G/K(G)$. Prove that Q is non-elementary hyperbolic and $K(Q) = \{1\}$.

Proposition 8.15. *Let G and H be non-elementary hyperbolic groups. Then G and H have a common quotient Q which is non-elementary hyperbolic.*

Proof. By exercise 8.14, we can assume $K(G) = K(H) = \{1\}$. Then G and H are suitable subgroups of $G * H$. Hence we can apply Theorem 8.9 twice

$$\alpha_1\colon G * H \twoheadrightarrow Q_1, \alpha_2\colon Q_1 \twoheadrightarrow Q_2$$

such that $\alpha_1|_G$ is surjective and $\alpha_2|_H$ is surjective. □

Corollary 8.16. *Every non-elementary hyperbolic group has a non-elementary hyperbolic quotient with Property (T).*

Proof. Let H be a non-elementary hyperbolic group with property (T). Then for any non-elementary hyperbolic group G, there is a non-elementary hyperbolic group Q which is a quotient of both G and H. Since quotients of property (T) groups have property (T), we are done. □

By first replacing G with a non-elementary hyperbolic quotient Q with property T and then applying Theorem 8.10 or Theorem 8.11 to Q, we get:

Corollary 8.17. *The groups constructed in Theorems 8.10 and 8.11 can be chosen to have Property (T) and hence be non-amenable. In particular, there exists a non-amenable group which does not contain F_2.*

8.3 SQ-universality

The following is a classical theorem of Higman-Neumann-Neumann.

Theorem 8.18. *Every countable group embeds into a 2-generator group.*

Definition 8.19. A group G is called SQ-universal if every countable group embeds into a quotient of G.

Hence the theorem of Higman-Neumann-Neumann is equivalent to the statement that F_2 is SQ-universal. Olshanskii showed that this property holds for all non-elementary hyperbolic groups.

Theorem 8.20. *[22] Every non-elementary hyperbolic group is SQ-universal.*

sketch. We sketch here the proof from [1].

Let G be a non-elementary hyperbolic group. By exercise 8.14, it suffices to assume $K(G) = \{1\}$. Since every countable group embeds into a finitely generated group, if we can show that every finitely generated group embeds into a quotient of G then we are done.

Let A be a group generated by a finite set $\{t_1, ..., t_n\}$. Then G is suitable in $G * A$, so we can add relations $t_1 = W_1, ..., t_n = W_n$, where each $W_i \in G$ satisfes suitable small cancellation conditions. Let Q be the resulting quotient. By construction, Q is a quotient of G; furthermore, and it can be shown that A embeds in Q.

□

Formally, since $G * A$ is not hyperbolic for many groups A, this approach uses the corresponding version of Theorem 8.9 for relatively hyperbolic groups, see [24]. Note that Olshanskii's original proof lives completely in the world of hyperbolic groups and uses a variation of the $C(\varepsilon, \mu, \lambda, c, \rho)$-condition.

Since there are uncountably many isomorphism types of finitely generated groups and every countable group contains only countably many finitely generated subgroups, we get the following:

Corollary 8.21. *Every non-elementary hyperbolic group has uncountably many normal subgroups.*

9 Relatively hyperbolic groups

The goal of this section is to give a very brief introduction to relatively hyperbolic groups. The general philosophy of relatively hyperbolic groups is that they behave like hyperbolic groups except for inside (cosets of) certain specified subgroups, called *peripheral subgroups.* The canonical examples of relatively hyperbolic groups are the following: if $G = A * B$, then G is hyperbolic relative to $\{A, B\}$, and if G is the fundamental group of a complete, finite volume Riemannian manifold with pinched negative sectional curvature, then G is hyperbolic relative to the collection of cusp subgroups.

As with hyperbolic groups, there are several equivalent characterizations of relatively hyperbolic groups. We will begin with a definition of Farb, which involves two parts. The first part is *weak relative hyperbolicity*, and the second is *bounded coset penetration.*

Definition 9.1 (Farb)**.** Let G be a group generated by a finite set S, and let $P_1, ..., P_n$ be subgroups of G. Let $\mathcal{P} = \sqcup_{i=1}^{n}(P_i \setminus \{1\})$. G is *weakly hyperbolic relative to* $\{P_1, ..., P_n\}$if $\Gamma(G, S \cup \mathcal{P})$ is a hyperbolic metric space.

$\Gamma(G, S \cup \mathcal{P})$ is called a *relative Cayley graph* of G.

Exercise 9.2. Prove that $\mathbb{Z}^2 = \langle a, b\rangle$ is weakly hyperbolic relative to $\langle a\rangle$.

It turns out that weak relative hyperbolicity is not strong enough to prove many natural analogues of properties of hyperbolic groups, so Farb introduced a second property called bounded coset penetration, often abbreviated as BCP.

Definition 9.3. The pair $(G, \{P_1, ..., P_n\})$ satisfies *bounded coset penetration* if there exist a constant C such that the following 2 conditions hold.

1. Suppose γ_1 and γ_2 are geodesics in $\Gamma(G, S \cup \mathcal{P})$ with the same endpoints. Suppose also that e is an edge of γ_1 with $\mathbf{Lab}(e) = h \in P_i$ for some $1 \leq i \leq n$, and $|h|_S \geq C$. Then γ_2 has an edge f such that $\mathbf{Lab}(f) \in P_i$ and $e_\pm$, $f_\pm$ all belong to the same left coset of P_i.

2. Suppose γ_1 and γ_2 are geodesics in $\Gamma(G, S \cup \mathcal{P})$ with the same endpoints, and e and f are edges of γ_1 and γ_2 respectively such that $e_\pm$, $f_\pm$ all belong to the same left coset of some P_i. Then $d_S(e_-, f_-) \leq C$ and $d_S(e_+, f_+) \leq C$.

Definition 9.4 (Farb)**.** G is hyperbolic relative to $\{P_1, ..., P_n\})$ if it is weakly hyperbolic relative to $\{P_1, ..., P_n\}$ and the pair satisfies BCP.

We turn now to an isoperimetric characterization of relatively hyperbolic groups due to Osin.

If G is generated by a finite set S and $P_1, ..., P_n$ are subgroups of G, there is a natural surjective homomorphism $F = F(S) * (*_{i=1}^n P_i) \twoheadrightarrow G$. Suppose $R \subset F(S) * (*_{i=1}^n P_i)$ such that the normal closure of R is equal to the kernel of the this map. Then we call

$$\langle S, \mathcal{P} \mid R\rangle \tag{10}$$

a *presentation of* G *relative to the subgroups* $\{P_1, ..., P_n\}$, or simply a *relative presentation* if the subgroups are understood. If R (and S) are finite, then (10) is called a *finite relative presentation.*

Given a word W in $S \cup \mathcal{P}$ such that $W =_G 1$, there exists some $r_1, ..., r_k \in R$ and $f_1, ..., f_k \in F$ such that

$$W =_F \Pi_{i=1}^k f_i^{-1} r_i f_i$$

We define $Area^{rel}(W)$ to be the minimal k such that W can be written in the above form. Equivalently, one if $\mathcal{O}_i$ is the set of words in P_i which represent the identity in P_i and $\mathcal{O} = \cup \mathcal{O}_i$, then G has an ordinary (infinite) presentation $\langle S \cup \mathcal{P} \mid ; \mathcal{O} \cup R\rangle$. Then for a van Kampen diagram Δ over this presentation, we define $Area^{rel}(\Delta)$ be the the number of 2-cells of Δ whose lable belongs to R (called R-cells). Then $Area^{rel}(W) = \min\{Area^{rel}(\Delta) \mid \mathbf{Lab}(\partial\Delta) \equiv W\}$.

The relative Dehn function $\delta^{rel}(n)$ is defined as the maximal relative area of all words in $S \cup \mathcal{P}$ of length at most n. As with the ordinary Dehn function, the relative Dehn function is independent (up to a natural equivalence relation) of the choice of finite sets S and R.

Definition 9.5 (Osin)**.** G is hyperbolic relative to $\{P_1, ..., P_n\}$ if G is finitely presented relative to $\{P_1, ..., P_n\}$ and the corresponding relative Dehn function is linear.

Examples 9.6.
1. G is hyperbolic if and only if G is hyperbolic relative to $\{1\}$. More generally, a hyperbolic group G is hyperbolic relative to a subgroup P if and only if P is quasi-convex and malnormal, that is $g^{-1}Pg \cap P - \{1\}$ for all $g \in G \setminus P$.

2. If $G = A * B$, then G is hyperbolic relative to $\{A, B\}$.

3. If G splits over a finite group, then G is hyperbolic relative to the factor groups. By theorems of Stallings and Dunwoody, it follows that every non-elementary finitely presented group is hyperbolic relative to one-ended subgroups.

4. If G is the fundamental group of a complete, finite volume Riemannian manifold with pinched negative sectional curvature, then G is hyperbolic relative to the collection of cusp subgroups.

5. If G is hyperbolic relative to P and P is hyperbolic, then G is hyperbolic.

We now survey a few results for relatively hyperbolic groups, which are natural extensions of corresponding results for hyperbolic groups. We mention here only a few examples, for more see [1, 3, 8, 9, 12, 14, 23, 24].

Let G be hyperbolic relative to $\{P_1, ..., P_n\}$.

Theorem 9.7. *[23] If each P_i is finitely presented then G is finitely presented.*

Theorem 9.8. *[9] If each P_i has solvable word problem, then G has solvable word problem.*

Theorem 9.9. *If each P_i has decidable conjugacy problem, then G has decidable conjugacy problem.*

Proposition 9.10. *$g \in G$ acts elliptically on $\Gamma(G, S \cup \mathcal{P})$ if and only if g has finite order or g is conjugate into one of the subgroups P_i. Otherwise, g acts loxodromically on $\Gamma(G, S \cup \mathcal{P})$ and g is contained in a unique, maximal elementary subgroup $E(g) = \{x \in G \mid x^{-1} g^n x = g^{\pm n}$ for some $n = n(x) \in \mathbb{Z}\}$.*

Theorem 9.11. *[1] If each P_i is a proper subgroup of G, then G is SQ-universal.*

10 Open questions

Finally, we finish by mentioning a few well-known open questions related to hyperbolic groups. These problems and more can be found on Bestvina's list [2]

The first two questions we have discussed previosly.

Question 10.1. *Is every group of type F which contains no Baumslag-Solitar subgroups hyperbolic?*

Conjecture 10.2 (Cannon Conjecture)**.** *Suppose G is a hyperbolic group and $\partial G \cong S^2$. Then G acts properly and cocompactly on $\mathbb{H}^3$.*

The next question was asked by Gromov, it is commonly refered as the *surface subgroup question.* This question was motivated by the virtual Haken conjecture (now proved by Agol) which says that every compact, irreducible 3-manifold M with infinite fundamental group has a finite cover which contains a 2-sided imcompressible surface S. In particular, this implies that $\pi_1(M)$ contains a surface subgroup, that is $\pi_1(S) \leq \pi_1(M)$.

Question 10.3. *Does every one-ended hyperbolic group contain a surface subgroup?*

A group G is *residually finite* if for all $g \in G$, there exists a fintie group H and a homomorphism $f\colon G \to H$ such that $f(g) \neq 1$. Equivalently, G is residually finite if and only if

$$\bigcap_{H \leq G, [G:H] < \infty} H = \{1\}$$

Loosely speaking, residually finite groups can be approximated by fintie groups. These approximations give can be used to deduce properties of the group, for example residually finite groups always have solvable word problem.

It is a classical result that free group and surface groups are residually finite. Topologically, the fact that surface groups are residually finite means that for every curve on a surface S, there is a cover of S where the lift of this curve is not closed. In addition, all 3-manifold groups are residually finite by work of Hempel and the geometrization theorem.

The following is probably the most famous open question in geometric group theory:

Question 10.4. *Is every hyperbolic group residually finite?*

Recall that a group is *linear* if it is isomorphic to a subgroup of $Gl(n, K)$ for some field K (usually $K = \mathbb{C}$).

Exercise 10.5. Prove that $Sl(n, \mathbb{Z})$ is residually finite.

It is a classical theorem of Mal'cev that all finitely generated linear groups are residually finite. So one way to prove a group is residually finite is to show that it is linear. However, M. Kapovich showed that there exist hyperbolic groups which are not linear, so this approach will not apply.

A group is *Hopfian* if every surjective homomorphism $f\colon G \to G$ is also injective, or in other words G is not a proper quotient of itself. It is also a theorem of Mal'cev that residually finite groups are always Hopfian. To show a group is not Hopfian one only has to find a single map $f\colon G \to G$ violating the condition, so it is often easier to show a group is not Hopfian then not residually fintie. For example a direct calculation can be used to show that $BS(2, 3)$ is not Hopfian, and hence not residually finite. However, again this approach will one be usefully if one wants to find a hyperbolic group which is not residually finite. The following theorem is a deep result of Sela, who stated the result only for torsion-free groups. It was extended to groups with torsion by Reinfeldt-Weidmann.

Theorem 10.6. *Every hyperbolic group is Hopfian.*

Let M_0 denote the standard Euclidian plane and M_{-1} denote the standard hyperbolic plane. For $\kappa < 0$, let M_κ denote the space obtained from the hyperbolic plane by scaling the metric until the sectional curvature is κ. Given a geodesic metric space X and three points $x_1, x_2, x_3 \in X$ spanning a triangle T, a triangle T' in M_κ with vertices x_1', x_2', x_3' is called a *comparison triangle* if $d_{M_\kappa}(x_i', x_j') = d_X(x_i, x_j)$ for all $1 \leq i, j \leq 3$. Then for each point a on T, there is point a' on T' which naturally corresponds to a. X is called a $CAT(\kappa)$-space if for all geodesic triangles T in X and any two points a, b on distinct sides of T,

$$d_X(a, b) \leq d_{M_\kappa}(a', b')$$

If $\kappa < 0$, then clearly every $CAT(\kappa)$ space is hyperbolic. However, the converse is not true since δ-hyperbolicity does not give any restrictions on triangles whose diameter is less then δ.

Question 10.7. *Does every hyperbolic group admit a proper, cocompact action on a $CAT(\kappa)$ space for some $\kappa < 0$?*

Finally we close with the following which is currently one of the main themes of geometric group theory.

Question 10.8 (meta-question)**.** *Which properties of hyperbolic groups extend to other classes of groups which admit natural actions on negatively or non-positively curved metric spaces, specifically relatively hyperbolic groups, $CAT(0)$ groups, semi-hyperbolic groups, $MCG(S)$, $Out(F_n)$, 3-manifold groups, etc.*

The remaining sections were contributed by students.

11 Dimension and Boundary by Daniel Ingebretson

In this section, we will give a short proof that the boundary of a δ hyperbolic group in the sense of Gromov has finite Hausdorff dimension with respect to the visual metric.

11.1 Hausdorff dimension

Here we review some of the basics of Hausdorff dimension for a metric space (X, d).

Definition 11.1. Let $s \geq 0$ and $\varepsilon > 0$ be real numbers. The *s-dimensional Hausdorff measure of X with respect to ε* is denoted by $H^s_\varepsilon(X)$, and is given by the following formula.

$$H^s_\varepsilon(X) = \inf\left\{\sum_{i=0}^{\infty} \operatorname{diam}(U_i)^s : \{U_i\} \text{ open cover of } X \text{ with diam } \{U_i\} < \varepsilon\right\}$$

Clearly H^s_ε decreases as ε decreases, so the following limits exists, although it may be (and often is) 0 or ∞.

$$H^s(X) = \lim_{\varepsilon \to 0} H^s_\varepsilon(X)$$

This limit is called the *s-dimensional Hausdorff measure of X* and induces an outer measure on X. If $X = \mathbb{R}^n$, this measure generalizes the product Lebesgue measure on $\mathbb{R}^n$. Specifically, for $A \subset \mathbb{R}^n$, $H^n(A)$ is a constant multiple of the Lebesgue measure of A.

Fact 11.2. If $s, t, u \in \mathbb{R}^+$ such that $s < t < u$ and $0 < H^t(X) < \infty$, then $H^u(X) = 0$ and $H^s(X) = \infty$

This implies that there is a critical value of s at which $H^s(X)$ jumps from ∞ to 0. At this point, $H^s(X)$ maybe 0, ∞, or finite. This critical value is called the *Hausdorff dimension* $\dim_H(X)$ of X and is given by

$$\dim_H(X) = \inf\{s \geq 0 : H^s(X) = 0\} = \sup\{s \geq 0 : H^s(X) = \infty\},$$

where we observe the convention that $\inf \emptyset = 0$.

11.2 Boundary of hyperbolic space

Let (X, d) be a geodesic δ hyperbolic metric space. Let $R(X)$ denote the set of all geodesic rays in X. We say two rays γ and ξ are asymptotically equivalent (and write $\gamma \sim \xi$) if the Hausdorff distance between their images is finite. A better (and equivalent) way of saying this is that

$$\sup_{0 \leq t < \infty} d(\gamma(t), \xi(t)) < \infty.$$

The boundary ∂X is defined to be $R(X)/\sim$.

It is also convenient to view the boundary as a limit of equivalence classes of sequences. Let $S(X)$ be the set of all sequences $x_n \in X$ that *diverge to infinity*, that is, satisfying the requirement

$$\liminf_{m,n \to \infty} (x_n | x_m)_{x_0} = \infty.$$

for any $x_0 \in X$. We say two such sequences x_n, y_m are asymptotically equivalent (and write $x_n \sim y_m$ if they satisfy the following identity.

$$\liminf_{m,n \to \infty} (x_n | y_m)_{x_0} = \infty$$

The set of all equivalence classes of sequences diverging to infinity is denoted by $S(X)/\sim$. An easy way to manufacture such sequences is to evaluate a unit-speed geodesic ray at integer points:

$$[\gamma] \mapsto [\{\gamma(n)\}_{n=0}^{\infty}]$$

Similarly, we may order a sequence of points diverging to infinity and connect them pairwise by geodesics. This is asymptotic to a geodesic ray diverging to infinity. In this way, we establish a bijective correspondence between $R(X)/\sim$ and $S(X)/\sim$. We will use these interchangeably to describe the boundary ∂X. Particularly, the topology on ∂X is the topology of pointwise convergence (to infinity) of equivalence classes of sequences in $S(X)/\sim$.

11.3 Visual boundary

To compute the Hausdorff dimension of ∂X, it is necessary to define a metric on ∂X that is compatible with the topology introduced above. First, let's extend the notion of Gromov product to the boundary ∂X. We will use the notation x_∞ to refer to an element of ∂X, and interchangeably describe x_∞ as an equivalence class of the image of a geodesic ray, or of a sequence diverging to infinity.

Definition 11.3. If x_∞ and y_∞ are elements of ∂X, we define their *Gromov product* with respect to any basepoint $x_0 \in X$ to be

$$(x_\infty | y_\infty)_{x_0} = \sup_{\substack{x_n \to x_\infty \\ y_m \to y_\infty}} \liminf_{m,n \to \infty} (x_n | y_m)_{x_0}.$$

Since the previous definitions of the boundary were independent of basepoint, so is the above Gromov product. For that reason we will hereafter suppress the basepoint x_0 that appears in the above definition.

Now let $\varepsilon > 0$; we will refer to this as the *visual parameter*. We make the following definition.

$$\rho_\varepsilon(x_\infty, y_\infty) = e^{-\varepsilon(x_\infty|y_\infty)}$$

Note that ρ_ε enjoys many properties of a metric- it is clearly symmetric with respect to x_∞ and y_∞, and since $x_\infty = y_\infty$ if and only if $(x_\infty|y_\infty) = \infty$ (by Definition 3.1 and the description of ∂X via $S(X)$ in the previous section), we have $\rho_\varepsilon(x_\infty, y_\infty) = 0$ if and only if $x_\infty = y_\infty$. However, ρ_ε has the defect of failing the triangle inequality. Note that the extension to ∂X of the Gromov product inherits the usual identity

$$(x_\infty|y_\infty) \geq \min\{(x_\infty|z_\infty), (z_\infty|y_\infty)\} - \delta,$$

which implies that ρ_ε satisfies the following quasi-ultrametric property.

$$\rho_\varepsilon(x_\infty, y_\infty) \leq e^{-\varepsilon\delta} \max\{\rho_\varepsilon(x_\infty, z_\infty), \rho_\varepsilon(z_\infty, y_\infty)\}$$

To remedy this situation, we will define the visual metric d_ε as the infimum over all chains of distances ρ_ϵ joining two points on the boundary. In this way we trade simplicity for the necessary triangle inequality. Let's make this precise:

Definition 11.4. If $x_\infty, y_\infty \in \partial X$, define the *visual metric* d_ε by the following formula.

$$d_\varepsilon(x_\infty, y_\infty) = \inf\left\{\sum_{i=0}^{N-1} \rho_\varepsilon(x_\infty^i, x_\infty^{i+1}) : x_\infty^i \in \partial X,\ x_\infty^0 = x_\infty \text{ and } x_\infty^N = y_\infty\right\}$$

It can be shown that for any $\varepsilon > 0$, d_ε induces the topology on ∂X by the following inequalities.

$$(3 - 2e^{-\varepsilon})\rho_\varepsilon(x_\infty, y_\infty) \leq d_\varepsilon(x_\infty, y_\infty) \leq \rho_\varepsilon(x_\infty, y_\infty)$$

Furthermore, ∂X is compact in the metric d_ε.

11.4 Dimension of the boundary

The previous discussion applies to any δ hyperbolic geodesic metric space. Now we will apply these results to a δ hyperbolic group G that is generated by a finite symmetric set Γ. Let $S(n)$ be the sphere of radius n in the Cayley graph of G.

$$S(n) = \{g \in G : d(e, g) = n\}$$

Notice that $\#S(n) \leq (\#\Gamma)^n$, with equality if and only if G is a free group. Cover $S(n)$ by the collection of open unit balls $B(n) = \{B_1^n(z) : z \in Z_n\}$, where $z \in Z_n \subset S(n)$, and this Z_n is chosen so that $B(n)$ has minimal cardinality with respect to the covering property. Putting a unit ball around each element of $S(n)$ will certainly result in a cover of $S(n)$, so because we are defining $B(n)$ to have minimal cardinality, we obtain $\#Z_n \leq (\#\Gamma)^n$. We will define an open cover of the boundary $U(n)$ in terms of $B(n)$ in the following way. For each $n \in \mathbb{N}$ and $z \in Z_n$, define $U_z^n \subset \partial X$ as follows.

$$U_z^n = \{x_\infty \in \partial G : \text{ there exists } \gamma \text{ representing } x_\infty \text{ such that im } \gamma \cap B_1^n(z) \neq \emptyset\}$$

Then define $U(n) = \{U_z^n : z \in Z_n\}$. Notice here we are using the description of ∂G as equivalence classes of rays in $R(G)$.

To bound the Hausdorff dimension of ∂G, we will need the following argument.

Lemma 11.5. *If $x_\infty, y_\infty \in U_z^n$, then $(x_\infty | y_\infty) \geq n-3$.*

Proof. Since $x_\infty, y_\infty \in U_z^n$, then there exist rays γ and ξ representing x_∞ and y_∞, respectively, such that $\operatorname{im}\gamma \cap B_1^n(z) \neq \emptyset$ and $\operatorname{im}\xi \cap B_1^n(z) \neq \emptyset$. Reparametrize these rays by arc length if necessary, so that there exist s and t such that $\gamma(s), \xi(t) \in B_1^n(z)$. Furthermore, since the diameter of the unit ball $B_1^n(z)$ is 2, we know that $n-2 \leq s, t \leq n+2$.

$$\begin{aligned} d(\gamma(n+i), \xi(n+j)) &\leq d(\gamma(n+i), \gamma(s)) + d(\gamma(s), \xi(t)) + d(\xi(t), \xi(n+j)) \\ &\leq (i+2) + 2 + (j+2) \\ &= i+j+6 \end{aligned}$$

Then using the definition of the Gromov product with respect to some basepoint x_0, we have

$$\begin{aligned} (\gamma(n+i)|\xi(n+j))_{x_0} &= \frac{1}{2}\left[d(\gamma(n+i), x_0) + d(\xi(n+j), x_0) - d(\gamma(n+i), \gamma(n+j))\right] \\ &\geq \frac{1}{2}\left[(n+i) + (n+j) - (i+j+6)\right] \\ &= \frac{1}{2}(2n-6) \\ &= n-3. \end{aligned}$$

Taking $i, j \to \infty$ we obtain the desired inequality. $\square$

Using the fact that $d_\varepsilon(x_\infty, y_\infty) \leq \rho_\varepsilon(x_\infty, y_\infty)$, and the above computation, we know that if $x_\infty, y_\infty \in U_z^n$,

$$d_\varepsilon(x_\infty, y_\infty) = e^{-\varepsilon(x_\infty|y_\infty)} \leq e^{-\varepsilon(n-3)}.$$

This demonstrates that the diameter of each U_z^n is less than $e^{\varepsilon(3-n)}$ in the visual metric d_ε.

Proposition 11.6. *Let G be a δ hyperbolic group generated by finite symmetric Γ. Suppose $\varepsilon > 0$ is a visual parameter for the metric inducing the topology of ∂G. Then the Hausdorff dimension of ∂G in this metric is bounded above by $\dfrac{\log \#\Gamma}{\varepsilon}$.*

Proof. Using the cover $U(n) = \{U_n^z : z \in Z_n\}$ defined above, we have

$$\begin{aligned} \sum_{z \in Z_n} \operatorname{diam}(U_z^n)^s &\leq \#Z_n e^{\varepsilon(3-n)s} \\ &\leq (\#\Gamma)^n e^{\varepsilon(3-n)s} \\ &= e^{3s\varepsilon} e^{(\log \#\Gamma - s\varepsilon)n}. \end{aligned}$$

Observe that this quantity goes to 0 as $n \to \infty$ if and only if $s > \frac{\log \#\Gamma}{\varepsilon}$. Since $\dim_H(\partial G) = \inf\{s \geq 0 : H^s(\partial G) = 0\}$, we have $\dim_H(\partial G) \leq \dfrac{\log \#\Gamma}{\varepsilon}$. $\square$

In the above argument, the only group structure that was used was the fact that the number of unit balls needed to cover the sphere of radius n in the Cayley graph of G was exponential in n. The above arguments applies to more general δ hyperbolic metric spaces (that includes such groups) that are said to *grow at most exponentially.* Specifically, for each fixed $x_0 \in X$, let $S(r) = \{x \in X : d(x_0, x) = r\}$. If $S(r)$ contains a finite set Z_r such that $\{B_1(z)\}_{z \in Z_r}$ covers $S(r)$ and $\#Z_r \sim e^r$, then *mutatis mutandis*, the above arguments show that the visual boundary of X will have finite Hausdorff dimension. Examples of metric spaces that grow at most exponentially include simply connected complete Riemannian manifolds of negative sectional curvature.

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